J. Austral. Math. Soc. (*Series A*) 36 (1984), 279–286

THE VON NEUMANN KERNEL OF A LOCALLY COMPACT GROUP

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(Received 16 December 1982)

Communicated by W. Moran

Abstract

For a locally compact group G, the von Neumann kernel, $n(G)$, is the intersection of the kernels of the finite dimensional (continuous) unitary representations of G. In this paper we calculate $n(G)$ explicitly for a general connected locally compact group and for certain classes of non-connected groups.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): 22 D 10.

Introduction

For a locally compact group G, the von Neumann kernel, $n(G)$, is the intersection of the kernels of the finite dimensional continuous complex unitary representations of G. Rothman [5] has calculated $n(G)$ when G is a connected Lie group. Every such group has a Levi decomposition $G = RL$, where R is the radical and $L = KS$ is the decomposition of the Levi factor into compact and noncompact parts. Rothman first considers the group $G' = G/[R, R]^-$, which again has such a decomposition $G' = R'K'S'$, but now R' is abelian. He shows that R' can be written as $V_f^\perp \times V_f \times T$, where T is compact and fixed by the action of $K'S'$, V_f is a vector group fixed by the action of $K'S'$, and $V_f^\perp$ is a vector group stabilized by the action of $K'S'$. Furthermore, every element of $V_f^\perp$ except the identity has infinite $K'S'$-orbit. Then the von Neumann kernel of G' is $V_f^\perp S'$, and the von Neumann kernel of G itself is the preimage of $V_f^\perp S'$ in G. That is, if $\pi: G \to G'$ is the projection, $n(G) = \pi^{-1}(V_f^\perp \cdot \pi(S))$. Unless otherwise mentioned, results in the Lie case are to be found in [5].

https://doi.org/10.1017/S1446788700024708 Published online by Cambridge University Press

If G is a connected locally compact group it has a similar decomposition $G = RKS$, R the radical, K a compact semisimple connected group, and S a connected semisimple Lie group with no compact factors [3]. In general, both K and R may be infinite dimensional. We show that there is a subgroup $V_f^{\perp}$ of $\pi(G) = G/[R, R]^-$ with the same properties as discussed above, and that in general

THEOREM. *Let G be a locally compact connected group. Then* $n(G) = \pi^{-1}(V_f^{\perp}\, \pi(S))$.

As a consequence it is shown that if L is a connected semisimple locally compact group and π a representation of L in a separable vector space V where V has an L-fixed subspace of finite co-dimension then π is completely reducible. In the concluding section there are some extensions to non-connected groups.

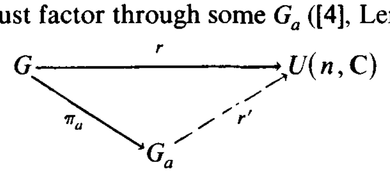
1

Every connected locally compact group G is the projective limit of Lie groups. More specifically, there exists a set of compact normal subgroups N_a of G such that each $G_a = G/N_a$ is a Lie group, and if $N_a \subset N_b$ there exists a surjective homomorphism f_{ba}: $G_a \to G_b$. Furthermore the intersection of the groups N_a is trivial. Then G can be identified to the closed subgroup of the direct product $\prod G_a$ consisting of the points (x_a) for which $f_{ba}(x_a) = x_b$ for every a and b with $N_a \subset N_b$, and we write $G = \lim G_a$. The projection from G to G_a is denoted π_a. If $N_a \subset N_b$ we say that $a < b$.

LEMMA 1.1. *Let $G = \lim G_a$ be a locally compact group. Then x belongs to $n(G)$ if and only if $\pi_a(x)$ belongs to $n(G_a)$ for every a.*

PROOF. Let x belong to $n(G)$. Any representation r_a of G_a can be pulled back to one of G by composing with π_a. Then $(r_a \cdot \pi_a)(x) = e$, so $\pi_a(x)$ belongs to $n(G_a)$.

Let $\pi_a(x)$ belong to $n(G_a)$ for every a and let r be a finite dimensional unitary representation of G. There is an N_a in the kernel of r since the image of G has no small subgroups. If G is connected N_a is compact, but in general N_a is some normal subgroup which may be taken to be contained in any neighborhood of the identity of G. Now r_a must factor through some G_a ([4], Lemma 2.2):



Since $\pi_a(x)$ belongs to $n(G_a)$, $r(x) = e$, so x belongs to $n(G)$.

