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On the stability of ring relative equilibria in the N-body problem on $\mathbb{S}^2$ with Hodge potential

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Abstract. In this paper, we study the stability of the ring solution of the N-body problem in the entire sphere $\mathbb{S}^2$ by using the logarithmic potential proposed in Boatto et al. (2016, *Proceedings of the Royal Society of London. Series A. Mathematical, Physical and Engineering Sciences* 472, 20160020) and Dritschel (2019, *Philosophical Transactions of the Royal Society of London. Series A. Mathematical, Physical and Engineering Sciences* 377, 20180349), derived through a definition of central force and Hodge decomposition theorem for 1-forms in manifolds. First, we characterize the ring solution and study its spectral stability, obtaining regions (spherical caps) where the ring solution is spectrally stable for $2 \leq N \leq 6$, while, for $N \geq 7$, the ring is spectrally unstable. The nonlinear stability is studied by reducing the system to the homographic regular polygonal solutions, obtaining a 2-d.o.f. Hamiltonian system, and therefore some classic results on stability for 2-d.o.f. Hamiltonian systems are applied to prove that the ring solution is unstable at any parallel where it is placed. Additionally, this system can be reduced to 1-d.o.f. by using the angular momentum integral, which enables us to describe the phase portraits and use them to find periodic ring solutions to the full system. Some of those solutions are numerically approximated.

1 Introduction

The study of N-body problems over curved spaces has its origins in the works of Bolyai [4] and Lobachevsky, on the theory of parallels, published in German in 1849 (see part of this work translated into English and published over a century later in [9]). In [11], the gravitational potential is extended to the sphere $\mathbb{S}^2$, where the potential obtained is of cotangent type. Using the cotangent potential leads to some nonintuitive behavior at a physical level, since it corresponds to a potential defined for a sphere without a point, namely, the punctured sphere $\mathbb{S}^2_p$, which gives rise to antipodal singularities (see, e.g., [2]).

This paper considers the potential derived by considering the unit sphere from an intrinsic geometry point of view and using the Hodge decomposition theorem to derive the central force extension to the various surfaces of interest. In the case of the sphere, the resulting potential is a logarithmic one and it is defined over the entire

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sphere $\mathbb{S}^2$. For more details, see [1, 2]. Additionally, the corresponding Hodge potential for closed hyper-surfaces is derived by Dritschel in [5].

Thus, for the unit sphere $\mathbb{S}^2$, a system of N masses $m_1, m_2, \ldots, m_N$ with positions $r_1, r_2, \ldots, r_N$, respectively, where $r_j = (\varphi_j, \theta_j)$ (with φ_j the longitude and θ_j the co-latitude), the potential energy of the system is a function $U(r)$, with

$$U(r) = \gamma \sum_{j=1}^{N} \sum_{k>j}^{N} m_j m_k \ln(1 - d_{jk}), \tag{1.1}$$

where $r = (r_1, r_2, \ldots, r_N)$, $d_{jk} = \cos\theta_j \cos\theta_k + \sin\theta_j \sin\theta_k \cos(\varphi_j - \varphi_k)$ and γ is the gravitational constant.

We will focus on the analysis of stability of a ring of bodies lying on a fixed parallel and rotating uniformly with respect to the z-axis. For this purpose, we consider N bodies with identical masses $m_1 = \cdots = m_N = 1$ in the Hamiltonian system associated with the potential (1.1). We will show that the regular N-gon configuration is a solution with position $\varphi_j = \nu t + \phi_j$, $\phi_j = \frac{2\pi(j-1)}{N}$, $\theta_j = \theta_0 \in (0, \pi)$, for $j = 1, \ldots, N$ and $\nu \neq 0$, if and only if, the angular velocity is taken as

$$\nu = \frac{\sqrt{N-1}}{\sin\theta_0}.$$

In addition to the aforementioned articles, we can find more recent works in which similar problems defined in curved surfaces are studied for both the N-vortex problem and the N-body problem. The linear stability of a ring-poles configuration with total vorticity equal to zero is studied in [3], whereas the linear stability of a ring solution in an infinite cylinder is studied in [1]. Furthermore, the problem of determining the stability of a ring of bodies in the sphere $\mathbb{S}^2$ with a cotangent potential has been recently studied in [8], obtaining results about spectral instability for a ring close to the poles and close to the equator. For logarithmic potential, this problem has not been studied previously and it will be totally characterized in this paper.

1.1 Equations of motion

We consider a system of N masses $m_1, m_2, \ldots, m_N$ with positions $r_1, r_2, \ldots, r_N$, respectively, where $r_j = (\varphi_j, \theta_j)$ (with φ_j the longitude and θ_j the co-latitude). The potential energy of the system is the function $U(r)$, where $r = (r_1, r_2, \ldots, r_N)$ and $U(r)$ is the potential energy obtained in [2] and given in (1.1). On the other hand, if $v_1, v_2, \ldots, v_N$ are the velocity vectors, then the kinetic energy is defined by

$$\mathcal{K} = \frac{1}{2} \sum_{j=1}^{N} m_j v_j^T g_j v_j, \tag{1.2}$$

where g is the metric tensor

$$g_j = \begin{pmatrix} \sin^2\theta_j & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.3}$$

Thus, the Lagrangian of the system is $\mathcal{L} = \mathcal{K} - U$ and the Legendre transformation allows to write the velocities in terms of the momentum $p_j = (p_{\varphi_j}, p_{\theta_j})$. In fact,

$$p_j = \frac{\partial \mathcal{K}}{\partial v_j} = m_j v_j^T g_j \Rightarrow v_j = \frac{1}{m_j}(g_j^{-1})^T p_j^T,$$

and the kinetic energy in terms of the momentum takes the form

$$\mathcal{K} = \sum_{j=1}^{N} \frac{1}{2m_j} p_j (g_j^{-1})^T p_j^T = \sum_{j=1}^{N} \frac{1}{2m_j}(p_{\varphi_j}^2 \csc^2\theta_j + p_{\theta_j}^2). \tag{1.4}$$

The Hamiltonian function is given by

$$H(Q,P) = \mathcal{K}(Q,P) + U(Q), \tag{1.5}$$

with $Q = (r_1, r_2, \dots, r_N)$ and $P = (p_1, p_2, \dots, p_N)$. Thus, the corresponding Hamiltonian equations take the form

$$\begin{aligned}
\dot\varphi_j &= \frac{\partial H}{\partial p_{\varphi_j}} = \frac{1}{m_j}\csc^2\theta_j p_{\varphi_j},\\
\dot\theta_j &= \frac{\partial H}{\partial p_{\theta_j}} = \frac{1}{m_j} p_{\theta_j},\\
\dot p_{\varphi_j} &= -\frac{\partial H}{\partial \varphi_j} = -\gamma \sum_{k=1,k\neq j}^{N} \frac{m_j m_k \sin\theta_j \sin\theta_k \sin(\varphi_j - \varphi_k)}{1 - d_{jk}},\\
\dot p_{\theta_j} &= -\frac{\partial H}{\partial \theta_j} = \frac{1}{m_j}\cot\theta_j \csc^2\theta_j p_{\varphi_j}^2\\
&\quad - \gamma \sum_{k=1,k\neq j}^{N} \frac{m_j m_k \left(\cos\theta_k \sin\theta_j - \cos\theta_j \sin\theta_k \cos(\varphi_j - \varphi_k)\right)}{1 - d_{jk}}.
\end{aligned} \tag{1.6}$$

1.2 Symmetries

The Hamiltonian (1.5) is invariant under the action of $SO(3)$, in particular, by rotations of $SO(2)$ about the z-axis, which implies the conservation of the total φ component of the angular momentum $P_\varphi = \sum p_{\varphi_j}$. Additionally, it is easy to see that the Hamiltonian function (1.5) also has the translation time-dependent symmetry $\varphi_j \mapsto \varphi_j + \nu t$.

