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# ON RINGS WHICH ARE SUMS OF SUBRINGS

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Abstract. There are presented some generalizations and extensions of results for rings which are sums of two or tree subrings. It is provided a new proof of the well-known Kegel's result stating that a ring being a sum of two nilpotent subrings is itself nilpotent. Moreover, it is proved that if $R$ is a ring of the form $R = A + B$, where $A$ is a subgroup of the additive group of $R$ satisfying $A^d \subseteq B$ for some positive integer $d$ and $B$ is a subring of $R$ such that $B \in S$, where $S$ is $N$-radical contained in the class of all locally nilpotent rings, then $R \in S$.

## 1. Introduction

In this paper, we study the rings of the form $R = R_1 + \cdots + R_n$ where $R_i$ are subrings or additive subgroups of $R$ and $R_1 + \cdots + R_n$ denotes the set of all sums $r_1 + \cdots + r_n$ with $r_1 \in R_1, \ldots, r_n \in R_n$. We focus mainly on the case $n = 2$ and $n = 3$. We also consider somewhat more general situation in which $R = A + B$, $B$ is a subring of $R$, and $A$ is a subgroup of the additive group of $R$.

Many interesting results concerning the subject of this paper were focused on rings which are sums of two subrings (e.g. [3, 6, 8, 10, 12, 11, 14, 15]). In particular, Kegel proved that a ring which is a sum of two nilpotent subrings is nilpotent itself (see [6]). In his next paper [7] Kegel presented various generalizations of this theorem. Among others, he showed that the prime radical $\beta$ of a ring that is a sum of two subrings contains a product of the hyperanihilators of these subrings (Theorem 1), while assuming that one of the subrings is a $\beta$ radical he showed that the Levitzki radical of this ring contains the hyperanihilator of the second subring (Theorem 3). These results were generalized in [11]. In addition, answers to most of the questions raised in [7] are presented in [11].

It turned out (see [10]) that if $R = R_1 + R_2$ is a ring, $R_1$ and $R_2$ are nil of bounded index subrings of $R$ (i.e. they satisfy an identity $x^n = 0$), then so is $R$. In addition, in [1], it was shown that if $R_i$ are left (right) $T$-nilpotent, then $R$ is left (right) $T$-nilpotent, too. However, in [9, 16] Kelarev and independently Salwa provided examples of a rings that are the sums of two $\beta$-radical subrings, which itself are not $\beta$-radical.

Another direction of research concerning the Kegel's Theorem is related to the class of $PI$ rings. In [13] it was shown that a ring that is the sum of two $PI$ rings is a $PI$ ring. A more effective combinatorial proof in the special case of this fact can be found in the paper [5], in which Felzenszwalb, Giambruno and Leal showed that if $R = A + B$, $A$ and $B$ are $PI$ rings and for some positive integer $n \geq 1$ either $(AB)^n \subseteq A$ or $(BA)^n \subseteq A$, then $R$ is a $PI$ ring. In this paper using the concept of one-sided accessible subrings defined in the Section 2 (cf. [4]) we prove

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that $R \in \mathcal{M}$ if and only if $A^nB^m \in \mathcal{M}$ for some positive integers $n, m$, where $\mathcal{M}$ is a fairly general left (right) strong class of rings containing the class of nilpotent rings (Section 2). Moreover we will present a new proof of Kegel's result and some it's extensions.

On the other hand, Bokut' proved in [2] that every algebra over a field can be embedded into a simple algebra which is a sum of three nilpotent subalgebras of index three, which shows that the cases related to the sums of two and three subrings are totally different. We will present a generalization of the result from [14] stating that a ring is nilpotent if it is a sum of three subrings with zero multiplication.

In [8], it was proved that if $R = A + B$, $A^2 \subseteq B$, $B$ is locally nilpotent subring of $R$ and $A$ is a subgroup of the group $R^+$, then the ring $R$ is locally nilpotent. We generalize this result proving the claim under the assumption $A^d \subseteq B$ for some positive integer $d$ and $B \in S$, where $S$ is a $N$-radical contained in the class of all locally nilpotent rings. We will complete the article with open problems for further studies.

All the rings considered in this paper are associative, but are not assumed to have an identity. By $R^1$ we denote a ring $R$ with an identity adjoined. We follow the convention that if $S \subseteq R$, then $S^0 = \{1\}$. If $A$ is an ideal (left ideal or right ideal) of a ring $R$, then we write $A \lhd R$ ($A <_l R$ or $A <_r R$). If it is not necessary to distinguish the side of a one-sided ideal $A$ of $R$, then we simple write $A < R$. For any subset $C$, the symbols $l_R(C)$ and $r_R(C)$ stand for the left and a right anihilator of $C$ in $R$, respectively.

