the universe by measurement of its emission of radio waves in the region of 1420 Mc/s. The presence of considerable masses of hydrogen had been long suspected from considerations of dark clouds and estimates of the mass of the galaxy; it was now proved by direct experiment. Displacement of the fundamental radio-frequency by the Doppler effect enabled us to measure the motion of various parts of the galaxy and threw a flood of light on its structure. Radio reception could also teach us a great deal about the evolution of the stellar universe and might ultimately enable us to decide between the discontinuous and the steady-state theories of the origin of matter. There was also a plan to use the large radio-telescope as a radar transmitter, and in particular to obtain echoes from the planet Venus, which should help us to elucidate the surface conditions obtaining there.

The meeting concluded with a light hearted discussion "That Geometry ought to be abolished."

CORRESPONDENCE

Angles and Numbers

To the Editor of the *Mathematical Gazette*

Dear Sir,

All teachers will surely applaud Mr Hope-Jones' vigorous and amusing exposure of the disastrous consequences of over-emphasis on the degree as a unit of angle. (If anyone would like a simple confirmation of this, let him try marking the angles of a 30°, 60°, 90° triangle $\frac{1}{6}\pi$, $\frac{1}{3}\pi$, $\frac{1}{2}\pi$ and observe the sensation it causes). The error to which he draws attention, however, is also due to an equally serious misdirection of emphasis at a higher level.

Think, first, of all the calculus books you know which have sections on differentiating $\sin x^\circ$, and then consider the harm these may have done. No mathematician endeavours to differentiate *t seconds*, or x^2 *cm*^{2}, for the very good reason that such a process would be as impossible to justify logically as is the algebra error of '$s = 10$ *miles*.' Yet a degree is an arbitrary unit of a similar type, and it is difficult to see that there is sufficient difference between these examples to make the attention given to differentiating $f(x^\circ)$ desirable. Instead of merely stressing that the angle must be measured in radians before any attempt is made to differentiate a circular function and leaving it at that, these unnecessary and possibly misleading formulae have been perpetuated by successive textbook rehashers.

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The radian (or, for that matter, the straight angle) is, of course, a *natural* unit of angle in a sense in which no unit of time or length can ever be said to be natural. But the main reason why such natural units are *possible* for angles and not for most other quantities is seldom made clear to students. It is merely that angle is a dimensionless quantity, and teachers will be wise to drive this fact home often. The old habit (mercifully obsolescent) of adding a stupid little ^{c} to indicate radian measure also helped to conceal this important feature.

The real problem, however, lies much deeper than this. From the point of view of higher mathematics, it is most unfortunate, even if inevitable, that pupils first meet the circular functions as functions of an *angle* rather than a *number*, and that this happens when they are at a very impressionable age. The real reason that these functions are important in mathematics is simply that they are periodic functions of their argument. But some students never seem able to forget that they first met sine as 'opposite over hypotenuse.' I have occasionally been astonished to find that this attitude has survived even among scientists capable of taking S Level and meeting, say, cis $\theta = \exp(i\theta)$ for the first time. Close investigation has sometimes revealed that a boy has been puzzled by some such formula, not because of difficulties with the complex numbers as such, but because he has ONLY been able to think of the θ on the left side as an *angle*. If one tries to define $\cos(a + ib)$ with such a person, he will immediately begin worrying about how $(a + ib)$ can be an angle, even though he may be reasonably happy about other complex functions, such as $\operatorname{ch}(a + ib)$. This means that during all the years in which he has been writing $\sin^{-1} x$ for $\int (1 - x^2)^{-\frac{1}{2}}\, dx$, he has never properly grasped the fact that these functions have exactly the same numerical status as the $\ln |x|$ he has been obtaining for $\int x^{-1}\, dx$. It is an alarming commentary on the success of one's teaching, I can assure you.

The solution obviously lies in taking every opportunity to speak of the tangent of a *number*, rather than the tangent of an angle, and so on. The very *first* time the radian is discussed, it should be pointed out that, with this unit, $\sin x$ becomes a function of the *number* x, just as much as x^3 or $\sqrt{x}$. This dogma should be reaffirmed before any attempt is made to differentiate any circular function, again when the inverse functions are introduced, again before obtaining the series expansions for $\sin x$ and $\cos x$, and so on. One might even venture to suggest that manufacturers of trigonometric equations should sometimes ask their victims to find all the *numbers* between 0 and 2π which are solutions. $\text{Sin}^{-1} x$ should sometimes be read as 'the number whose sine is x' and *never* as 'the angle whose sine is x.' If Mr Hope-Jones' candidate had

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received this sort of training, he would have been less likely to have evaluated his integral in square degrees!