1.3 Ring solution

Let us consider a system of N bodies with identical masses $m_1 = \cdots = m_N = m$. By this assumption, the Hamiltonian function becomes simpler. Moreover, we may assume without lost of generality that $m = \gamma = 1$. Indeed, it is achieved by introducing the $\left(\frac{1}{c}\right)$–symplectic change of coordinates $(Q,P) \mapsto (Q, cP)$ and the scaling $x \mapsto \frac{m}{c}x$ in time and energy, with $c = \gamma^{1/2} m^{3/2}$. Thus, we arrive to the Hamiltonian function

$$\mathcal{H} = \sum_{j=1}^{N} \frac{1}{2}(p_{\varphi_j}^2 \csc^2\theta_j + p_{\theta_j}^2) + \sum_{j=1}^{N}\sum_{k>j}^{N} \ln(1 - d_{jk}). \tag{1.7}$$

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Now, we consider a polygonal configuration formed by N identical masses placed at the vertices of a regular polygon, rotating with respect to its normal vector with constant angular velocity. Due to the $SO(3)$-symmetry, we can consider without loss of generality that the polygon is rotating in a fixed latitude around the z-axis. Such a particular configuration must be of the form

$$\varphi_j = \nu t + \phi_j, \quad \theta_j = \theta_0 \in (0, \pi), \quad p_{\varphi_j} = \nu \sin^2\theta_0, \quad p_{\theta_j} = 0, \tag{1.8}$$

where $\phi_j = \frac{2\pi(j-1)}{N}$ and ν is the constant angular velocity. It is verified that if $\theta_0 \neq \frac{\pi}{2}$, then the configuration (1.8) will be a relative equilibrium of the system associated with (1.7), if and only if the angular velocity satisfies

$$\nu^2 = \frac{N-1}{\sin^2\theta_0}. \tag{1.9}$$

For $\theta_0 = \frac{\pi}{2}$, the formula (1.9) is no longer valid and we get the nonisolated solution

$$\varphi_j = p_\varphi t + \phi_j, \quad \theta_j = \frac{\pi}{2}, \qquad p_{\varphi_j} = p_\varphi, \quad p_{\theta_j} = 0. \tag{1.10}$$

The structure of this work is as follows. In Section 1, we describe the equations of motion for the N-body problem over the entire sphere $\mathbb{S}^2$ and for identical masses $m_1 = \cdots = m_N$, we can find a particular solution called *ring solution*. In Section 2, the spectral stability of the ring solution of the complete problem is studied, obtaining spectral stability for $2 \le N \le 6$ and spectral instability for $N \ge 7$. In Section 3, on the space of homothetic solutions, we reduce the system to two and one degrees of freedom, respectively. Additionally, we calculate the equilibrium solutions of both reduced systems and study the phase portraits near the equilibria of the reduced system with one degree of freedom. To examine the stability of the ring solution, we normalize the Hamiltonian to terms of degree three via the Lie algorithm, and we determine that the ring solution is unstable in the Lyapunov sense. Finally, in Section 4, we give conditions for the existence of periodic orbits in the complete system along with some examples.

2 Spectral stability of the ring solution

From now on, we move to a co-rotating frame associated with the solution (1.8) ($\theta_0 \neq \pi/2$). Hence, it becomes a fixed equilibrium of the new Hamiltonian system. To analyze the spectral stability of the ring solution, we compute the Hessian matrix evaluated at the solution (1.8)

$$D_z^2\mathcal{H}(z) = \begin{pmatrix} S & 0_N & 0_N & 0_N \\ 0_N & R & \alpha I_N & 0_N \\ 0_N & \alpha I_N & \beta I_N & 0_N \\ 0_N & 0_N & 0_N & I_N \end{pmatrix}, \tag{2.1}$$

where $z = (\varphi_1, \dots, \varphi_N, \theta_1, \dots, \theta_N, p_{\varphi_1}, \dots, p_{\varphi_N}, p_{\theta_1}, \dots, p_{\theta_N})$, $\alpha = -2\nu\cot\theta_0$, $\beta = \csc^2\theta_0$, 0_N, and I_N are the zero and identity matrix of order $N \times N$, respectively, and

$$S = sI_N + P_N, \quad R = rI_N - \beta P_N,$$

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with

$$s = -\frac{1}{6}(N^2-1), \quad r = \nu^2\left(2+\cos 2\theta_0\right) + \frac{\beta}{6}(N-5)(N-1), \quad P_N = \left(\rho_{ij}\right)_{N\times N},$$

$$\rho_{ij} = \begin{cases} \frac{1}{1-\cos\left(\frac{2\pi(i-j)}{N}\right)}, & i \neq j, \\ 0, & i = j. \end{cases}$$

The spectral stability of the solution (1.8) is determined through the eigenvalues of the matrix $JD^2\mathcal{H}$, where J is the standard symplectic matrix. The characteristic polynomial is

$$p(\lambda) = \det[D^2\mathcal{H} + \lambda J] = \beta^N \det\left[-\beta P_N^2 + \delta P_N + g(\lambda) I_N\right],$$

where $\delta = -\alpha^2/\beta + r - \beta s$, $g(\lambda) = 1/\beta\lambda^4 + (r/\beta + s)\lambda^2 + rs - \alpha^2 s/\beta$ and P_N is a circulant matrix

$$P_N = \begin{pmatrix} p_1 & p_2 & p_3 & \cdots & p_{N-2} & p_{N-1} & p_N \\ p_N & p_1 & p_2 & \cdots & p_{N-3} & p_{N-2} & p_{N-1} \\ p_{N-1} & p_N & p_1 & \cdots & p_{N-4} & p_{N-3} & p_{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ p_2 & p_3 & p_4 & \cdots & p_3 & p_2 & p_1 \end{pmatrix} := Circ\left[p_1, p_2, \ldots, p_N\right],$$

with $p_1 = 0$ and $p_k = \rho_{k1}$, $k = 2, \ldots, N$. Furthermore, taking into account the following relation:

$$p_j = p_{N-j+2}, \qquad j = 2, \ldots, N, \tag{2.2}$$

we have that P_N is a symmetric circulant matrix. Therefore, the matrix $\Lambda_N := -\beta P_N^2 + \delta P_N + g(\lambda) I_N$ is also symmetric circulant, i.e.,

$$\Lambda_N := Circ\left[x_1, x_2, \ldots, x_N\right], \text{ with } x_j = x_{N-j+2}, \ j = 2, \ldots, N.$$

Thus, ignoring the constant factor β^N, we get the characteristic polynomial

$$p(\lambda) = \det \Lambda_N = \prod_{j=1}^{N}\left(x_1 + x_2\omega_j + x_3\omega_j^2 + \cdots + x_N\omega_j^{N-1}\right), \tag{2.3}$$

where ω_j is the jth root of unity. The eigenvalues of the matrix P_N are given by

$$\tau_j = p_1 + p_2\omega_j + p_3\omega_j^2 + \cdots + p_N\omega_j^{N-1}, j = 1, \ldots, N,$$

and denoting $P_N^2 = Circ\left[q_1, q_2, \ldots, q_N\right]$, it is verified that the eigenvalues of P_N^2 are given by

$$\sigma_j = q_1 + q_2\omega_j + q_3\omega_j^2 + \cdots + q_N\omega_j^{N-1} = \tau_j^2.$$

See [6] for more details about circulant matrices results. Now, from the definition of Λ_N it follows that

$$x_1 = -\beta q_1 + g(\lambda), \text{ and } x_k = -\beta q_k + \delta p_k, \quad k = 2, \ldots, N. \tag{2.4}$$

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By replacing the quantities (2.4) into (2.3), we arrive at

$$p(\lambda) = \prod_{j=1}^{N} \left(g(\lambda) + \delta\tau_j - \beta\tau_j^2 \right). \tag{2.5}$$

Note that from condition (2.2), using the fact that $\omega_j^{N-k+1} = \overline{\omega}_j^{k-1}$ and in virtue of the formula given in [7, p. 271], we obtain an explicit expression for the eigenvalues of P_N, namely

$$\tau_j = \sum_{m=1}^{N-1} p_{m+1} \cos\left(\frac{2\pi m j}{N}\right) = \sum_{m=1}^{N-1} \frac{\cos\left(\frac{2\pi m j}{N}\right)}{1-\cos\left(\frac{2\pi m}{N}\right)} = \frac{1}{6}(N^2-1) - j(N-j).$$

Thus, we obtain that the $4N$ eigenvalues read as follows:

$$\lambda_{1j} = \frac{1}{\sqrt{2}}\sqrt{a+\sqrt{b_j}},\ \lambda_{2j} = \frac{1}{\sqrt{2}}\sqrt{a-\sqrt{b_j}},\ \lambda_{3j} = -\lambda_{1j},\ \lambda_{4j} = -\lambda_{2j}, \tag{2.6}$$

with $j = 1, \ldots, N$, and

$$\begin{aligned} a &= \ -r - s\beta, \\ b_j &= \ 4\alpha^2 s + (r-\beta s)^2 + 4\tau_j\left(\alpha^2 + \beta^2\tau_j - \beta r + \beta^2 s\right). \end{aligned} \tag{2.7}$$

By replacing the expressions $r, s, \alpha, \beta, \tau_j$ in terms of θ_0, N, and j, we obtain

$$\lambda_{1j} = \sqrt{-(j-1)(j-N+1)-(N-1)\cos 2\theta_0}\,\csc\theta_0, \quad \lambda_{2j} = i\sqrt{j(N-j)}\,\csc\theta_0,$$

where we observe the following properties:

(i) $\lambda_{2j} \in i\mathbb{R}$, for all $j = 1, \ldots, N-1$.
(ii) $\lambda_{2N} = 0$.
(iii) $\lambda_{pj} = \lambda_{p,N-j}$ for all $j = 1, \ldots, \lfloor \frac{N-1}{2} \rfloor$, $p = 1, 2$.