The prime, Levitzki, and Jacobson radicals are denoted by $\beta$, $\mathcal{L}$, $\mathcal{J}$, respectively. The symbol $\mathcal{N}$ stands for the class of all nilpotent rings. The set of all positive integers is denoted by $\mathbb{N}$.

## 2. New propositions

Let's start with following simple observations.

*Remark* 2.1. Suppose that $R = L + B$, where $L <_l R$ and $B$ is a subring of the ring $R$. If the rings $L$ and $B$ are nilpotent, then so is $R$. Indeed, there exists $n \in \mathbb{N}$ such that $L^n = \{0\}$. Let $I = L + LR$. Then $I \lhd R$ and $I = LR^1$, whence $I^n = \{0\}$. Moreover, $R/I = (B + I)/I \cong B/(B \cap I)$ is a nilpotent ring. Consequently, the ring $R$ is nilpotent too.

*Remark* 2.2. Let $A$ and $B$ be nilpotent subrings of a ring $R$ such that $R = A + B$. If $S$ is a subring of the ring $R$ such that $A <_l S$, then $S$ is also a nilpotent ring. Indeed, from the modularity of the lattice of subgroups of the group $(R, +)$ and the fact that $A \subseteq S$ and $R = A + B$, we get $S = A + (S \cap B)$. But the rings $A$ and $S \cap B$ are nilpotent and $A <_l S$, so if follows from Remark 2.2 that the ring $S$ is nilpotent.

Let $A$ and $B$ be subrings of the ring $R$ such that $R = A + B$. By repeatedly using Remark 2.2 we infer that if $A^n = \{0\}$, then $A <_l A + RA^{n-1} <_l A + RA^{n-2} <_l \ldots <_l A + RA <_l R$. Therefore, if subrings $A$ and $B$ of the ring $R = A + B$ are nilpotent, then $R$ is also nilpotent. Thus we obtain a simple proof of the well-known Kegel Theorem from [6].

This provides the motivation for us to consider subrings one-sided accessible. We say that a subring $A$ of a ring $R$ is *one-sided accessible* in $R$ if there exist subrings $A_0 = A, A_1, \ldots, A_n = R$ such that $A_i < A_{i+1}$ for $i = 0, 1, \ldots, n-1$. It is equivalent to $A^mRA^k \subseteq A$ for some $m, k \in \mathbb{N} \cup \{0\}$. Hence, for example, every nilpotent subring of the ring $R$ is one-sided accessible in $R$.

**Lemma 2.3.** *Let $A$ and $B$ be subrings of a ring $R$ such that $R = A + B$. Then $S = A^{k_1}B^{l_1} \dots A^{k_s}B^{l_s}$ and $T = A^{k_1}B^{l_1} \dots A^{k_s}B^{l_s}A^{k_{s+1}}$ are one-sided accessible in $R$ for all $s, k_1, \dots, k_s, k_{s+1}, l_1, \dots, l_s \in \mathbb{N}$. Moreover, $SRS \subseteq S^2$ and $TRT \subseteq T$.*

*Proof.* Since $S^2 \subseteq A^{k_1}B^{l_1} \cdot \dots \cdot A^{k_{s-1}}B^{l_{s-1}}RA^{k_s}B^{l_s} \subseteq A^{k_1}B^{l_1} \cdot \dots \cdot A^{k_{s-1}}B^{l_{s-1}}(A+ +B)A^{k_s}B^{l_s} = A^{k_1}B^{l_1} \cdot \dots \cdot A^{k_s+1}B^{l_s} + A^{k_1}B^{l_1} \cdot \dots \cdot B^{l_{s-1}+1}A^{k_s}B^{l_s} \subseteq S$, $S$ is a subring of $R$. Moreover, $SRS = S(A + B)S = SAS + SBS \subseteq S^2$, because of $AS \subseteq S$ and $SB \subseteq S$. Hence $S <_r S + RS <_l R$, which means that $S$ is one-sided accessible ring in $R$.

