Narayana's integer sequence revisited

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An essential feature of the Fibonacci sequence (F_n): 1, 1, 2, 3, 5, 8, … is an explicit expression for its nth term F_n (Binet's formula) and the corollary that $F_n = \left\{\frac{\phi_n}{\sqrt{5}}\right\}$ where $\phi = \frac{1}{2}(1 + \sqrt{5})$ is the golden ratio and $\{x\}$ is the nearest integer to x.

Therefore, from the Fibonacci sequence we can construct a rectangle which is an approximation to a golden rectangle (which has short side unity and long side ϕ) (see Figure 1).

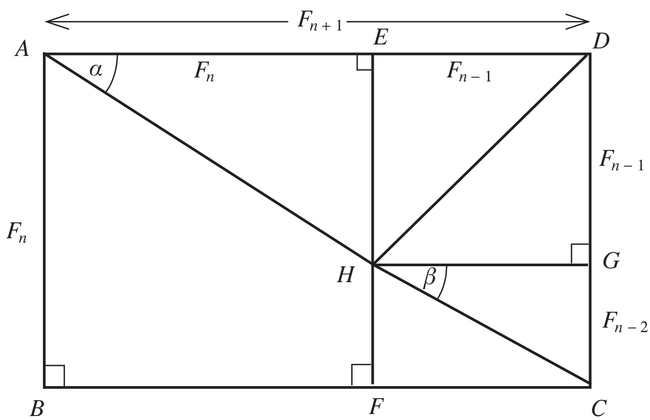

FIGURE 1: Approximation to the golden rectangle

Are there corresponding formulas and approximations for Narayana's cows sequence, the integer sequence ascribed to the fourteenth century Indian mathematician Narayana Panditha? In this sequence rabbit population used to illustrate the Fibonacci sequence is replaced by a cow population which has an extra generation inserted between the young and the mature [1, 2, 3].

1. *Introduction*

Narayana's integer sequence is

$$(a_n) : 1, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28, 41, 60, 88, 129, 189, 277, 406, 595, \ldots \quad (1)$$

defined by $a_1 = a_2 = a_3 = 1$ and $a_{n+3} = a_{n+2} + a_n$ for $n = 1, 2, 3, \ldots$.

A leading property, analogous to the Fibonacci property $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \phi$, is that:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \psi, \qquad n = 1, 2, 3, \ldots \quad (2)$$

where ψ is the supergolden ratio, see [4].

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Since $a_{n+3} = a_{n+2} + a_n$,

$$\frac{a_{n+3}}{a_{n+2}} = 1 + \frac{a_n}{a_{n+2}} = 1 + \frac{1}{\frac{a_{n+2}}{a_{n+1}} \frac{a_{n+1}}{a_n}}.$$

Taking limits of both sides, as $n \to \infty$, $\psi = 1 + \frac{1}{\psi^2}$, so that ψ is a solution of the *cubic* equation:

$$x^3 - x^2 - 1 = 0. \tag{3}$$

This compares with the *quadratic* equation, $x^2 - x - 1 = 0$, for the Fibonacci sequence. Many of the 'nice properties' of the Fibonacci sequence depend on the equation being quadratic, and we might wonder what can be done for the Narayana's sequence.

The cubic equation can be solved by numerical techniques, but it also has a closed form solution:

$$\psi = \frac{1}{3}\left(1 + \sqrt[3]{\frac{29 + 3\sqrt{93}}{2}} + \sqrt[3]{\frac{29 - 3\sqrt{93}}{2}}\right).$$

The decimal value is

$$\psi = 1.46557123187677\ldots\ .$$

We don't need to compute the other roots numerically for we have the factorisation:

$$x^3 - x^2 - 1 = (x - \psi)(x^2 + \psi^{-2}x + \psi^{-1}). \tag{4}$$

The cubic has one real root ψ and because the discriminant of the quadratic part

$$\Delta = \text{‘}b^2 - 4ac\text{’} = \psi^{-4} - 4\psi^{-1} = \psi^{-1}(\psi^{-3} - 4)$$

is negative, the other two roots are complex numbers. Thus, we can write the roots of the cubic (3) as $\psi, \omega, \bar{\omega}$ where $\omega, \bar{\omega}$ are the roots of the quadratic equation part of (4) and $\bar{\omega}$ is the conjugate of ω.