Note that property (iii) implies that for N even, $\pm\lambda_{p,\frac{N}{2}}$, $\pm\lambda_{1N}$ are the only eigenvalues with multiplicity one, while the remaining $4N-6$ eigenvalues have multiplicity two. For N odd, $\pm\lambda_{1N}$ are the only eigenvalues with multiplicity one, while the remaining $4N-2$ eigenvalues have multiplicity two.

Finally, the spectral stability depends on the eigenvalues λ_{1N}. It is easy to verify that

$$\lambda_{max} := \sqrt{\max_j \lambda_{1j}^2} = \lambda_{1,\lfloor\frac{N}{2}\rfloor} = \begin{cases} \sqrt{\left(1-\frac{N}{2}\right)^2 - (N-1)\cos 2\theta_0}\,\csc\theta_0, & \text{if } N \text{ is even}, \\ \frac{1}{2}\sqrt{(N-1)(N-3-4\cos 2\theta_0)}\,\csc\theta_0, & \text{if } N \text{ is odd}, \end{cases}$$

and it follows that $\lambda_{max}^2 \le 0$, if and only if, $\theta_0 \in (0, \Theta_N] \cup [\pi - \Theta_N, \pi)$ with

$$\Theta_N = \frac{1}{2}\begin{cases} \arccos\left(\frac{(N-2)^2}{4(N-1)}\right), & \text{if N is even}, \\ \arccos\left(\frac{N-3}{4}\right), & \text{if N is odd}, \end{cases}$$

which is a real number only for $2 \le N \le 6$.

Therefore, we may conclude the following result.

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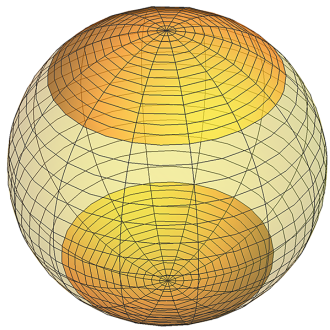

Figure 1: Shaded regions correspond to zones of $\mathbb{S}^2$ where the ring solution is spectrally stable. These regions exist only for $2 \le N \le 6$.

Theorem 2.1. *For $2 \le N \le 6$, there exists $\Theta_N \in (0, \pi)$ such that the ring solution is spectrally stable for any $\theta_0 \in (0, \Theta_N) \cup (\pi - \Theta_N, \pi)$ (see Figure 1) if $N = 2, 3$ and any $\theta_0 \in (0, \Theta_N] \cup [\pi - \Theta_N, \pi)$ if $N = 4, 5, 6$. For $N \ge 7$, the ring solution is unstable at any parallel where the ring is located.*

The stability of the ring placed at the equator must be studied separately. Since the set $\theta_1 = \cdots = \theta_N = \pi/2$ is an invariant set, we can restrict the system to the equator. The equations of motion restricted to the equator and written in a co-rotating frame are given by the N-d.o.f. Hamiltonian system:

$$\begin{aligned} \dot{\phi}_j &= p_{\phi_j} - \nu, \\ \dot{\theta}_j &= 0, \\ \dot{p}_{\phi_j} &= -\sum_{k=1, k \ne j}^{N} \frac{\sin(\phi_j - \phi_k)}{1 - \cos(\phi_j - \phi_k)}, \\ \dot{p}_{\theta_j} &= 0. \end{aligned} \tag{2.8}$$

Solution (1.10) corresponds to the fixed equilibrium $\phi_j = 2\pi(j-1)/N$, $p_{\phi_j} = \nu$. The matrix of the linearization at this equilibrium is given by

$$\mathcal{A} = \begin{pmatrix} R_N & 0_N \\ 0_N & I_N \end{pmatrix},$$

with $R_N = sI_N - P_N$, where s and P_N are defined in the previous case.

Similarly to the case $\theta_0 \ne \pi/2$, we get the eigenvalues

$$\lambda_j = \pm\sqrt{\frac{1}{3}(N^2 - 1) - j(N - j)} \in \mathbb{R},\ j = 1, \ldots, N.$$

Thus, we conclude the following result.

Theorem 2.2. *For $\theta_0 = \pi/2$ and $N \ge 2$, the ring solution* (1.10) is unstable.

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Remark 2.3 The study stability of the ring relative equilibria for cotangent potential has been recently treated in [8], where the authors prove spectral instability for a ring close to the poles for any $N \geq 2$, which differs with the results obtained in Theorem 2.1 for $2 \leq N \leq 6$.

3 Nonlinear stability of the ring solution in the space of homothetic ring solutions

3.1 Reduction

In order to study the nonlinear stability, we consider identical masses such that at any time they form a regular polygon contained in a plane parallel to the xy plane. Thus, we consider solutions with positions given by

$$r_j = (\varphi_j, \theta_j) = \left(\varphi + \frac{2(j-1)\pi}{N}, \theta\right), \; j = 1, \ldots, N. \tag{3.1}$$

By replacing (3.1) into (1.6) with the Hamiltonian given in (1.7), we obtain that φ is a cyclic variable and the system may be reduced by one degree of freedom:

$$\begin{aligned} \dot{\varphi} &= p_\varphi \csc^2\theta, \\ \dot{\theta} &= p_\theta, \\ \dot{p}_\varphi &= 0, \\ \dot{p}_\theta &= \cot\theta \csc^2\theta p_\varphi^2 - (N-1)\cot\theta. \end{aligned} \tag{3.2}$$

Thus, the reduction is carried out by fixing the integral of motion $p_\varphi = c$, and the reduced Hamiltonian function read as follows:

$$\mathcal{H}_c = \frac{1}{2}(p_\theta^2 + c^2\csc^2\theta) + (N-1)\ln(\sin\theta), \tag{3.3}$$

with associated Hamiltonian system:

$$\begin{aligned} \dot{\theta} &= p_\theta, \\ \dot{p}_\theta &= c^2\cot\theta\csc^2\theta - (N-1)\cot\theta. \end{aligned} \tag{3.4}$$

3.2 Equilibria and periodic solutions

The equilibrium points of (3.4) give rise to periodic solutions of the two degrees of freedom system (3.2) of the form

$$z(t) = (vt + \varphi_0, \theta_0, v\sin^2\theta_0, 0),$$

where v is either a constant depending on (θ_0, N) (for a nonequatorial ring) or an arbitrary value (for an equatorial ring). Furthermore, if $\theta_0 \neq \frac{\pi}{2}$, then this periodic solution coincides with the ring solution defined in Section 1.3. Such a particular periodic solution can also be obtained as a fixed equilibrium of the system (3.2)

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provided a co-rotating frame, i.e., by introducing the change of coordinates $\varphi = \nu t + \phi$ to obtain the new Hamiltonian

$$\mathcal{H} = \frac{1}{2}(\csc^2\theta p_\phi^2 + p_\theta^2) - \nu p_\phi + (N-1)\ln(\sin\theta), \tag{3.5}$$

with Hamiltonian equations:

$$\begin{aligned} \dot{\phi} &= p_\phi \csc^2\theta - \nu, \\ \dot{\theta} &= p_\theta, \\ \dot{p}_\phi &= 0, \\ \dot{p}_\theta &= \cot\theta\left(p_\phi^2\csc^2\theta - (N-1)\right). \end{aligned} \tag{3.6}$$

To characterize the equilibria of (3.6), we distinguish the following three cases:

(1) General equilibrium: $\nu > 0$ and $\theta \neq \pi/2$ given by

$$X_0: \ \phi = 0, \quad \theta = \theta_0 \in (0,\pi)\backslash\{\pi/2\}, \ p_\phi = \sqrt{(N-1)}\sin\theta_0, \ \text{and } p_\theta = 0. \tag{3.7}$$

(2) Equatorial equilibrium 1: $\nu \neq 0$ and $\theta = \pi/2$:

$$X_0: \ \phi = 0, \quad \theta = \theta_0 = \pi/2, \quad p_\phi = \nu, \ \text{and} \quad p_\theta = 0. \tag{3.8}$$