Furthermore, $T = SA^{k_{s+1}}$ i $A^{k_{s+1}}S \subseteq S$, so $T^2 \subseteq S^2A^{k_{s+1}} \subseteq T$ and $TRT \subseteq SRSA^{k_{s+1}} \subseteq SA^{k_{s+1}} = T$, whence $T$ is one-sided accessible ring in $R$ too. □

Throughout the paper, by $\mathcal{M}$ we denote an arbitrary homomorphically closed class of rings which satisfy the following conditions: $\mathcal{N} \subseteq \mathcal{M}$, $\mathcal{M}$ is closed under extensions, i.e. if $I \lhd H$ and $I$, $H/I \in \mathcal{M}$, then $H \in \mathcal{M}$, $\mathcal{M}$ is hereditary for subrings, i.e. if $I$ is a subring of $H$ and $H \in \mathcal{M}$, then $I \in \mathcal{M}$ and $I$ is one-sided strong, i.e. if $I < H$ and $I \in \mathcal{M}$, then the ideal of $H$ generated by $I$ belongs to $\mathcal{M}$. The classes $\mathcal{N}$, $\beta$, and $\mathcal{L}$ can be considered as $\mathcal{M}$.

Recall that a ring $A$ is said to be *left $T$-nilpotent* if for each sequence $(a_n)$ of elements of $A$, there exists $n \in \mathbb{N}$ such that $a_1 \cdot a_2 \cdot \dots \cdot a_n = 0$. Let $l_0(A) = \{0\}$ and $l_\alpha(A) = \{x \in A \,|\, xA \subseteq \bigcup_{\beta<\alpha} l_\beta(A)\}$ for any ordinal number $\alpha > 0$. It is well known that a ring $A$ is left $T$-nilpotent if and only if $l(A) = A$, where $l(A)$ is the union $\bigcup_{\alpha\geq 0} l_\alpha(A)$. The ideal $l(A)$ is collded a *left hiperanihilator* of $A$. The *right hiperanihilator* $r(A)$ of $A$ is defined analogously. It is easy to see that a ring $A$ is left $T$-nilpotent if and only if every non-zero homomorphic image of $A$ has non-zero left annihilator. Clearly, the class of all left $T$-nilpotent rings is homomorphically closed, hereditary for subrings, closed under extensions, and by [1, Lema 2.4], one-sided strong. Therefore, the class of all left $T$-nilpotent rings can be considered as $\mathcal{M}$.

We obtain the following

**Proposition 2.4.** *Let $R_1$ and $R_2$ be subrings of a ring $R$ such that $R = R_1 + R_2$. If $R_1$ is a nilpotent ring and $R_2 \in \mathcal{M}$, then $R \in \mathcal{M}$.*

*Proof.* Clearly, $R_1 <_l R_1 + RR_1^{n-1} <_l R_1 + RR_1^{n-2} <_l \dots <_l R_1 + RR_1 <_l R$, where $n$ is the index of nilpotency of $R_1$. Hence $R_1$ is one-sided accessible in $R$. As in Remark 2.2, we infer that $R \in \mathcal{M}$. □

**Proposition 2.5.** *Let $A$ and $B$ be subrings of a ring $R$ such that $R = A + B$. If $A \in \mathcal{M}$ and $B \in \mathcal{M}$, then the following subrings of $R$ belong to $\mathcal{M}$: $Id_l(A) = \{x \in R : xA \subseteq A\}$, $A + l_R(A^n)$, and $A + Rl_n(A)$ for $n \in \mathbb{N}$. Moreover, $Id_r(B) = \{x \in R : Bx \subseteq B\}$, $B + r_R(B^n)$, and $B + r_n(B)R \in \mathcal{M}$ for $n \in \mathbb{N}$.*

*Proof.* Clearly, $S = Id_l(A)$ is a subring of $R$ and $A <_l S$. Furthermore $S = A + B \cap S$, $B \cap S \in \mathcal{M}$, and $\mathcal{M}$ is one-side strong, so $S \in \mathcal{M}$. Since $L = l_R(A^n) <_l R$ and $(LA)A^n \subseteq LA^n = 0$, we infer that $L + A$ is a subring of $R$. But $(L + A)A^n = A^{n+1} \subseteq A$, so $A$ is one-sided accessible in $L + A$. Hence $L + A \in \mathcal{M}$, and consequently $l_R(A) \in \mathcal{M}$. Moreover for every $n \in \mathbb{N}$, we have $l_n(A)A^n = \{0\}$, whence $(A + Rl_n(A))A^n = A^{n+1} \subseteq A$, i.e. $A$ is one-sided accessible in $A + Rl_n(A)$. Therefore $A + Rl_n(A) \in \mathcal{M}$. The rest of the proof runs analogously. □