Because this quadratic equation part $x^2 + \psi^{-2}x + \psi^{-1} = 0$ the roots are expressible as functions of ψ. The solution ω is

$$\omega = \frac{-\psi^{-2} + \sqrt{-\Delta}\, i}{2}.$$

The product of the roots of the quadratic equation being ψ^{-1} we note that $|\omega| = \sqrt{\omega\bar{\omega}} = \psi^{-\frac{1}{2}}$. If we switch to the polar form

$$\omega = |\omega| e^{i\theta} = |\omega|(\cos\theta + i\sin\theta),$$

we have

$$\sin\theta = \frac{\sqrt{4\psi^2 + 3}}{2\psi^{\frac{3}{2}}}, \tag{5}$$

and we shall use this later.

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For calculation it is convenient to begin Narayana's sequence (1) with the terms $a_{-1} = 0, a_0 = 0$ and $a_1 = 1$.

2. *Matrix algebra*

From (1) we define the sequence of vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots$ by

$$\mathbf{u}_n = \begin{pmatrix} a_{n-2} \\ a_{n-1} \\ a_n \end{pmatrix}, \qquad n = 1, 2, 3, \ldots \tag{6}$$

and the matrix

$$S = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

noting that

$$S\mathbf{u}_n = \mathbf{u}_{n+1}.$$

The cubic equation (3) reappears as the characteristic polynomial of S since

$$charpoly(S) = \det(xI - S) = x^3 - x^2 - 1 = f(x).$$

The eigenvalues of S are therefore $\psi, \omega, \bar{\omega}$ with corresponding linearly independent eigenvectors $\mathbf{e}_\psi, \mathbf{e}_\omega, \mathbf{e}_{\bar{\omega}}$,

$$\mathbf{e}_\psi = \begin{pmatrix} 1 \\ \psi \\ \psi^2 \end{pmatrix}, \mathbf{e}_\omega = \begin{pmatrix} 1 \\ \omega \\ \omega^2 \end{pmatrix}, \mathbf{e}_{\bar{\omega}} = \begin{pmatrix} 1 \\ \bar{\omega} \\ \bar{\omega}^2 \end{pmatrix}.$$

These constitute a basis for $\mathbb{C}^3$ the vector space of triples of complex numbers.

We next express the initial vector $\mathbf{u}_1$ in terms of this basis:

$$\mathbf{u}_1 = \begin{pmatrix} a_{-1} \\ a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \alpha\mathbf{e}_\psi + \beta\mathbf{e}_\omega + \gamma\mathbf{e}_{\bar{\omega}} \tag{7}$$

and compute the values of the coefficients α, β, γ by solving the three equations derived from (7):

$$\begin{aligned} \alpha &= \frac{1}{(\omega - \psi)(\bar{\omega} - \psi)} = \frac{\psi}{\psi^2 + 3}, \\ \beta &= \frac{1}{(\omega - \bar{\omega})(\omega - \psi)}, \\ \gamma &= \frac{1}{(\bar{\omega} - \omega)(\bar{\omega} - \psi)}. \end{aligned} \tag{8}$$

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3. *Binet's formula*

By applying the matrix S^{n-1} to $\mathbf{u}_1$ we obtain

$$S^{n-1}\mathbf{u}_1 = \mathbf{u}_n = \alpha S_{n-1}\mathbf{e}_\psi + \beta S^{n-1}\mathbf{e}_\omega + \gamma S^{n-1}\mathbf{e}_{\bar{\omega}}, \tag{9}$$

and, using the fact that $\mathbf{e}_\psi$, $\mathbf{e}_\omega$, $\mathbf{e}_{\bar{\omega}}$ are eigenvectors,

$$\mathbf{u}_n = \alpha\psi^{n-1}\mathbf{e}_\psi + \beta\omega^{n-1}\mathbf{e}_\omega + \gamma\bar{\omega}^{n-1}\mathbf{e}_{\bar{\omega}}. \tag{10}$$

As a byproduct of (10) we have an expression for a_n:

$$a_n = \alpha\psi^{n+1} + \beta\omega^{n+1} + \gamma\bar{\omega}^{n+1}.$$

Therefore, (using (8), we have a 'Binet formula' for Narayana's sequence (1):

$$a_n = \frac{\psi^{n+2}}{\psi^2 + 3} + \frac{\omega^{n+1}}{(\omega - \bar{\omega})(\omega - \psi)} + \frac{\bar{\omega}^{n+1}}{(\bar{\omega} - \omega)(\bar{\omega} - \psi)}. \tag{11}$$