(3) Equatorial equilibrium 2: $\nu = 0$:

$$X_0: \ \phi = 0, \quad \theta = \theta_0 = \pi/2, \quad p_\phi = 0, \quad \text{and} \quad p_\theta = 0. \tag{3.9}$$

3.3 Dynamics in the reduced 1-d.o.f. system

Notice that the equilibrium points of the 1-d.o.f. reduced system (3.4) are of the form $(\theta, 0)$, where $\theta \in (0,\pi)$ is a zero of the function $f_c(\theta) = \cot\theta(c^2\csc^2\theta - (N-1))$, with $c = p_\phi$. Therefore, it follows that the equilibria are given by:

- The equatorial equilibrium:

$$p_\theta = 0, \quad \theta = \pi/2, \ \forall c \in \mathbb{R}. \tag{3.10}$$

- The nonequatorial equilibria:

$$p_\theta = 0, \quad \theta = \pm\arcsin\left(\frac{c}{\sqrt{N-1}}\right) \in (0,\pi), \ \forall\, 0 < |c| < \sqrt{N-1}. \tag{3.11}$$

When we consider $c = 0$, the only type of equilibria is the equatorial one (see Figure 2). In the case of nonequatorial equilibria, we have that the eigenvalues are given by

$$\lambda_{ne}^\pm = \frac{\sqrt{2(N-1)}}{c}\sqrt{c^2 - (N-1)},$$

while in the equatorial equilibrium, the eigenvalues read as follows:

$$\lambda_e^\pm = \pm\sqrt{N-1-c^2}.$$

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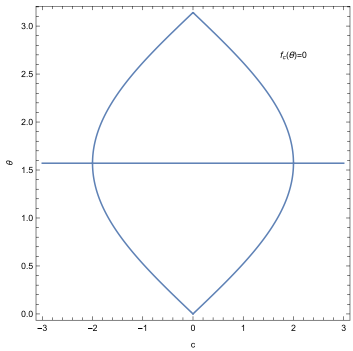
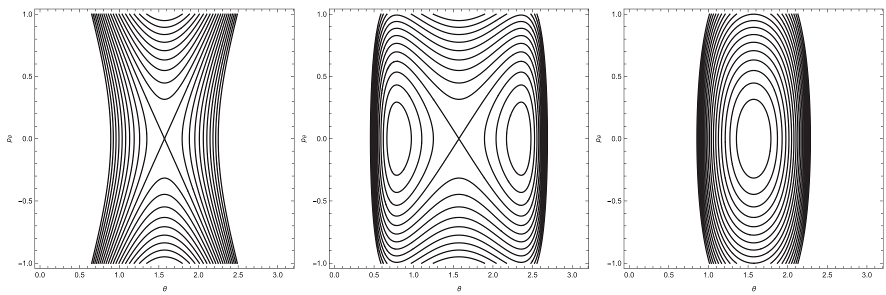


Figure 2: Equilibria bifurcation diagram of system (3.4).



(a) $c = 0$ (b) $0 < |c| < \sqrt{N-1}$ (c) $|c| \geq \sqrt{N-1}$

Figure 3: Phase portrait associated with the reduced Hamiltonian system (3.4).

Thus, if $c = 0$, the only equilibrium is the equatorial one, which is a saddle. If $0 < |c| < \sqrt{N-1}$, then the equatorial equilibrium is a saddle and the nonequatorial equilibria are centers. If $|c| \geq \sqrt{N-1}$, then the only equilibrium is the equatorial one and it is a center. Note that the values $c = \pm\sqrt{N-1}$ correspond to Hamiltonian pitchfork bifurcation values (see Figure 3).

Remark 3.1 Note that the nonequatorial relative equilibria are centers in the 1-d.o.f. reduced system, which implies that they are orbitally stable within homothetic ring solutions. On the other hand, in [8], the authors prove for the cotangent potential that the orbital stability in the reduced manifold is guaranteed only for θ in a neighborhood of $\pi/2$, $\theta \neq \pi/2$, and N odd.

3.4 Nonlinear stability of the ring solution for $\nu > 0$ and $\theta_0 \in (0, \pi/2)$

It is clear that the nonlinear stability in the 1-d.o.f. system does not guarantee the nonlinear stability in the full system (1.6), but the instability in the reduced one is, in fact, a sufficient condition to assure instability in the full system.

In this section, we make a study of the nonlinear stability of the ring solution described in case (3.7).

The linearization matrix associated with the Hamiltonian system (3.6) along the equilibrium solution defined in (3.7) is given by

$$A = JHess(\mathcal{H}(X_0)) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ -2\sqrt{N-1}\cot\theta_0\csc\theta_0 & 0 & 0 & \csc^2\theta_0 \\ -2(N-1)\cot^2\theta_0 & 0 & 0 & 2\sqrt{N-1}\cot\theta_0\csc\theta_0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

whose eigenvalues are obtained as

$$\lambda_{1,2} = 0, \quad \lambda_{3,4} = \pm i\sqrt{2(N-1)}\cot\theta_0, \tag{3.12}$$

with $N \geq 2$ and $\theta_0 \in (0, \pi/2)$ is the colatitude angle of the ring. Although this solution is stable for the reduced system (3.4), we can not ensure that the full system will inherit this property (3.6).

Let us denote by a_2 and a_4 the eigenvector and associated generalized eigenvector, respectively, associated with the null eigenvalue. Similarly, we denote by $a_1 = r_1 + is_1$ the eigenvector associated with the eigenvalue $i\omega_1$. A simple computation shows that a_2, a_4, r_1, and s_1 are given by

$$a_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \; a_4 = \begin{pmatrix} -\frac{\sin\theta_0\tan\theta_0}{\sqrt{N-1}} \\ 0 \\ 0 \\ -\sin^2\theta_0 \end{pmatrix}, \; r_1 = \begin{pmatrix} 0 \\ \frac{\sec\theta_0}{\sqrt{N-1}} \\ 1 \\ 0 \end{pmatrix}, \; s_1 = \begin{pmatrix} -\frac{\tan\theta_0}{\sqrt{2(N-1)}} \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

By using the algorithm provided in [10], we construct the symplectic matrix

$$N = [\delta_1\kappa_1 r_1 \quad \delta_2\kappa_2 a_2 \quad \kappa_1 s_1 \quad \kappa_2 a_4],$$

where $\delta_1 = sgn(\{r_1, s_1\}) = 1$, $\delta_2 = sgn(\{a_2, a_4\}) = -1$, $\kappa_1 = \frac{1}{\sqrt{|\{r_1,s_1\}|}}$, and $\kappa_2 = \frac{1}{\sqrt{|\{a_2,a_4\}|}}$, where $\{\cdot,\cdot\}$ denotes the Poisson bracket between vectors, that is, $\{u, v\} = u^T Jv$. By introducing the symplectic linear change of coordinates induced by the above matrix into the Hamiltonian (3.5), we obtain that the quadratic part assumes its normal form

$$K_2 = \frac{\omega_1}{2}(x^2 + p_x^2) - \frac{p_y^2}{2}, \quad \omega_1 = \sqrt{2(N-1)}\cot(\theta_0), \tag{3.13}$$

with its corresponding transpose quadratic term

$$K_2^T = \frac{y^2}{2} - \frac{\omega_1}{2}(x^2 + p_x^2). \tag{3.14}$$

Remark 3.1 For the case where $\nu > 0$ and $\theta \in (\pi/2, \pi)$, the same steps of the previous case are followed and we obtain that

$$K_2 = -\frac{\omega_1}{2}(x^2 + p_x^2) - \frac{p_y^2}{2} \text{ and } K_2^T = \frac{y^2}{2} + \frac{\omega_1}{2}(x^2 + p_x^2).$$

We are now assuming that the Hamiltonian (3.5) is written in the coordinates that normalize the quadratic terms of the Taylor expansion. We proceed to normalize the cubic terms by introducing a near identity symplectic coordinates obtained through a suitable generating function (see Appendix B) and taking into account the Lie equation $\{G_3, K_2^T\} = 0$, we get that the normalized third-order terms are given by

$$G_3 = y(x^2 + p_x^2)g_{2100} + y^3 g_{0300}, \tag{3.15}$$

with coefficients

$$g_{0300} = -\frac{\sqrt[4]{8(N-1)}(4 + \cos(2\theta_0))\csc^4\theta_0 \sec\theta_0}{3\sqrt{\tan\theta_0}},$$
$$g_{2100} = -\frac{\sec^3\theta_0 + \sqrt{2(N-1)^5}(4 + \cos(2\theta_0))\csc^3(\theta_0)}{\sqrt[4]{8(N-1)^5}\tan^{\frac{3}{2}}\theta_0}$$

(see Appendix B.1 for the explicit coefficients of the generating function).