**Proposition 2.6.** *Let $A$ and $B$ be subrings of a ring $R$ such that $R = A + B$, $A \in \mathcal{M}$, and $B \in \mathcal{M}$. Then the following conditions are equivalent:*
*(i) $A^nB^m \in \mathcal{M}$ for some $n, m \in \mathbb{N}$,*
*(ii) $AB \in \mathcal{M}$,*
*(iii) $R \in \mathcal{M}$.*

*Proof.* $(i) \Rightarrow (ii)$. Let $S = A^nB^m$. Then by Lemma 2.3 we have $SRS \subseteq S^2$. Hence $S <_r S + RS$, so $S + RS^2 \in \mathcal{M}$. Consequently, $RS^2 \in \mathcal{M}$. Moreover, $(RS)^2 = R(SRS) \subseteq RS^2$, so $RS \in \mathcal{M}$. Furthermore, $RS \lhd S + RS$, whence $S + RS \in \mathcal{M}$. Thus $I = S + RS + SR + RSR \in \mathcal{M}$ whereas $I \lhd R$. Let $\overline{R} = R/I$, $\overline{A} = (A + I)/I$, and $\overline{B} = (B + I)/I$. Then $\overline{R} = \overline{A} + \overline{B}$, whereas $\overline{A}$ is left $T$ nilpotent and $\overline{B} \in \mathcal{M}$. Moreover, $\overline{A}^n\overline{B}^m = 0$, so $\overline{A}^n\overline{B} \subseteq l_{\overline{R}}(\overline{B}^{m-1})$. Therefore, by Proposition 2.5, $\overline{A}^n\overline{B} = (A^nB + I)/I \in \mathcal{M}$. But $I \in \mathcal{M}$, so $A^nB \in \mathcal{M}$. The analogous reasoning based on the the right-side version of Proposition 2.5 leads to the conclusion that $AB \in \mathcal{M}$.

$(ii) \Rightarrow (iii)$. Let $P = A + AR$. Then $P <_r R$ and $P = A + AB$, since $R = A + B$. Notice that $AB <_l P$, so combining this with $A \in \mathcal{M}$, we infer that $P \in \mathcal{M}$. Consequently, $R \in \mathcal{M}$.

The implication $(iii) \Rightarrow (i)$ is obvious. □

**Proposition 2.7.** *Let $A \in \mathcal{M}$ and $B \in \mathcal{M}$ be subrings of a ring $R$ such that $R = A + B$. If there exist positive integers $s, k_1, \ldots, k_s, k_{s+1}, l_1, \ldots, l_s$ such that $S = A^{k_1}B^{l_1} \ldots A^{k_s}B^{l_s} \in \mathcal{M}$ or $T = A^{k_1}B^{l_1} \ldots A^{k_s}B^{l_s}A^{k_{s+1}} \in \mathcal{M}$, then $R \in \mathcal{M}$.*

*Proof.* Suppose that $S \in \mathcal{M}$, and define $n = \max\{k_1, l_1, \ldots, k_s, l_s\}$. Then $(A^nB^n)^s \subseteq S$, so $(A^nB^n)^s \in \mathcal{M}$, whence $A^nB^n \in \mathcal{M}$. Therefore, it follows from Proposition 2.6, that $R \in \mathcal{M}$.

Now suppose that $T \in \mathcal{M}$. In view of Lemma 2.3, $TRT \subseteq T$, so $T <_l T + TR$. It follows $T + T^2R \in \mathcal{M}$. Hence $T^2R \in \mathcal{M}$. In particular, $T^2B \in \mathcal{M}$. Substituting $T^2B$ for $S$ in the first part of the proof, we obtain $R \in \mathcal{M}$. □

In [1, Theorem 2.9], it was shown that a ring which is a sum of two left $T$-nilpotent subrings is left $T$-nilpotent. The question arises whether this result can be generalized like Kegel's theorem in the Proposition 2.4.

**Problem 1.** Let $A$ and $B$ be subrings of a ring $R$ such that $R = A + B$. Is it true that if $A$ is a left $T$-nilpotent and $B \in \mathcal{M}$, where $\mathcal{M}$ contains the class of all left $T$-nilpotent rings, then $R \in \mathcal{M}$?

As was mentioned before, by a result of Bokut', Kegel's theorem cannot be generalized to rings that are sums of more than two nilpotent subrings. Nevertheless, in [14, Theorem 3.1] it was proved that a ring which is a sum of three subrings with zero multiplication is nilpotent. Now we will show, on the basis of the results from [2], that it is not possible to increase a sum of three to a sum of four subrings with zero multiplication or index nilpotency two to three in [14, Theorem 3.1].