The main proposition

Considering the tail of (11), we define

$$T_n = \frac{\omega^{n+1}}{(\omega - \bar{\omega})(\omega - \psi)} + \frac{\bar{\omega}^{n+1}}{(\bar{\omega} - \omega)(\bar{\omega} - \psi)} \tag{12}$$

and we can write (11) as

$$a_n = \frac{\psi^{n+2}}{\psi^2 + 3} \pm |T_n|. \tag{13}$$

Proposition: For Narayana's cows sequence (1):

$$a_n = \left\{\frac{\psi^{n+2}}{\psi^2 + 3}\right\}$$

recalling that $\{x\}$ means the integer nearest to x. There are other formulae for a_n but this new one appears to be the simplest [1]. Let us see how it works, using the (approximate) value of $\psi = 1.46557123187677$.

n	$\frac{\psi^{n+2}}{\psi^2 + 3}$	$\left\{\frac{\psi^{n+2}}{\psi^2 + 3}\right\}$	T_n
1	0.61149	1	+0.38851
2	0.89618	1	+0.10382
3	1.31342	1	-0.31342
4	1.92491	2	+0.07509
5	2.82110	3	+0.17890
6	4.13452	4	-0.13452
7	6.05943	6	-0.05943
8	8.880538	9	+0.11946
9	13.01506	13	-0.01506
10	19.07449	19	-0.07449

TABLE 1

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Proof of the proposition

If we can prove $|T_n| < \frac{1}{2}$ for all positive integer values n, the Proposition will follow. From (12) we rewrite T_n as

$$T_n = \frac{1}{(\omega - \bar{\omega})}\left(\frac{\omega^{n+1}}{(\omega - \psi)} - \frac{\bar{\omega}^{n+1}}{(\bar{\omega} - \psi)}\right). \tag{14}$$

Although T_n involves complex numbers ω and $\bar{\omega}$, T_n is a real number since a_n is real.

We now evaluate T_n. Switching to the polar form $\omega = |\omega| e^{i\theta}$, and using (5):

$$\frac{1}{(\omega - \bar{\omega})} = \frac{i}{-2|\omega| \sin\theta} = -\frac{\psi^2}{\sqrt{4\psi^2 + 3}}\, i.$$

By writing

$$\frac{1}{(\omega - \psi)} = r_\psi e^{i\theta_\psi}$$

$$\frac{\omega^{n+1}}{(\omega - \psi)} = r_\psi e^{i\theta_\psi} |\omega|^{n+1} e^{i(n+1)\theta} = r_\psi |\omega|^{n+1} e^{i(\theta_\psi + (n+1)\theta)}$$

and similarly

$$\frac{\omega^{n+1}}{(\omega - \psi)} = r_\psi e^{-i\theta_\psi} |\omega|^{n+1} e^{-i(n+1)\theta}.$$

Writing $\gamma_n = \theta_\psi + (n + 1)\theta$, (14) gives

$$T_n = \frac{i}{-2|\omega| \sin\theta} r_\psi |\omega|^{n+1} \left(e^{i\gamma_n} - e^{-i\gamma_n}\right).$$

Since $e^{i\gamma_n} - e^{-i\gamma_n} = 2i \sin\gamma_n$,

$$T_n = r_\psi \frac{\sin\gamma_n}{|\sin\theta|} |\omega|^n. \tag{15}$$

Hence, from (15), and (5),

$$|T_n| \leqslant \frac{r_\psi}{\sin\theta} |\omega|^n. \tag{16}$$

And from $r_\psi = \dfrac{1}{|\omega - \psi|}$,

$$r_\psi = \sqrt{\frac{\psi}{\psi^2 + 3}}.$$

We thus obtain:

$$|T_n| \leqslant \frac{r_\psi}{|\sin\theta|} |\omega|^n = \sqrt{\frac{\psi}{\psi^2 + 3}}\, \frac{2\psi^{3/2}}{\sqrt{4\psi^2 + 3}} |\omega|^n, \qquad n = 1, 2, 3, \ldots . \tag{17}$$

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From this and using $|\omega| = \psi^{-1/2}$, we obtain:

$$|T_n| \leqslant \frac{1}{\left(\sqrt{1 + \frac{3}{\psi^2}}\sqrt{1 + \frac{3}{4\psi^2}}\right)}\psi^{-n/2} < \psi^{-n/2}, \qquad n = 1, 2, 3, \ldots . \quad (18)$$

Since $\psi^{-n/2} < \frac{1}{2}$ for $n \geqslant 4$, $|T_n| < \frac{1}{2}$ for these positive integers. Finally, we verify the formula holds for the initial values $n = 1, 2$ and 3 (Table 1) or directly using (18). Therefore $|T_n| < \frac{1}{2}$ for all positive integer values n, and the Proposition is proved.