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Assume for the time being that G is connected. If $G = RKS$ is a Levi decomposition for G then $G_a = \pi_a(R)\pi_a(K)\pi_a(S) = R_a K_a S_a$ is a Levi decomposition for G_a. Note that π_a is a closed map.

PROPOSITION 1.2. *S is closed in G and $S = \lim S_a$.*

PROOF. The noncompact part of a Levi factor of a connected Lie group is closed ([5], Theorem 1.4), so we have $\pi_a(S^-) = \pi_a(S)^- = \pi_a(S)$, which is semisimple with no compact factors. The radical of S^- is contained in the intersection of the kernels of the maps π_a and so is trivial, so S^- is semisimple with no compact factors. Since S^- is closed and $\pi_a\colon S^- \to S_a$ surjective, $S^- = \lim S_a$. Now $G = RKS$ and $G = RK(S^-)$ are both Levi decompositions, so S and S^- are conjugate ([3], Theorem 3), so $S = S^-$.

Let $[R, R]^-$ be the closure of the commutator subgroup of R in G. Then $G' = G/[R, R]^-$ has abelian radical, and $n(G) = \pi^{-1}(n(G'))$, where $\pi\colon G \to G/[R, R]^-$ (see [5] for details). Consequently, we may assume that G has abelian radical. Using Rothman's characterization for the von Neumann kernel of a connected Lie group ([5], Theorem 1.2), we have $n(G_a) = V_{af}^{\perp} S_a$. We know that $S = \lim S_a$. The following section shows that $V_f^{\perp} = \lim V_{af}^{\perp}$ exists and $n(G) = V_f^{\perp} S$.

Since G_a is Lie it has the decomposition $G_a = (V_{af}^{\perp} \times V_{af} \times T_a)K_a S_a$. Since $n(G_a) = V_{af}^{\perp} S_a$, the group $V_{af}^{\perp}$ belongs to $R_a \cap n(G_a)$. The latter group is $V_{af}^{\perp} \times D_a$, where $D_a = S_a \cap R_a$ is a discrete group in the center of G.

LEMMA 1.3. *$f_{ba}(V_{af}^{\perp})$ belongs to $V_{bf}^{\perp}$.*

PROOF. The map f_{ba} takes R_a onto R_b and $n(G_a)$ into $n(G_b)$, so $f_{ba}(V_{af}^{\perp})$ belongs to $V_{bf}^{\perp} \times D_b$. Since $V_{af}^{\perp}$ is connected, its image belongs to $V_{bf}^{\perp}$, the connected component of $V_{bf}^{\perp} \times D_b$.

Because R is abelian it has a decomposition $R = V \times P$, where P is a maximal compact subgroup and V is a (finite dimensional) vector group of dimension n. The dimension does not depend on the choice of V. Since the kernel of π_a is compact, the restriction of π_a to V is injective, so R_a may be decomposed as $\pi_a(V) \times \pi_a(P)$ with $\dim \pi_a(V) = n$. The maps f_{ba} also have compact kernel and so are injective when restricted to a vector subgroup of G_a.

LEMMA 1.4. *$f_{ba}(V_{af}^{\perp}) = V_{bf}^{\perp}$.*

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PROOF. Consider the groups $V_{af} \times T_a$. Each of them is actually $R_a \cap Z(G_a)$, $Z(G_a)$ being the center of G_a, so $f_{ba}(V_{af} \times T_a)$ belongs to $V_{bf} \times T_b$, and $\dim V_{af} \leqslant \dim V_{bf}$. From Lemma 1.3, $\dim V_{af}^{\perp} \leqslant \dim V_{bf}^{\perp}$. Together with the above discussion, this implies that $\dim V_{af}^{\perp} = \dim V_{bf}^{\perp}$, so $V_{af}^{\perp}$ maps onto $V_{bf}^{\perp}$.

PROPOSITION 1.5. *The projective limit of the* $V_{af}^{\perp}$ *exists.*

PROOF. Take the elements (x_a) in the direct product of the groups G_a which have x_a in $V_{af}^{\perp}$ for every a. Then, given an a, $f_{ba}(x_a) = x_b$ for every $a < b$, by Lemma 1.4. The projective limit consists of the (x_a).

Denote the projective limit by $V_f^{\perp}$. It is a subgroup of G and the projections π_a are just the restrictions of those of G.