**Definition 2.8.** Let $K$ be associative algebra over a field $k$ of dimension $\dim_k K = \alpha \geqslant \aleph_0$. We say that $K$ satisfies a condition $(*)$ if there exists in $K$ a countable sequence of subalgebras $0 \subset K^{(1)} \subset K^{(2)} \subset \ldots$ such that $K = \bigcup_{n=1}^{\infty} K^{(n)}$ and $\dim K^{(n)} = \dim K^{(n+1)}/K^{(n)} = \alpha$ for every $n \geqslant 1$.

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**Example 2.9.** Let $K$ be an algebra over a field $k$ of dimension $\aleph_0$ with zero multiplication. We will show that $K$ satisfies the condition $(*)$ for $\alpha = \aleph_0$. Indeed, let $X$ be an arbitrary base of algebra $K$. Since $|X| = \aleph_0$, there exists pairwise disjoint countable subsets $X_1, X_2, \dots$ of $X$ such that $X = \bigcup_{n=1}^{\infty} X_n$. Let $K^{(n)}$ be a $k$-subspace of a linear space $K$ generated by $X_1 \cup \ldots \cup X_n$ for $n \geq 1$. Then $K^{(n)}$ is a subalgebra of an algebra $K$ and $\dim K^{(n)} = \dim K^{(n+1)}/K^{(n)} = \aleph_0$ for every $n \geqslant 1$. Clearly, $K = \bigcup_{n=1}^{\infty} K^{(n)}$.

**Theorem 2.10.** ([2, Theorem 2]) *Let $K_1$, $K_2$, $K_3$, $K_4$ be algebras over a field $k$ of dimension $\alpha \geq |k|$, where $\alpha \geqslant \aleph_0$ which satisfy the condition $(*)$. Then any associative $k$-algebra $A$ of dimension $\leq \alpha$ can be embedded into a certain simple associative $k$-algebra $\mathcal{A}$ such that $\mathcal{A} = A_1 + A_2 + A_3 + A_4$ for some its subalgebras $A_i \cong K_i$, where $i = 1, 2, 3, 4$.*

Using Theorem 2.10 and Example 2.9 we get straight away the following

**Corollary 2.11.** *Let $p$ be a prime number and let $A$ be a $\mathbb{Z}_p$-algebra with zero multiplication of dimension $\aleph_0$. Then, every at most countable ring $R$ such that $pR = 0$ can be embedded in some simple ring $\mathcal{A}$ such that $p\mathcal{A} = 0$ and $\mathcal{A}$ is a sum of four subrings $A_i \cong A$. In particular, there exist a simple idempotent countable ring $P$ such that $pP = 0$ and $P = P_1 + P_2 + P_3 + P_4$ for some its countable subrings $P_1, P_2, P_3, P_4$ with zero multiplication.*

Now we will present in more detail mentioned in the introduction very important result of Bokut'.

**Theorem 2.12.** ([2, Theorem 3]) *Any associative $k$-algebra can be embedded into a simple $k$-algebra being an algebraic sum of its three nilpotent $k$-subalgebras, each of which is a free nilpotent algebra of index nilpotency three.*

**Corollary 2.13.** *Let $p$ be a prime number and let $A$ be a free nilpotent $\mathbb{Z}_p$-algebra of index nilpotency three and dimension $\aleph_0$. Then, every at most countable ring $R$ such that $pR = 0$ can be embedded in some countable simple ring $\mathcal{A}$ such that $p\mathcal{A} = 0$ and $\mathcal{A}$ is a sum of three subrings $A_i \cong A$. In particular, there exist an idempotent countable simple ring $P$ such that $pP = 0$ and $P = P_1 + P_2 + P_3$ for some nilpotent countable subrings $P_1, P_2, P_3$ (each of which is isomorphic to $A$).*

It turns out that the following generalization of [14, Theorem 3.1] holds.

**Proposition 2.14.** *Let $R_1$, $R_2$, and $R_3$ be subrings of a ring $R$ such that $R = R_1 + R_2 + R_3$. If $R_1$ is a nilpotent ring, $R_2^2 = \{0\}$, and $R_3^2 = \{0\}$, then $R$ is a nilpotent ring.*

*Proof.* We proceed by induction with respect to the index $n$ of nilpotency of $R_1$. For $n = 1$, we get $R_1 = \{0\}$, so the assertion follows from [6]. Suppose that $n > 1$ and the assertion holds for any smaller index of nilpotency.