Therefore, we have that $g_{0300} \neq 0$, for all $\theta_0 \in (0, \pi/2)$ and, according to Sokol'skii's Theorem (A.1), we conclude that the equilibrium is unstable in the Lyapunov sense.

Remark 3.2 In the case where $\nu > 0$ and $\theta_0 \in (\pi/2, \pi)$, we have that the terms of degree three in their normal form is as in (3.15), where g_{0300} is given by

$$g_{0300} = \frac{\sqrt[4]{8(N-1)}(4 + \cos(2\theta_0))\csc^4\theta_0 \sec\theta_0}{3\sqrt{|\tan\theta_0|}} \neq 0.$$

Proposition 3.1 *The ring solution is Lyapunov unstable for any ν and $\theta_0 \in (0, \pi)\backslash\{\pi/2\}$.*

4 Periodic orbits

In this section, we will study some periodic solutions of the N-body problem on the sphere $\mathbb{S}^2$ emerging from the ring solution. More precisely, we look for periodic solutions that preserve the regular polygon configuration of the bodies, but not necessarily rotating uniformly in a fixed parallel. For this purpose, we first note that in the reduced 1-d.o.f. system, the nonequatorial equilibrium point $(\theta_0, 0)$ given in (3.11) is surrounded by periodic orbits whenever $c \in (0, \sqrt{N-1})$. Let $\eta(t) = (\theta(t), p_\theta(t))$ one of these periodic solutions with $\eta(0) = (\theta, 0)$ and period $\tau(\theta)$, then we need to find a suitable initial value $\theta(0) = \theta \in (\pi/2, \theta_0)$ such that the variable φ in (3.2) verifies the equation

$$\varphi(n\tau(\theta)) = \varphi(0) \bmod (2\pi), \tag{4.1}$$

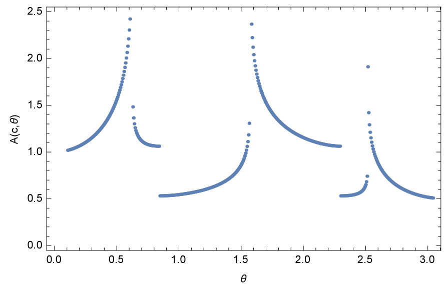


Figure 4: Graph of the function $f(\theta) = A(c, \theta)$ for $N = 2$ and $c = (2 + \sqrt{3})/5$.

$N°$	m	n	θ^*	T_F
1	2	1	0.5759770806073893‘	10.401954398957935‘
2	2	1	0.624322279750267‘	13.34487228193203‘
3	3	2	0.4812090229194145‘	6.362189077257438‘
4	3	2	0.6289478386427696‘	9.080108087002557‘
5	3	2	1.7091952021830619‘	9.080063873702084‘
6	3	2	2.5217269786955137‘	9.195817485260322‘
7	2	2	2.517270315553412‘	6.672579176853166‘
8	2	2	2.565615608913528‘	5.200975192175653‘

Table 1: Initial conditions θ^* and periods T_F that give rise to periodic orbits in the complete system for $N = 2$ and $c = (2 + \sqrt{3})/5$.

for some $n \in \mathbb{N}$. By using the integral form of $\varphi(t)$

$$\varphi(t) = \varphi_0 + c \int_0^t \csc^2 \theta(s) ds, \tag{4.2}$$

and taking into account that $\theta(s)$ is a $\tau(\theta)$-periodic function, we get that equation (4.1) assumes the form

$$\frac{c}{2\pi} \int_0^{\tau(\theta)} \csc^2 \theta(s, \theta) ds = \frac{m}{n}. \tag{4.3}$$

Now, we define

$$A(c, \theta) = \frac{c}{2\pi} \int_0^{\tau(\theta)} \csc^2 \theta(s, \theta) ds, \tag{4.4}$$

and note that if we fix c in (4.4), we can take $A(c, \theta)$ as a function depending only on θ, namely, $f(\theta)$. Next, we define the function $g(x, \theta) = f(\theta) - x$. If $\theta = \theta_0$ is a

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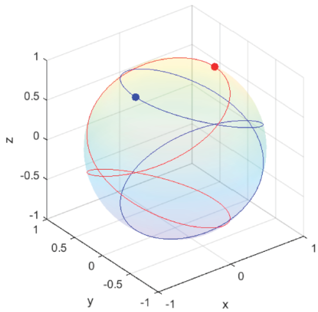
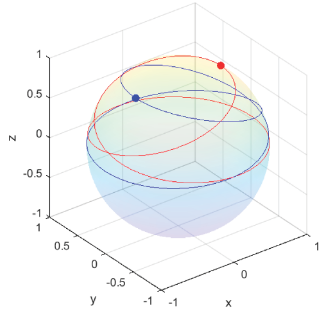
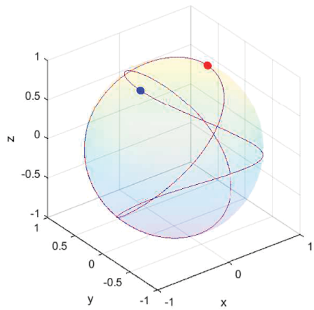


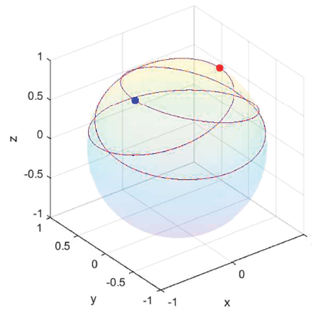
(a) Fig. 1



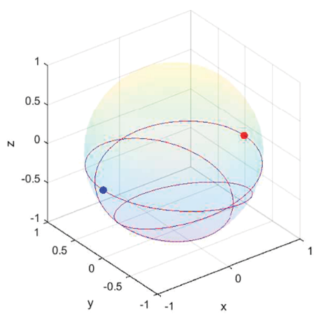
(b) Fig. 2



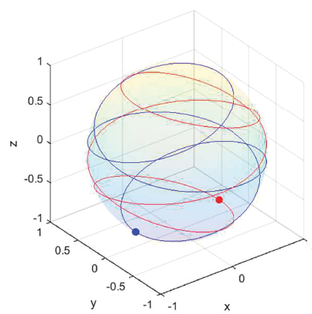
(c) Fig. 3



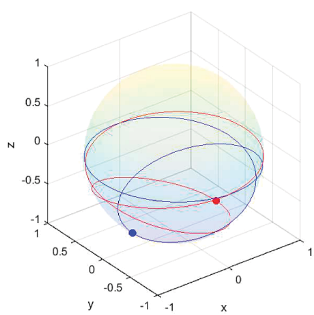
(d) Fig. 4



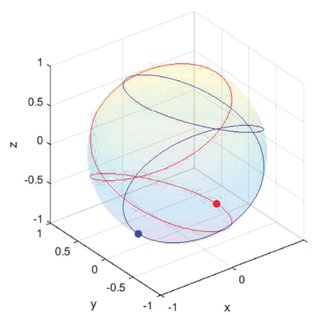
(e) Fig. 5



(f) Fig. 6



(g) Fig. 7



(h) Fig. 8

Figure 5: Periodic orbits on the sphere $\mathbb{S}^2$ for $N = 2$, $c = (2 + \sqrt{3})/5$ with initial condition given in Table 1.

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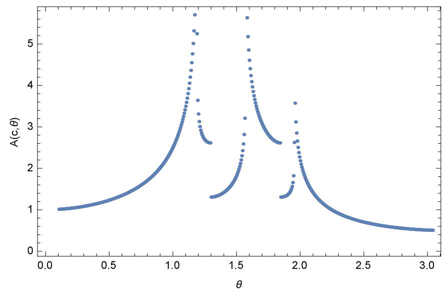


Figure 6: Graph of the function $f(\theta) = A(c,\theta)$ for $N = 3$ and $c = 1 - \sqrt{3}/5$.

N°	m	n	θ^*	T_F
1	1	2	1.3770293531731441‘	2.159805256919048‘
2	1	2	2.8582707622368555‘	2.159822954834652‘
3	1	1	0.28332190140248037‘	4.319619500151175‘
4	1	1	1.7645618100195024‘	4.319582499689864‘
5	3	2	0.2553629252331931‘	4.757886479607715‘

Table 2: Initial conditions θ^* and periods T_F that give rise to periodic orbits in the complete system for $N = 3$ and $c = 1 - \sqrt{3}/5$.

