A NOTE ON p-CYCLIC MATRICES AND DIGRAPHS

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We use the terminology of [1]. Let D be a strongly connected digraph on n points and containing m lines, and let $A = A(D)$ be the corresponding adjacency matrix, so that A is an n x n 0-1 matrix containing m unit elements. We recall that A and D are said to be p-cyclic if p is the greatest common divisor of the lengths of all directed cycles of D. Clearly, the larger the value of p, the smaller the value of m must be; in this note we make the latter and related statements precise.

Let $U_1, \ldots, U_p$ denote the cyclic components of D containing, respectively, $n_1, \ldots, n_p$ points. These can be defined as the equivalence classes induced by the equivalence relation wherein two points are equivalent if any (and hence every) directed path joining them has length which is some multiple of p. They are ordered such that points of U_r can only be adjacent to points in U_{r+1}, the subscript being taken modulo p. We then have:

(1) $m \leq g(n_1, \ldots, n_p) = n_1 n_2 + n_2 n_3 + \ldots + n_{p-1} n_p + n_p n_1$

where

(2) $n_1 + n_2 + \ldots + n_p = n$

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LEMMA 1. *If D is a p-cyclic, strongly connected digraph on n points and containing m lines, and in which*

$$q = \min n_i$$

where the i-th cyclic component contains n_i points, then

$$(3) \qquad m \le \left[\frac{n-pq}{2}\right] \left[\frac{n-pq+1}{2}\right] + 2nq - pq^2 .$$

and this inequality is sharp.

Proof. $[x]$ denotes the largest integer $\le x$. Clearly $q \le \left[\frac{n}{p}\right]$. Now from (1),

$$(4) \qquad m \le (n_1 - q)(n_2 - q) + (n_2 - q)(n_3 - q) + \ldots + (n_p - q)(n_1 - q) + 2nq - pq^2 .$$

Case 1: p *even*. From (4),

$$\begin{aligned} m \le \{(n_1 - q) + (n_3 - q) + \ldots + (n_{p-1} - q)\} \{(n_2 - q) + (n_4 - q) \\ + \ldots + (n_p - q)\} + 2nq - pq^2 \\ = X\{n - pq - X\} + 2nq - pq^2 \\ \le \left[\frac{n-pq}{2}\right] \left[\frac{n-pq+1}{2}\right] + 2nq - pq^2, \end{aligned}$$

where $X = (n_2 - q) + (n_4 - q) + \ldots + (n_p - q)$.

Case 2: p *odd*. Without loss of generality, we may assume that $n_p = q$. Then, from (4),

$$\begin{aligned} m \le \{(n_1 - q) + (n_3 - q) + \ldots + (n_p - q)\} \{n_2 - q) + (n_4 - q) \\ + \ldots + (n_{p-1} - q)\} + 2nq - pq^2, \end{aligned}$$

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the term $(n_p - q)(n_1 - q)$ being zero. The proof now follows that of Case 1.

The inequality (3) is sharp if we let $n_1 = n_2 = \ldots = n_{p-3} = n_{p-2} = q$,

$$n_{p-1} = \left[\frac{n-pq}{2}\right], \quad n_p = \left[\frac{n-pq+1}{2}\right]$$

and maximise the number of lines used in deriving (1).

LEMMA 2. <u>Let, in Lemma 1,</u>

(5) $$f(q) = \left[\frac{n-pq}{2}\right]\left[\frac{n-pq+1}{2}\right] + 2nq - pq^2 .$$

Then, if $p \geq 4$ and $1 \leq q_1 < q_2 \leq \left[\frac{n}{p}\right]$,

(6) $$f(q_1) \geq f(q_2) .$$

<u>Proof</u>. $f(q_1) - f(q_2) \geq 1/4 \{ (n-pq_1)^2 - 1\} + 2nq_1 - pq_1^2$

$$- 1/4 \{ (n-pq_2)^2 \} - 2nq_2 + pq_2^2$$

$$= 1/4 \{ (q_2 - q_1)(p-4)(2n-p(q_1 + q_2)) - 1\}$$

$$\geq - 1/4.$$

But $f(q_1) - f(q_2)$ is an integer, and the lemma follows.

LEMMA 3. <u>If</u> $p \geq 4$, <u>then Lemma 1 still holds if</u>

$$q \leq \min n_i$$

<u>Proof</u>. Follows immediately from Lemmas 1 and 2.

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THEOREM. *If* D *is a p-cyclic, strongly connected digraph on* n *points and containing* m *lines, then*

$$m \le \begin{cases} n(n-1) & , \quad p=1 \quad (a) \\ 2\left[\frac{n}{2}\right]\left[\frac{n+1}{2}\right] & , \quad p=2 \quad (b) \\ \left\{2n-3\left[\frac{n+1}{3}\right]\right\}\left[\frac{n+1}{3}\right] & , \quad p=3 \quad (c) \\ \left[\frac{n-p}{2}\right]\left[\frac{n-p+1}{2}\right] + (2n-p), & \quad p \ge 4 \quad (d) \end{cases}$$

Furthermore, these bounds are best possible

Proof.

(a) Obvious

(b) $m \le 2n_1 n_2 \le 2\left[\frac{n}{2}\right]\left[\frac{n+1}{2}\right]$, this being sharp

for $n_1 = \left[\frac{n}{2}\right]$, $n_2 = \left[\frac{n+1}{2}\right]$.

(c) From Lemma 1,

$$m \le \max\left\{\left[\frac{n-3q}{2}\right]\left[\frac{n-3q+1}{2}\right] + 2nq - 3q^2\right\}$$

$$1 \le q \le \left[\frac{n}{3}\right]$$

and the result follows from a tedious, but straightforward argument. The result is sharp for

$$n_1 = n_2 = r + \left[\frac{s}{2}\right], \quad n_3 = r + \left(\left[\frac{s+1}{2}\right] - \left[\frac{s}{2}\right]\right)$$

where $n = 3r + s$, $0 \le s \le 2$.

(d) Follows immediately from Lemma 3 with $q = 1$. This bound is, of course, sharp for

$$n_1 = n_2 = \ldots = n_{p-3} = n_{p-2} = 1$$

$$n_{p-1} = \left[\frac{n-p}{2}\right], \quad n_p = \left[\frac{n-p+1}{2}\right]$$

COROLLARY. If D is a strongly connected digraph on n points (n > 2) containing m lines and

$$m > 2 \left[\frac{n}{2}\right] \left[\frac{n+1}{2}\right],$$

then D is primitive.

Proof. Immediate.

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REFERENCE

1. Harary, F., Norman R.Z. and D. Cartwright: Structural models: an introduction to the theory of directed graphs, Wiley, New, 1965.

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