Consider the subring $S = R_1 + R_1^{n-1}R$ of $R$. By the modularity of the lattice of subgroups of the group $R^+$ we obtain that $S = S \cap R = R_1 + (R_2 + R_3) \cap S$. Now we show that $S$ is nilpotent. Since $R_1$ is a nilpotent right ideal of $S$, the ideal $I$ of $S$ generated by $R_1$ is nilpotent too. Moreover, $S = I + (R_2 + R_3) \cap S$. First we prove that $((R_2 + R_3) \cap S)^m = \{0\}$ for some positive integer $m$. Let $m = n + 3$ and let $a_i = b_i + c_i$ be elements of $(R_2 + R_3) \cap S$, where $b_i \in R_2$, $c_i \in R_3$ and $1 \leq i \leq m$.

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Notice that the condition $R_1a_i \subseteq R_1^2$ implies that for every $r \in R_1$, there exists $\overline{r} \in R_1^2$ such that

$$rb_i = -rc_i + \overline{r}. \tag{2.1}$$

To show that $a_1a_2\cdots a_m = 0$, it is enough to prove that

$$x_1y_2x_3y_4\cdots z_m = 0, \tag{2.2}$$

where $z_m = x_m$ if $m = 2k+1$ or $z_m = y_m$ if $m = 2k$ for some $k \in \mathbb{N}$. We have the following two cases related to (2.2)

(1) $x_1, x_3, \ldots \in R_2$ and $y_2, y_4, \ldots \in R_3$,
(2) $x_1, x_3, \ldots \in R_3$ and $y_2, y_4, \cdots \in R_2$.

First assume (1). Clearly, $y_2x_3 = t_1 + x + y$ for some $t_1 \in R_1$, $x \in R_2$, and $y \in R_3$. Furthermore, by (2.1) we have $t_1y_4 \in t_1R_2 + R_1^2$. Hence $x_1y_2x_3y_4\cdots z_m = x_1(y_2x_3)y_4\cdots z_m = x_1t_1y_4x_5\cdots z_m \in x_1(t_1R_2^2 + R_1^2x_5)\cdots z_m = x_1R_1^2x_5y_6\cdots z_m$. Therefore, $x_1y_2x_3y_4\cdots z_m = x_1t_2x_5y_6\cdots z_m$ for some $t_2 \in R_1^2$. Consequently, by (2.1), $t_2x_5 \in t_2R_3 + R_1^3$, so $x_1y_2x_3y_4\cdots z_m = x_1t_3y_6\cdots z_m$ for some $t_3 \in R_1^3$. Similarly, after $n$ steps, we get $x_1y_2x_3y_4\cdots z_m = x_1t_nz_m$, where $t_n \in R_1^m$, which gives $x_1y_2x_3y_4\cdots z_m = 0$. The proof of (2.2) for (2) runs analogously.

It follows that $((R_2+R_3)\cap S)^m = \{0\}$, and consequently, $S$ is nilpotent. Therefore, a subring $T = R_1^{n-1} + R_1^{n-1}R \subseteq S$ is nilpotent too. But $T$ is a right ideal of $R$, so the ideal $J$ of $R$ generated by $T$ is nilpotent. Moreover, $R/J = (R_1+J)/J+(R_2+J)/J+(R_3+J)/J$. It is easy to see that $((R_1+J)/J)^{n-1} = \{0\}$. Now we apply the induction hypothesis to infer that the ring $R/J$ is nilpotent and consequently $R$ is nilpotent too. □

In the context of the presented results, the following question arises.

**Problem 2.** Let $R_1$, $R_2$, and $R_3$ be subrings of a ring $R$ such that $R = R_1+R_2+R_3$. Is it true that if $R_1$ and $R_2$ are nilpotent rings, $R_3^2 = \{0\}$, then $R$ is a nilpotent ring?

By above proposition and [14, Theorem 3.1] we obtain the following

**Proposition 2.15.** *Let $R_1$, $R_2$, and $R_3$ be subrings of a ring $R$ such that $R = R_1 + R_2 + R_3$. If $R_1$ is a nil PI ring, $R_2^2 = \{0\}$ and $R_3^2 = \{0\}$, then $R \in \beta$.*

The next two problems are naturally associated with foregoing propositions.

**Problem 3.** Let $R_1$, $R_2$, and $R_3$ be subrings of a ring $R$ such that $R = R_1+R_2+R_3$. Is it true that if $R_1$ is a nil $PI$ ring, $R_2^2 = \{0\}$, and $R_3^2 = \{0\}$, then $R$ is a nil $PI$ ring?