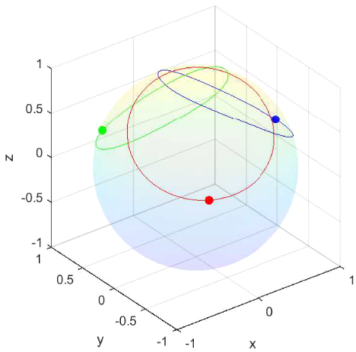
nonequatorial equilibrium and denoting by $\tau_0 = \tau(\theta_0)$, with τ_0 a value that will be suitable chosen later, then it follows that

$$g(x,\theta_0) = \frac{\nu\tau_0}{2\pi} - x \quad \text{and} \quad \varphi(t) = \varphi_0 + \nu t, \tag{4.5}$$

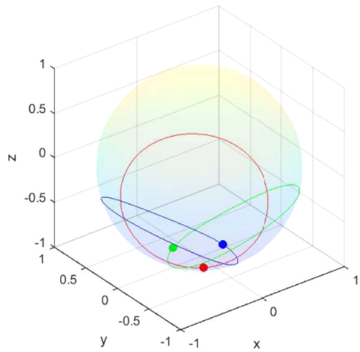
with $\nu = c\csc^2\theta_0$. Since φ is $\frac{2\pi}{\nu}$-periodic, we can choose $\tau_0 = \frac{2\pi}{\nu}$ and obtain that $g(x,\theta_0) = 1 - x$. Now, we can easily see that

$$g(1,\theta_0) = 0 \quad \text{y} \quad \left.\frac{\partial g(x,\theta)}{\partial x}\right|_{(1,\theta_0)} = -1 \neq 0.$$

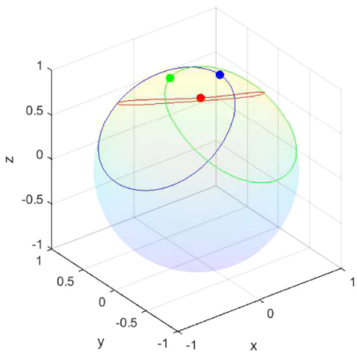
Then, by the Implicit Function Theorem, we have that there are open intervals $I_\varepsilon = (\theta_0 - \varepsilon, \theta_0 + \varepsilon)$ and $J = (1-\delta, 1+\delta)$ such that $g(x,\theta) = 0$, for all $(x,\theta) \in J \times I_\varepsilon$. Furthermore, there exists a C^k class function, $x : I_\varepsilon \to J$, such that $g(x(\theta),\theta) = 0$ and $x(\theta_0) = 1$. Then, by density of $\mathbb{Q}$ en $\mathbb{R}$, we have that $J \cap \mathbb{Q} \neq \varnothing$. Therefore, the set $\Upsilon = x^{-1}(J \cap \mathbb{Q}) \neq \varnothing$, is such that given $m/n \in J \cap \mathbb{Q}$, there exists $\theta^* \in \Upsilon$ such that $g(m/n, \theta^*) = 0$. Therefore, we conclude that there are initial conditions θ^* that generate periodic orbits in the complete system.



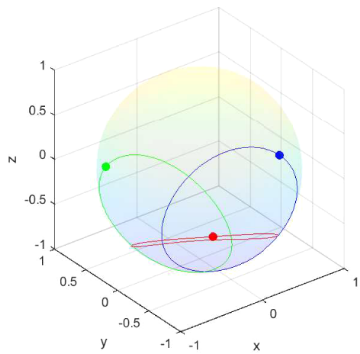
(a) Fig. 1



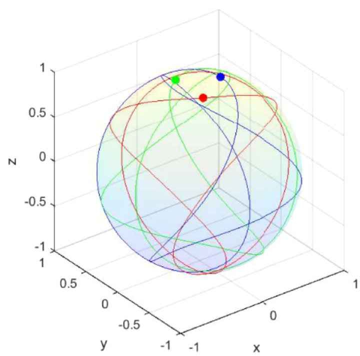
(b) Fig. 2



(c) Fig. 3



(d) Fig. 4



(e) Fig. 5

Figure 7: Periodic orbits on the sphere $\mathbb{S}^2$ for $N = 3$, $c = 1 - \sqrt{3}/5$ with initial condition given in Table 2.

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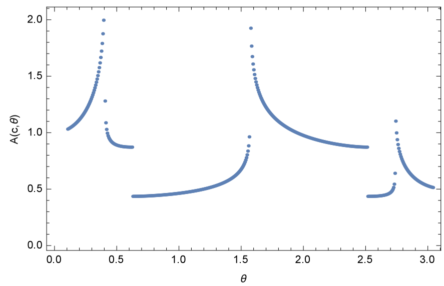

Figure 8: Graph of the function $f(\theta) = A(c,\theta)$ for $N = 4$ and $c = 2\sqrt{5+\sqrt{2}}/5$.

$N°$	m	n	θ^*	T_F
1	1	2	1.1963809469081823‘	1.4724533519714116‘
2	1	2	2.7190241813008105‘	1.4724503122514878‘
3	1	1	0.42256876046583774‘	2.944897452098061‘
4	1	1	1.9452119521433546‘	2.9448981256444933‘
5	2	2	2.74727275996211‘	3.2124622727718513‘
6	3	2	1.6085027918855728‘	6.200496191217298‘

Table 3: Initial conditions θ^* and periods T_F that give rise to periodic orbits in the complete system for $N = 4$ and $c = 2\sqrt{5+\sqrt{2}}/5$.

4.1 Some examples of periodic orbits on the sphere $\mathbb{S}^2$

In this section, we present some examples of periodic orbits on the sphere $\mathbb{S}^2$ obtained through the method described above. Note that the initial conditions that generate polygonal periodic orbits are the zeros of the equation (4.3). Thus, for a fixed value of c, we can consider the graph of the function $f(\theta) = A(c,\theta)$ and approximate their intersections with horizontal lines of the form $y = m/n$.

For the case $N = 2$, the value of c must be chosen such that $|c| < 1$. In particular, for $c = (2+\sqrt{3})/5$ the graph of $A(c,\theta)$ is shown in Figure 4.

The following table gives initial conditions for some values of (m,n) and their corresponding period.

In Table 1, for different rationals m/n, we find initial conditions θ^* giving rise to periodic orbits in the complete system. Some of them are located close to the equilibrium $\theta_0 = 2.298941328617731$‘ and others are close to the homoclinic orbit shown in Figure 3b, which also correspond to periodic orbits that cross the equator. Similarly, near the symmetric equilibrium $\theta_0 = 0.8426513249720621$‘, we can also find initial conditions that give periodic solutions in the whole system (Figure 5).

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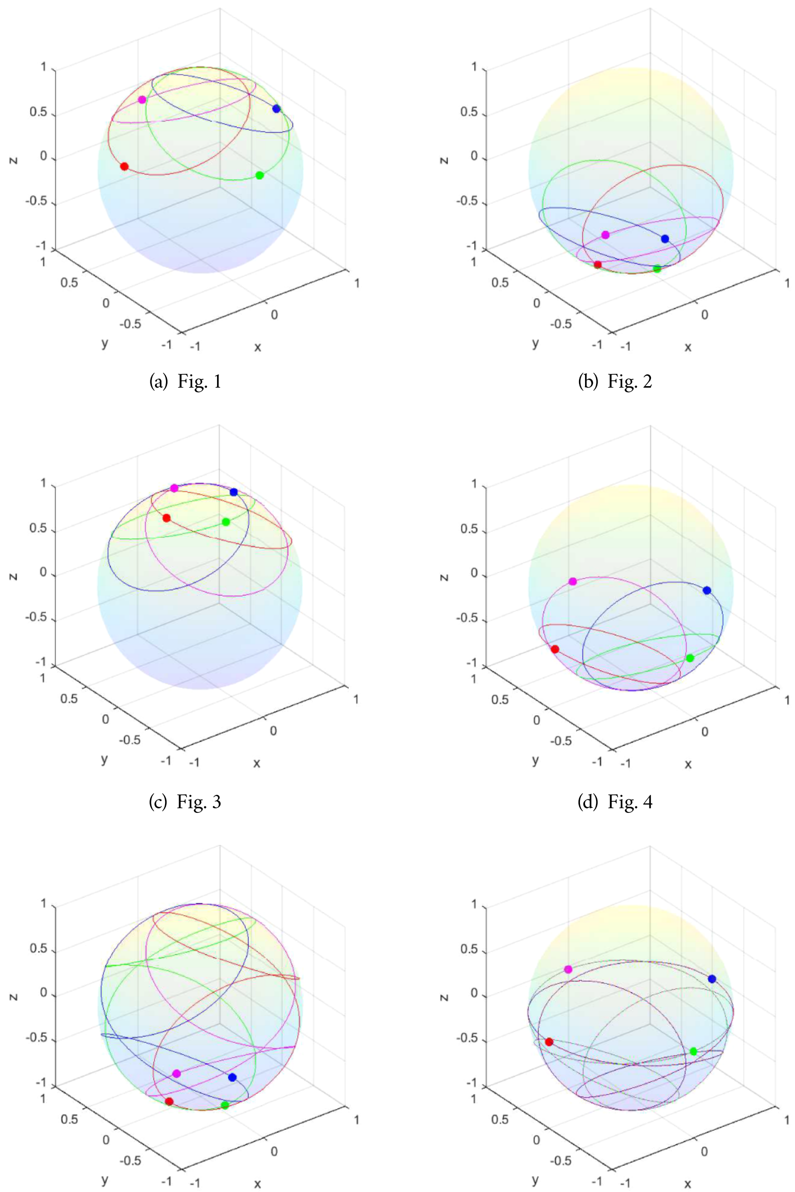