Incidentally, I should like to take this opportunity to disagree profoundly with another writer in the same issue of the *Gazette* who regrets the British use of $\sin^{-1}$ and prefers the continental arcsin for these inverse functions. While I agree that there are a few purposes for which $\sin^{-1} x$ is not an ideal notation, I think that arcsin x is a far worse choice, (quite apart from its clumsiness and the fact that arcsh x by analogy seems rather absurd). Remember that is says 'the *arc* whose sine is x,' and though this approach has a considerable historic interest, it should be clear from the arguments above that by the time the modern pupil has gone far enough to require these inverse functions, he should not be thinking of the sine of an arc any more than of the sine of an angle, but only of the sine of a number While it is true that weaker pupils are often bewildered by $\sin^{-1}$, I cannot agree that, from a more farsighted viewpoint, our notation is as 'misleading' as is claimed, even though an appreciation of its subtlety does perhaps require a greater mathematical maturity than can be expected from the majority of schoolboys. The only satisfactory justification is to present sin as an operator, with inverse $\sin^{-1}$. These operators satisfy the identities $\sin \sin^{-1} x \equiv x$ (for $-1 \leqslant x \leqslant 1$) and $\sin^{-1} \sin x \equiv x$ (for $-\frac{1}{2}\pi \leqslant x \leqslant \frac{1}{2}\pi$). Such manipulations ($TT^{-1} x \equiv T^{-1}T\, x \equiv x$) of transformations, mappings, permutations, automorphisms, etc. are, however, commonplace in modern algebra, and it probably won't be too many years before this flourishing subject has penetrated the school curriculum. Our maligned $\tan^{-1}$ notation, therefore, may eventually prove to be of considerable anticipatory value to our students. (In the meantime, the most urgent reform needed at the school level is a *consistent* policy by examiners to reserve the symbol with the lower case initial letter exclusively for the principal value of the inverse function, and not to use it indiscriminately for either the principal or the general value, as happens with at least one Examining Board.)

For a similar reason, I cannot share the same writer's enthusiasm for the continental and American habit of deliberately confusing an arc length with an angle, however many examples history may provide to sanction this procedure. (There should be no dismay at this; many interesting early treatments, e.g. of series or of complex numbers, are considered inadequate by modern standards.) Having recently had the opportunity of teaching geometry according to this scheme, and having been obliged as a result to tell frequent lies to save contradicting the textbook *too* often, I challenge the contention that it gives 'clearer insight and a valuable generalization.' No confusion between quantities of different mathematical significance

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(and in this case of different dimensions, even) can ever be considered to have clarified the presentation. In Britain, it would only accumulate a wealth of trouble for future work in mechanics and differential geometry, where our syllabus requires an easy familiarity with transformations of the type $s = a\theta$, $v = \dot{s} = a\dot{\theta}$, etc. The formulation this writer wishes us to adopt is, in fact, almost as pernicious as the definitions of the trigonometric functions used by some American writers, who draw a circle of unit radius and *define* their circular functions as *lengths*!

Yours etc., ROGER F. WHEELER

Hymers College, Hull

THE DEFINITION OF A LOCUS

To the Editor of the *Mathematical Gazette*

DEAR SIR,

Mr. Wheeler's criticism (*Gazette*, XLII, 61) of the too-prevalent kinematical account of loci (not 'dynamic,' surely, even in Clifford's idiom) is well directed, and it can only strengthen his case to protest that his definition of a locus has been orthodox for a very long time.

It is not quite the original definition, for in the Greek the stress when the new word was introduced seems to have been on the theorem, not on the set of points identified in the theorem. The locus classicus, so to speak, is a sentence in the *Commentaries* of Proclus. The first English translation, by Thomas Taylor in 1792, runs (Vol. II, 177) "I call those (theorems) local, to which the same symptom happens in a certain place." Few of us would interpret this sentence with any confidence, even with the help of an entry in Thomas Walter's *Mathematical Dictionary*, 1762; "Local problem, such a problem as is capable of an infinite number of solutions, and all different." But Heath in his *Euclid*, 1908 (vol. I, 329) gives us an intelligible version, "I call those (theorems) locus-theorems in which the same property is found to exist on the whole of some locus," thus claiming in effect that the idea of a locus as a propertied class is classical.

This is not to suggest that this idea had the same generality long ago as it has to-day. Until recently there has always been a tacit assumption that only relations of certain kinds were recognized in polite society To L'Hôpital and Maclaurin in the first half of the 18th century, for example, dazzled by the invention of coordinates, a locus is the locus *of an equation*. Again, in the sentence "To every property in relation to each other which points can have, there corresponds some locus, which consists of all the points possessing the property," A. Whitehead in his *Introduction to Mathematics*, 1911 (p. 121), seems to be making no reservations, and it is a shock to find him in the preceding sentence asserting

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