**Problem 4.** Let $R_1$, $R_2$, and $R_3$ be subrings of a ring $R$ such that $R = R_1+R_2+R_3$. Is it true that if $R_1$ is a left (right) $T$-nilpotent, $R_2^2 = \{0\}$, and $R_3^2 = \{0\}$, then $R$ is a left (right) $T$-nilpotent?

To present another result recall some standard properties concerning radicals. A radical $\mathcal{R}$ is called *left hereditary* or *right hereditary* if for every ring $S \in \mathcal{R}$, $I <_l S$ or $I <_r S$ implies $I \in \mathcal{R}$. Furthermore, $\mathcal{R}$ is called *left strong* or *right strong* if for every ring $S$, $\mathcal{R}(S)$ contains all the left or right $S$-ideals of $S$, respectively. A radical $\mathcal{R}$ containing $\beta$, left (or right) hereditary, and left (or right) strong is called *$N$-radical* (for detals see [17]). Examples of $N$-radicals are $\beta$, $\mathcal{L}$ and $\mathcal{J}$.

**Lemma 2.16.** *Let $B$ and $A$ be an $\mathcal{R}$-radical subring of a ring $R$, where $\mathcal{R}$ is an $N$-radical, and a subgroup of the additive group of $R$, respectively. If $R = A + B$ and $A^d \subseteq B$ for some positive integer $d$, and $\mathcal{R}(R) = 0$, then $AB \subseteq A$ and $BA \subseteq A$.*

*Proof.* Since $\mathcal{R}(R) = 0$, $r_R(R)^2 = \{0\}$ and $r_R(R) <_r R$ we have $r_R(R) = \{0\}$. Note that if $AB \nsubseteq A$, then there exists $x \in AB$ such that $x = a + b$, where $a \in A$ and $b \in B$. Hence $A^{d-1}b \subseteq B$. Therefore, $A\bar{b} \subseteq B$ for some $0 \neq \bar{b} \in B$. Consequently, $R\bar{b} \subseteq B$, whence $R\bar{b} \in \mathcal{R}$ because of $R\bar{b} <_l B$. But $R\bar{b} <_l R$ and $\mathcal{R}(R) = 0$, so $Rb = \{0\}$, and consequently $r_R(R) \neq \{0\}$, a contradiction. Thus $AB \subseteq A$. Similarly, we show that $BA \subseteq A$. □

**Proposition 2.17.** *Let $A$ be a subgroup of the additive group of $R$, and $B$ be a $\mathcal{R}$-radical subring of $R$, where $\mathcal{R}$ is an $N$-radical such that $\mathcal{R} \subseteq \mathcal{L}$. If $R = A + B$ and $A^d \subseteq B$ for some positive integer $d$, then $R \in \mathcal{R}$.*

*Proof.* Without loss of generality we can assume that $\mathcal{R}(R) = 0$. Moreover, $A \neq \{0\}$ and $B \neq \{0\}$, so there exists $b \in B$ such that the left ideal $Rb = Ab + Bb$ of $R$ is non-zero. Furthermore, $\mathcal{R}(R) = 0$, whence $Rb \notin \mathcal{R}$. By Lemma 2.16, $Ab \subseteq A$, so $(Ab)^d \subseteq Bb$. Clearly, $Bb \in \mathcal{R}$. Suppose that $Bb \nsubseteq \mathcal{R}(Rb)$. Take any $x \in B$ such that $xb \notin \mathcal{R}(Rb)$. Since $\mathcal{R} \subseteq \mathcal{L}$, the subring $G$ generated by $\{b, x\}$ in $B$ is nilpotent. Hence there exists a positive integer $n \geq 1$ which is maximal with respect to the condition $G^n b \nsubseteq \mathcal{R}(Rb)$. As $b \in G$, we get $bG^n b \subseteq \mathcal{R}(Rb)$, and consequently $RbG^n b \subseteq \mathcal{R}(Rb)$. Therefore, if $S = Rb/\mathcal{R}(Rb)$, then $r_S(S) \neq \{0\}$, a contradiction. Hence $Bb \subseteq \mathcal{R}(Rb)$. Whence $S = (Ab + \mathcal{R}(Rb))/\mathcal{R}(Rb)$. But $(Ab)^d \subseteq Bb \subseteq \mathcal{R}(Rb)$, so $S$ is nilpotent. Therefore, $Rb \in \mathcal{R}$ once again gives a contradiction. □

In this way, we obtained some generalization and simpler proof of [8, Lemma 3.9], where the assertion of Proposition 2.17 has been proved for $\mathcal{R} = \mathcal{L}$ under the assumption $d = 2$, and [14, Proposition 2.4], where $\mathcal{R} = \beta$. Furthermore, the assertion of Proposition 2.17 has been obtained for $\mathcal{R} = \mathcal{J}$ in [15, Theorem 2.2], although is known that $\mathcal{L} \subseteq \mathcal{J}$.