(a) Fig. 1 (b) Fig. 2

(c) Fig. 3 (d) Fig. 4

(e) Fig. 5 (f) Fig. 6

Figure 9: Periodic orbits on the sphere $\mathbb{S}^2$ for $N = 4$, $c = 2\sqrt{5+\sqrt{2}}/5$ with initial condition given in Table 3.

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For the case $N = 3$, the value of c must be chosen such that $|c| < \sqrt{2}$. In particular, for $c = 1 - \sqrt{3}/5$ the graph of $A(c, \theta)$ is shown in Figure 6.

The following table gives initial conditions for some values of (m, n) and their corresponding period.

In Table 2, for different rationals m/n, we can find initial conditions θ^* that give rise to periodic orbits in the complete system. Some of them are close to the equilibrium $\theta_0 = 2.6611657363097923$' (or in the symmetric equilibrium $\theta_0 = 0.48042691728000075$'), and others are close to the homoclinic orbit shown in Figure 3b. Moreover, we chose all initial conditions except row 5 inside the homoclinic orbit (Figure 7).

For the case $N = 4$, the value of c must be chosen such that $|c| < \sqrt{3}$. In particular, for $c = 2\sqrt{5+\sqrt{2}}/5$ the graph of $A(c, \theta)$ is shown in Figure 8.

The following table gives initial conditions for some values of (m, n) and their corresponding period.

In Table 3, we find initial conditions θ^* leading to periodic orbits in the complete system for different rationals m/n. Some of them are close to the equilibrium $\theta_0 = 2.51685340624551$' (or in the symmetric equilibrium $\theta_0 = 0.6247392473442833$'), and others are close to the homoclinic orbit shown in Figure 3b. Moreover, we chose all initial conditions except row 5 inside the homoclinic orbit (Figure 9).

A Reduced system

A.1 Reduction to two degrees of freedom

In this section, we will see the details regarding the reduction of the system (1.6) to a Hamiltonian system with two degrees of freedom. For this, let us consider the Hamiltonian function defined in (1.7), then we see that from the first equation of the system (1.6) the moment p_{φ_j} is given by

$$p_{\varphi_j} = \dot{\varphi}_j \sin^2 \theta_j. \tag{A.1}$$

Differentiating with respect to time in this last relation and using the third equation defined in (1.6), we find the following equality:

$$\ddot{\varphi}_j \sin^2 \theta_j + 2\dot{\varphi}_j \sin\theta_j \cos\theta_j \dot{\theta}_j = -\sum_{k=1, j\neq k}^{N} \frac{\sin\theta_j \sin\theta_k \sin(\varphi_j - \varphi_k)}{1 - d_{jk}}, \tag{A.2}$$

and having into account that

$$\sum_{k=1, j\neq k}^{N} \frac{\sin\theta_j \sin\theta_k \sin(\varphi_j - \varphi_k)}{1 - d_{jk}} = -\sum_{k=1, j\neq k}^{N} \frac{\sin\theta_k \sin\theta_j \sin(\varphi_k - \varphi_j)}{1 - d_{kj}},$$

we finally get

$$\sum_{j=1}^{N} \sum_{k=1, j\neq k}^{N} \frac{\sin\theta_j \sin\theta_k \sin(\varphi_j - \varphi_k)}{1 - d_{jk}} = 0.$$

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Therefore, using this last relation together with (3.1) in (A.2), we have

$$\begin{aligned}0 &= \sum_{j=1}^{N} \ddot{\varphi}_j \sin^2\theta_j + 2\dot{\varphi}_j \sin\theta_j \cos\theta_j \dot{\theta}_j \\ &= \sum_{j=1}^{N} \ddot{\varphi} \sin^2\theta + 2\dot{\varphi}\sin\theta\cos\theta\,\dot{\theta} \\ &= N(\ddot{\varphi}\sin^2\theta + 2\dot{\varphi}\sin\theta\cos\theta\,\dot{\theta}) \\ &= \ddot{\varphi}\sin^2\theta + 2\dot{\varphi}\sin\theta\cos\theta\,\dot{\theta} \\ &= \frac{d}{dt}(\dot{\varphi}\sin^2\theta) = \dot{p}_\varphi.\end{aligned}$$

On the other hand, differentiating the second equation obtained in (1.6) and using the fourth equation of the system (1.6) together with (A.2), we obtain

$$\begin{aligned}\ddot{\theta}_j &= \cot\theta_j \csc^2\theta_j p_{\varphi_j}^2 - \sum_{k=1, j\neq k}^{N} \frac{\cos\theta_k \sin\theta_j - \cos\theta_j \sin\theta_k \cos(\varphi_j - \varphi_k)}{1 - d_{jk}} \\ &= \cot\theta_j \csc^2\theta_j (\dot{\varphi}_j \sin^2\theta_j)^2 - \sum_{k=1, j\neq k}^{N} \frac{\cos\theta_k \sin\theta_j - \cos\theta_j \sin\theta_k \cos(\varphi_j - \varphi_k)}{1 - d_{jk}} \\ &= \cos\theta_j \sin\theta_j\, \dot{\varphi}_j^2 - \sum_{k=1, j\neq k}^{N} \frac{\cos\theta_k \sin\theta_j - \cos\theta_j \sin\theta_k \cos(\varphi_j - \varphi_k)}{1 - d_{jk}}.\end{aligned}$$

Again, in this last relation, we can use the equations defined in (3.1) to get

$$\begin{aligned}\ddot{\theta} &= \cos\theta\sin\theta\,\dot{\varphi}^2 - \cos\theta\sin\theta \sum_{k=1, j\neq k}^{N} \frac{1 - \cos(\varphi_j - \varphi_k)}{1 - \cos^2\theta - \sin^2\theta\cos(\varphi_j - \varphi_k)} \\ &= \cos\theta\sin\theta\,\dot{\varphi}^2 - (N-1)\cot\theta. \\ &= \frac{\cos\theta}{\sin^3\theta}(\sin^2\theta\,\dot{\varphi})^2 - (N-1)\cot\theta. \\ &= \cot\theta\csc^2\theta\, p_\varphi^2 - (N-1)\cot\theta.\end{aligned}$$

Therefore, we obtain that our reduced Hamiltonian is a Hamiltonian with two degrees of freedom of the form

$$\mathcal{H} = \frac{1}{2}(\csc^2\theta p_\varphi^2 + p_\theta^2) + (N-1)\ln(\sin\theta).$$

A.2 Sokol'skii stability theorem 1977

Suppose that the Hamiltonian function in its normal form admits the following form:

$$H = \frac{\delta_1}{2} q_1^2 + \frac{\omega\delta_2}{2}(q_2^2 + p_2^2) + \sum_{s=3}^{M} \sum_{j=0}^{[s/2]} a_{s-2j,j}\, p_1^{s-2j}(q_2^2 + p_2^2)^j + H^{M+1} + \cdots, \tag{A.3}$$

where $a_{s-2j,j}$ are real constants. The normalization must be carried out up to terms of order M such that $a_{M,0}$ is different from zero.

