and the proof of the theorem. However, we should remind the reader that when two theorems, P, Q say, are equivalent (i.e, a proof of P can be deduced from Q and vice versa) they need not be generalisations of one another. For example, Pythagorean theorem and Law of Cosines are equivalent, and the latter is a generalisation of the former but not necessarily vice versa (note, no generalisation of Law of Cosines is known in the literature that is proved via Pythagorean theorem, up to now). We like to remind the reader that in [2, Prob. 15(c). p. 58] it is observed that Euler's theorem can be deduced from Fermat's theorem. However, in [2], and also in the literature, in general, the peculiar and simple fact that Fermat's theorem is, indeed, a generalisation of Euler's theorem too, is overlooked (note, as John Conway once said, see [1], [3]: All the easy things, at first sight, appear to have been said already, but you can find that they have not been said). Finally, we would like to record the following equivalence, also as a corollary to Theorem B.

Corollary 3: The following theorems are equivalent.

(1) Fermat's theorem.

(2) Euler's theorem.

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108.21 An amazing quartet of integrals

Introduction: Some time ago I stumbled upon the following four related integral representations of well-known mathematical constants:

$$\pi^2 = 2J(-2), \ \zeta(3) = \tfrac{2}{7}J(-1), \ \pi^3 = 8J(0), \ G = \tfrac{1}{4}J(1)$$

where

$$J(k) = \int_0^{\frac{\pi}{2}} \operatorname{arctanh}^2(\cos t)\cos^k t \ dt$$

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and G (Catalan's constant) and $\zeta(3)$ (Apéry's constant) are given by:

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}, \qquad \zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}.$$

Note that $\operatorname{arctanh} x = \frac{1}{2} \log \frac{1+x}{1-x}$.

In this Note we will prove these results in an Eulerian way, by turning the integrals into series which sum to the corresponding constants. This technique, using integrals to sum series and vice versa, was a favourite of Leonhard Euler. For instance, when he tried to solve the Basel problem, the problem of summing the series of the reciprocals of the square numbers (see [1])

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} + \ldots =?$$

he made use of the integral

$$\int \frac{\log(1-t)}{t}\, dt = \int \frac{1}{t}\left(t + \frac{t^2}{2} + \frac{t^3}{3} + \ldots + \frac{t^n}{n} + \ldots\right) dt$$
$$= t + \frac{t^2}{2^2} + \frac{t^3}{3^2} + \ldots + \frac{t^n}{n^2} + \ldots + C.$$

Solving the Basel problem made Euler famous, and during his lifetime he gave several proofs of the result. Note that this problem, which had been around since 1650, was probably tackled by every mathematician of that time. One of them was Niklaus I. Bernoulli, who used another approach, leading to a new kind of series. The idea was the following: why not square the well-known Leibniz series for $\frac{\pi}{4}$? See [2] for the whole story. In doing so he needed the sum of the series

$$1 - \frac{1}{2}\left(1 + \frac{1}{3}\right) + \frac{1}{3}\left(1 + \frac{1}{3} + \frac{1}{5}\right) - \ldots + \frac{(-1)^{n-1}}{n} \sum_{i=1}^{n} \frac{1}{2i-1} + \ldots$$

or, with the notation $h_n = \sum_{i=1}^{n} \frac{1}{2i-1}$, the sum of

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{h_n}{n}. \tag{1}$$

Leonhard Euler had no problem summing this series. In a letter to Johann Bernoulli, uncle of Niklaus, dated July 30, 1738, Euler gave the formula

$$2 \int \frac{\arctan x}{1+x^2}\, dx = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{h_n}{n} x^{2n} \tag{2}$$

which solves Niklaus' problem since the integral on the left-hand side is equal to $\arctan^2 x$. (Note that Euler didn't write the constant of integration.)

Similar sums to the one in (1) were later studied extensively by Euler, and are called Euler sums. Using the same method we are able to find Euler-

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like sums for the constants π, π^2, $\zeta(3)$ and G, starting from the value of the integrals $J(k)$ for $k = -2, -1, 0, 1$.

Calculating the integrals: We now prove the four results from the introduction. The calculation of these integrals proves to be an interesting and not too difficult exercise in integration. For $J(-2)$ the obvious partial integration leads to a cousin of $J(-1)$:

$$J(-2) = \int_0^{\frac{\pi}{2}} \frac{\operatorname{arctanh}(\cos t)}{\cos t}\, dt = \int_0^{\frac{\pi}{2}} \frac{\log \frac{1+\cos t}{1-\cos t}}{\cos t}\, dt$$

which, using the half-angle formulas, reduces to

$$-2 \int_0^{\frac{\pi}{2}} \frac{\log \tan \frac{t}{2}}{\left(1 - \tan^2 \frac{t}{2}\right) \cos^2 \frac{t}{2}}\, dt = -4 \int_0^1 \frac{\log x}{1 - x^2}\, dx.$$

We calculate this last integral using the substitution $v = \log x$ and the formula

$$\int v\, e^{av}\, dv = \frac{(av - 1)\, e^{av}}{a^2} + C.$$

Indeed, we have

$$\int_0^1 \frac{\log x}{1 - x^2}\, dx = \int_{-\infty}^0 \frac{v\, e^v}{1 - e^{2v}}\, dv = \int_{-\infty}^0 v\, e^v \sum_{n=0}^{\infty} e^{2nv}\, dv$$

$$= \int_{-\infty}^0 v \sum_{n=0}^{\infty} e^{(2n+1)v} dv = -\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}.$$

This last sum is equal to $-\frac{\pi^2}{8}$. Hence $J(-2) = \frac{\pi^2}{2}$.