This motivates the next following question:

**Problem 5.** Let $A$ be a subgroup of the additive group of a ring $R$, and let $B$ be an $\mathcal{R}$-radical subring of $R$, where $\mathcal{R}$ is an $N$-radical. Is it true that if $R = A + B$ and $A^d \subseteq B$ for some positive integer $d$, then $R \in \mathcal{R}$?

**Proposition 2.18.** *Let $A$ be a subgroup of the additive group of a ring $R$, and let $B$ be a non-zero left $T$-nilpotent subring of $R$. If $R = A + B$ and $l_R(R) = \{0\}$, then $l_R(A^n) \cap l_R(B) = \{0\}$ for every $n \in \mathbb{N}$.*

*Proof.* Since $l_R(R) = \{0\}$, it follows from Proposition 2.4 that $A^n \neq \{0\}$ for every $n \in \mathbb{N}$. Let $L = l_R(B)$. Since $B \neq \{0\}$ and $B$ is left $T$-nilpotent, $l_B(B) \neq \{0\}$, so $L \neq \{0\}$. For $0 \neq x \in L$ we have $0 \neq xR = xA$. Let $X$ be the set of all positive integers $n$ such that $xA^n \neq \{0\}$ for every $x \in L \setminus \{0\}$. Clearly, $1 \in X$.

Suppose that $X$ is finite. Then there exists the largest number $n$ in $X$. Hence for some non-zero $x \in L$, we have $xA^{n+1} = \{0\}$. But $xA^n \neq \{0\}$, $l_R(R) = \{0\}$, and $R = A + B$, so $xA^n B \neq \{0\}$. Hence $xA^n b_1 \neq \{0\}$ for some $b_1 \in B$. Suppose that for a certain $k \in \mathbb{N}$ there are elements $b_1, \ldots, b_k \in B$ such that $xA^n b_1 \cdot \ldots \cdot b_k \neq \{0\}$. Since $xA^n BA^n \subseteq x(A + B)A^n = xA^{n+1} = \{0\}$, $xA^n b_1 \ldots b_k A^n = \{0\}$, whence $xA^n b_1 \ldots b_k B \neq \{0\}$, that is $xA^n b_1 \ldots b_k b_{k+1} \neq \{0\}$ for some $b_{k+1} \in B$. Thus, we constructed a sequence $(b_k)$ of elements of $B$ such that $xA^n b_1 \ldots b_k \neq \{0\}$, and consequently, $b_1 \cdot \ldots \cdot b_k \neq 0$ for all $k \in \mathbb{N}$. This contradicts the fact that the ring $B$ is left $T$-nilpotent.

Thus the set $X$ is infinite. Take any $m \in \mathbb{N}$. Then there exists $k > m$ such that $k \in X$. Take any non-zero $x \in L$. Then $0 \neq xA^k \subseteq xA^m$, so $xA^m \neq \{0\}$, and consequently $m \in X$. Hence $X = \mathbb{N}$. This means that $L \cap l_R(A^m) = \{0\}$ for every $m \in \mathbb{N}$. □

As a consequence of Proposition 2.18 we get a simple proof of [1, Theorem 2.6]. Namely, we have the following

**Corollary 2.19.** *Let $B$ and $A$ be a left $T$-nilpotent subring of a ring $R$ and a subgroup of the additive group of $R$, respectively. If $R = A + B$ and $A^d \subseteq B$ for some positive integer $d$, then the ring $R$ is left $T$-nilpotent.*

*Proof.* Since $A^d \subseteq B$, we have $l_R(B) \subseteq l_R(A^d)$. But from the left $T$-nilpotence of $B$ it follows that $l_R(B) \neq \{0\}$, so $l_R(A^d) \cap l_R(B) \neq \{0\}$. By Proposition 2.18, $l_R(R) \neq \{0\}$. Now it is easy to see that every non-zero homomorphic image of $R$ has the non-zero left annihilator, so $R$ is a left $T$-nilpotent ring. □

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