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Theorem A.1 (A.G. Sokol'skii, 1977) *Suppose that a canonic system with two degrees of freedom has one zero frequency and multiple elementary divisors and that its Hamiltonian function has been reduced to form* (A.3). Then:

(1) If M is odd, the equilibrium position is unstable.
(2) If M is even and $\delta_1 a_{M,0} < 0$, the equilibrium position is unstable.
(3) If M is even and $\delta_1 a_{M,0} > 0$, the equilibrium position is Lyapunov-stable.

For more information, see [12].

B Normal form of high-order terms

Let us consider the Taylor expansion of H

$$H = H_2 + H_3 + H_4 + \cdots + H_s + \cdots,$$

with

$$H_s = \sum_{|k|+|l|=s} h_{k_1 k_2 l_1 l_2} q_1^{k_1} q_2^{k_2} p_1^{l_1} p_2^{l_2},$$

and H_2 in its normal form. The normalized Hamiltonian G of H is obtained via the Lie algorithm

$$G = G_2 + G_3 + G_4 + \cdots + G_s + \cdots,$$

where $G_2 = H_2$ and G_s are homogeneous polynomials of degree s

$$G_s = \sum_{|k|+|l|=s} g_{k_1 k_2 l_1 l_2} q_1^{k_1} q_2^{k_2} p_1^{l_1} p_2^{l_2}.$$

Similarly, we define the generating function of the symplectic transformation that becomes H into G, denoted by

$$W = q_1 p_1 + q_2 p_2 + S_3 + S_4 + \cdots + S_s + \cdots, \tag{B.1}$$

where S_s are homogeneous polynomials of degree s, so that

$$S_s = \sum_{|k|+|l|=s} s_{k_1 k_2 l_1 l_2} q_1^{k_1} q_2^{k_2} p_1^{l_1} p_2^{l_2}.$$

B.1 Normal form of the third-order terms with quadratic part as (3.13)

Since, in our case, the quadratic term is degenerated, the Lie equation takes the form $\{H_0^T, G_s\} = 0$. We start with the case $s = 3$; that is, we will determine the Lie normal form of the terms H_3 of degree three of the Hamiltonian H under the assumption that the quadratic part H_2 is already normalized. Furthermore, the matrix associated with the linear system is nondiagonalizable. Therefore, we must solve the system of equations generated by the following Lie equation:

$$\{K_2^T, G_3\} = 0,$$

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where K_2^T is defined in equation (3.14). From where we get that

$$g_{3000} = 0,\ g_{2010} = 0,\ g_{2001} = 0,\ g_{1200} = 0,\ g_{1110} = 0,\ g_{1101} = 0,\ g_{1020} = 0,$$
$$g_{1011} = 0,\ g_{1002} = 0,\ g_{0210} = 0,\ g_{0201} = 0,\ g_{0111} = 0,\ g_{0102} = 0,\ g_{0030} = 0,$$
$$g_{0021} = 0,\ g_{0012} = 0,\ g_{0003} = 0,\ g_{0120} = g_{2100}.$$

Therefore, the terms of degree three of the normalized Hamiltonian are given by

$$G_3 = y(x^2 + p_x^2)g_{2100} + y^3 g_{0300}. \tag{B.2}$$

Furthermore, according to the Lie triangle and the recurrence formula, we have

$$G_3 = H_3 + \{H_0^T, S_3\} \equiv H_2 + G_3\Big|_{\left(q_1, q_2, \frac{\partial W}{\partial q_1}, \frac{\partial W}{\partial q_2}\right)} = H_2 + H_3\Big|_{\left(\frac{\partial W}{\partial p_1}, \frac{\partial W}{\partial p_2}, p_1, p_2\right)}. \tag{B.3}$$

It is verified that equation (B.3) is satisfied for G_3 is as it appears in (B.2), for this purpose, we must choose the coefficients of S_3 in a convenient way. More precisely,

$$s_{3000} = \frac{3\sqrt[4]{2}\tan\theta_0 \sec^2\theta_0 + 2^{7/4}(N-1)^{5/2}(4+\cos(2\theta_0))\csc^2\theta_0}{9(N-1)^{3/4}\omega_1 \tan^{\frac{5}{2}}\theta_0},$$

$$s_{2100} = -\frac{(2+\cos(2\theta_0))\csc^3\theta_0 \sec\theta_0((N-1)\omega_1\cos(2\theta_0) + (N-1)\omega_1 - 2)}{2\sqrt{2}\omega_1},$$

$$s_{2010} = 0, \quad s_{2001} = 0, \quad s_{0102} = 0, \quad s_{0003} = 0,$$

$$s_{1200} = \frac{\sqrt[4]{8(N-1)^3}(4+\cos(2\theta_0))\csc^4\theta_0}{\omega_1\sqrt{\tan\theta_0}},$$

$$s_{1110} = \frac{\sqrt{2(N-1)^5}(4+\cos(2\theta_0))\csc^3\theta_0 - \sec^3\theta_0}{\sqrt[4]{8(N-1)^5}\omega_1\tan^{\frac{3}{2}}\theta_0},$$

$$s_{1101} = -\frac{2\sqrt{2}\csc\theta_0(2(N-1)\omega_1 + 3(1-(N-1)\omega_1)\csc^2\theta_0 + \sec^2\theta_0)}{\sqrt{N-1}\omega_1^2},$$

$$s_{1020} = \frac{2^{3/4}(N-1)^{7/4}(4+\cos(2\theta_0))\csc^2\theta_0}{3\omega_1\tan^{\frac{5}{2}}\theta_0},$$

$$s_{1011} = \frac{(2+\cos(2\theta_0))\csc^3\theta_0\sec\theta_0((N-1)\omega_1\cos(2\theta_0) + (N-1)\omega_1 - 2)}{2\sqrt{2}\omega_1^2},$$

$$s_{1002} = -\frac{\sqrt[4]{2}\left(2\omega_1\sec^2\theta_0 - (N-1)\omega_1^2 + 2\sqrt{2}(N-1)^{3/2}(4+\cos(2\theta_0))\csc^3\theta_0\sec\theta_0\right)}{(N-1)^{3/4}\omega_1^3\tan^{\frac{3}{2}}\theta_0},$$

$$s_{0300} = -\frac{\sqrt{2}}{3}\csc\theta_0(3\cot\theta_0\csc\theta_0 + \sec\theta_0),$$

$$s_{0210} = -\frac{\sqrt{2}\csc\theta_0(3\csc^2\theta_0 + \sec^2\theta_0)}{\sqrt{N-1}\omega_1},$$

$$s_{0201} = -\frac{\csc\theta_0}{\sqrt[4]{8(N-1)}\sqrt{\tan\theta_0}},$$

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$$
\begin{aligned}
s_{0120} &= -\frac{(2+\cos(2\theta_0))\csc^3\theta_0\sec\theta_0((N-1)\omega_1\cos(2\theta_0)+(N-1)\omega_1+2)}{2\sqrt{2}\omega_1},\\
s_{0111} &= -\frac{2\sqrt[4]{2}\csc^4\theta_0\left(\omega_1\sin^3\theta_0\sec\theta_0+\sqrt{2}(N-1)^{3/2}(4+\cos(2\theta_0))\right)}{(N-1)^{3/4}\omega_1^2\sqrt{\tan\theta_0}},\\
s_{0030} &= -\frac{\sqrt{2(N-1)}(2+\cos(2\theta_0))\csc^3\theta_0}{3\omega_1},\\
s_{0021} &= -\frac{\sqrt{2(N-1)^5}(4+\cos(2\theta_0))+4(N-1)\omega_1\sec\theta_0-\sec^3\theta_0}{2\sqrt{2(N-1)^5}\omega_1^2\tan^{\frac{3}{2}}\theta_0},\\
s_{0012} &= \frac{2\sqrt{2}\csc\theta_0(2(N-1)\omega_1+3(1-(N-1)\omega_1)\csc^2\theta_0+\sec^2\theta_0)}{\sqrt{N-1}\omega_1^3}.
\end{aligned}
$$

This choice of the coefficients of the generating function implies that the terms of degree three of the normalized Hamiltonian are given by (B.2), where

$$
\begin{aligned}
g_{0300} &= -\frac{\sqrt[4]{8(N-1)}(4+\cos(2\theta_0))\csc^4\theta_0\sec\theta_0}{3\sqrt{\tan\theta_0}},\\
g_{2100} &= -\frac{\sec^3\theta_0+\sqrt{2(N-1)^5}(4+\cos(2\theta_0))\csc^3(\theta_0)}{\sqrt[4]{8(N-1)^5}\tan^{\frac{3}{2}}\theta_0}.
\end{aligned}
$$

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