The value of $J(-1)$ is found in a similar way:

$$J(-1) = \int_0^{\frac{\pi}{2}} \log^2 \tan \frac{t}{2} \cdot \frac{1 + \tan^2 \frac{t}{2}}{1 - \tan^2 \frac{t}{2}}\, dt = 2 \int_0^1 \frac{\log^2 x}{1 - x^2}\, dx.$$

Again we use the substitution $v = \log x$ and the formula

$$\int v^2\, e^{av}\, dv = \frac{\left(a^2 v^2 - 2av + 2\right) e^{av}}{a^3} + C$$

leading to

$$\int_0^1 \frac{\log^2 x}{1 - x^2}\, dx = \int_{-\infty}^0 v^2 \sum_{n=0}^{\infty} e^{(2n+1)v} dv = \sum_{n=0}^{\infty} \frac{2}{(2n+1)^3}.$$

The last series sums to $\frac{7}{4}\zeta(3)$. This proves that $J(-1) = \frac{7}{2}\zeta(3)$.

In the same way we can rewrite $J(0)$ as

$$J(0) = 2 \int_0^1 \frac{\log^2 x}{1 + x^2}\, dx = 2 \int_{-\infty}^0 v^2 \sum_{n=0}^{\infty} (-1)^n e^{(2n+1)v} dv$$

$$= 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{8}.$$

Finally, for $J(1)$ we use partial integration to obtain

$$J(1) = \int_0^{\frac{\pi}{2}} \left(\log^2 \tan \frac{t}{2}\right) \cos t \, dt = -2 \int_0^{\frac{\pi}{2}} \log \tan \frac{t}{2} \, dt.$$

The same substitution as before leads to

$$\int_0^{\frac{\pi}{2}} \log \tan \tfrac{t}{2} \, dt = 2 \int_{-\infty}^{0} v \sum_{n=0}^{\infty} (-1)^n e^{(2n+1)v} dv = -2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = -2G.$$

We may ask ourselves if the integral $J(k)$ produces other fundamental constants for other integers k. This is not the case. For $k \leqslant -3$ the integral diverges. For $k \geqslant 1$ we use the following set of recurrences. Let $I(k)$ be defined by

$$I(k) = \int_0^{\frac{\pi}{2}} \log\left(\tan \tfrac{t}{2}\right) \cos^k t \, dt,$$

and note that $J(k)$ can be written as

$$J(k) = \int_0^{\frac{\pi}{2}} \log^2\left(\tan \tfrac{t}{2}\right) \cos^k t \, dt.$$

Then we have for $k = 1, 2, 3, \ldots$ that

$$I(k) = -\frac{1}{k} \int_0^{\frac{\pi}{2}} \cos^{k-1} t \, dt + \frac{k-1}{k} I(k-2)$$

$$J(k) = -\frac{2}{k} I(k-1) + \frac{k-1}{k} J(k-2).$$

These recurrences are reminiscent of the Wallis integral formulas, and it is an interesting basic calculus exercise to prove them. In addition we have $I(0) = -2G$, $I(1) = -\frac{\pi}{2}$ and $J(0) = \frac{\pi^3}{8}$. It now follows immediately that all $J(k)$, with $k \geqslant 0$ even, have values of the form $C_1\pi + C_2\pi^3$, and for k odd they have the form $C_3 + C_4 \cdot G$.

The corresponding Euler-like sums: Mimicking Euler we can associate with each of the four integrals an Euler-like sum. To do this we need a series for $\operatorname{arctanh}^2 x$,

$$\operatorname{arctanh}^2 x = \sum_{n=1}^{\infty} \frac{h_n}{n} x^{2n}$$

which is very similar to (2). To prove it, note that the derivative should satisfy

$$\frac{2 \operatorname{arctanh} x}{1 - x^2} = 2 \sum_{n=1}^{\infty} h_n x^{2n-1}$$

and this is easily checked by multiplying both sides by $1 - x^2$ and using the Maclaurin series

$$\operatorname{arctanh} x = \tfrac{1}{2} \log \frac{1+x}{1-x} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}.$$

We now use this series in the four integrals. For instance, if we take $J(-2)$, we can write

$$J(-2) = \int_0^{\frac{\pi}{2}} \frac{\operatorname{arctanh}^2(\cos t)}{\cos^2 t}\, dt = \int_0^{\frac{\pi}{2}} \sum_{n=1}^{\infty} h_n \cos^{2n-2} t\, dt.$$

Hence

$$J(-2) = \sum_{n=1}^{\infty} \frac{h_n}{n} \int_0^{\frac{\pi}{2}} \cos^{2n-2} t\, dt = \pi \sum_{n=1}^{\infty} \frac{h_n}{(2n-1)} \binom{2n}{n} \frac{1}{2^{2n}}$$

using Wallis' integral formulas, gives the first of the following four Euler-like sums

$$\pi = 2 \sum_{n=1}^{\infty} \frac{\binom{2n}{n} h_n}{(2n-1)\, 2^{2n}}, \qquad \zeta(3) = \frac{1}{7} \sum_{n=1}^{\infty} \frac{2^{2n} h_n}{n^2 \binom{2n}{n}},$$

$$\pi^2 = 4 \sum_{n=1}^{\infty} \frac{\binom{2n}{n} h_n}{n 2^{2n}}, \qquad G = \frac{1}{4} \sum_{n=1}^{\infty} \frac{2^{2n} h_n}{n(2n+1) \binom{2n}{n}}.$$

Note that the first appearance in the literature of this series for $\zeta(3)$ seems to be in 2018 [3]. The series for Catalan's constant appears to be new.

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