As a numerical example verifying this result we find:

$$a_{19} = \left\{\frac{\psi^{21}}{\psi^2 + 3}\right\} = \left\{594{\cdot}9980483329\ldots\right\} = 595.$$

5. *Geometric representation*

How can we represent the sequence (1) geometrically corresponding to Figure 1 for the Fibonacci sequence?

To construct such a diagram D_n, we take any three consecutive terms of (1), a_{n-2}, a_{n-1}, a_n for $n \geqslant 4$ (equivalent to choosing a vector $\mathbf{u}_n$) and follow the steps:

1. Let $ABFE$ be a square of side a_n.
2. Let $ED = a_{n-2}$ (and hence $AD = a_{n+1}$).
3. Let $DG = a_{n-1}$ (and hence $GC = a_{n-3}$). Join GH.

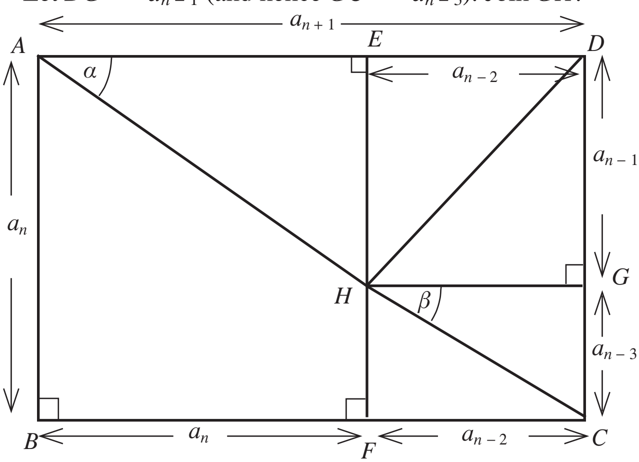

FIGURE 2: D_n

In the five consecutive terms a_{n-3}, a_{n-2}, a_{n-1}, a_n, a_{n+1} the diagram D_n is defined by the vector $\mathbf{u}_n$ and D_{n+1} is generated by $S\mathbf{u}_n = \mathbf{u}_{n+1}$.

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In the construction of D_n, the angles α, β are not equal; and AHC is not quite a straight line, but has a kink at H (just as there was a kink in the construction of the approximation diagram in the Fibonacci case (Figure 1)). As n increases, this kink is straightened and the limiting *form* of D_n is that of D, the supergolden rectangle (the diagram of the rectangle of short side 1 and long side ψ (Figure 3)).

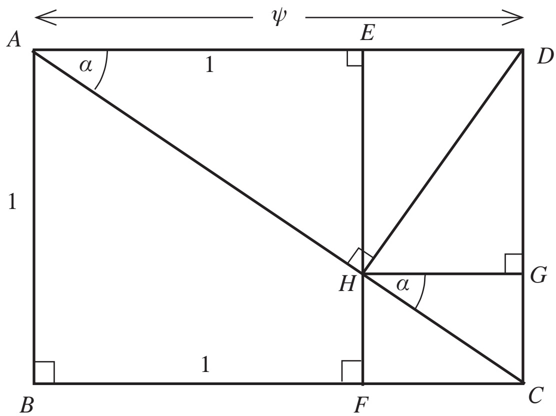

FIGURE 3: The supergolden rectangle

The supergolden rectangle D has the property that rectangle $EHGD$ is proportional to the whole rectangle $BCDA$, that is $\frac{x}{1} = \frac{y}{x - 1}$ and so $x(x - 1) = y$. Letting the horizontal length $AD = x$ and the vertical length $DG = y$, and using the fact that AHC is here a straight line,

$$\tan \alpha = \frac{y}{1} = \frac{1 - y}{x - 1}$$

whence $xy = 1$.

The two conditions

$$x(x - 1) = y,$$

$$xy = 1.$$

combine to give the cubic equation $x^3 - x^2 - 1 = 0$ and the side $x = \psi$.

The supergolden rectangle, with short side unity and long side ψ, has the special property that the angle AHD is a right angle, and we note that this is not the case for the golden rectangle (Figure 1).

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