LEMMA 1.6. $V_f^{\perp}$ *is a vector group of dimension* m *which is* L*-stable, and every element except the identity has infinite* L*-orbit.*

PROOF. $V_f^{\perp}$ is L-stable since $\pi_a(gxg^{-1}) = \pi_a(g)\pi_a(x)\pi_a(g^{-1})$ belongs to $V_{af}^{\perp}$ for every a, and $f_{ba}(\pi_a(gxg^{-1})) = \pi_b(gxg^{-1})$. Operations in the direct product $\prod V_{af}^{\perp}$ are componentwise, so $V_f^{\perp}$ is a vector group, and a basis is easily found by fixing an a, choosing a basis for $V_{af}^{\perp}$, $\{y_i\}$, and then taking the elements (x_a) of $V_f^{\perp}$ which have y_i in the ath place. The dimensions of all the groups $V_{af}^{\perp}$ and $V_f^{\perp}$ are the same, say m, because of the injectivity of the π_a when restricted to $V_f^{\perp}$. Now suppose that x in $V_f^{\perp}$ is L-fixed. Then $\pi_a(gxg^{-1}) = \pi_a(x)$, so that $\pi_a(x)$ is L-fixed, and in $V_{af}^{\perp}$ for every a. Consequently $\pi_a(x) = e_a$, the identity of $V_{af}^{\perp}$ for every a. Suppose that x is not L-fixed. Let L_x be the stabilizer of x in L. Then the orbit of x is homeomorphic to L/L_x. Since L is connected and $L \neq L_x$, the quotient L/L_x has an infinite number of points.

LEMMA 1.7. *Let* H, M, *and* HM *be closed subgroups of* G. *Then* $\lim H_a \cdot \lim M_a = \lim H_a M_a$ *if the limits exists.*

PROOF. $\pi_a(HM) = \pi_a(H)\pi_a(M) = H_a M_a$, showing surjectivity. Since HM is closed the lemma follows. (See [2], Theorem 2.3.)

LEMMA 1.8. *Let* H *be the subgroup of* R *consisting of* L*-fixed elements. Then* $R = V_f^{\perp} \times H$.

PROOF. Since H is abelian and locally compact, $H = V_f \times P$, where P is a compact group and V_f a vector group. The group $V_f^{\perp} \times V_f$ is closed in R, and therefore so is $V_f^{\perp} \times H$. Since $V_f^{\perp}$ and H are closed too, $R = \lim R_a = \lim V_{af}^{\perp} \times V_{af} \times T_a = \lim V_{af}^{\perp} \times H_a = \lim V_{af}^{\perp} \times \lim H_a = V_f^{\perp} \times H$.

Proposition 1.9. *$V_f^{\perp}\, S$ is closed in G.*

Proof. Choose a decomposition $H = V_f \times P$ for H. The group P is unique, and once chosen, V_f will remain fixed throughout the proof. Let $V = V_f^{\perp} \times V_f$.

Case 1: Assume that $Z(L)$ is finite. Then $V \cap PL$ is a discrete central subgroup of PL which must actually be trivial, since V has no nontrivial finite subgroups. Since $G = V \cdot PL$, and V is normal, G is the semidirect product of V and PL. In particular, as a space $G = V \times PL$. Since $V_f^{\perp}$ is closed in V and L is closed in G and therefore in PL, $V_f^{\perp} \cdot L$ is closed in G. Since S is closed in L [3], $V_f^{\perp}\, S$ is also closed in G.

Case 2: Now consider the case $Z(L)$ arbitrary. The semisimple group L acts linearly on $V_f^{\perp}$. Since a semisimple linear group has a finite center, there exists a subgroup Γ of finite index in $Z(L)$ which acts trivially on $V_f^{\perp}$. The groups V_f and P belong to $Z(G)$, and G is generated by $V_f^{\perp}$, V_f, P, and L, so Γ belongs to $Z(G)$. Let $\pi\colon G \to G/\Gamma$ be the canonical projection. Since $Z(L)$ is totally disconnected, π is a local isomorphism. Now Γ belongs to L, so $\pi^{-1}(\pi(V_f^{\perp}\, L)) = V_f^{\perp}\, L$, and therefore by the continuity of π it is enough to show that $\pi(V_f^{\perp})\pi(L)$ is closed in G/Γ. If we show that $\pi(V_f^{\perp})$ is the "$V_f^{\perp}$" of G/Γ, we are reduced to the first case, and the theorem is proven.

Lemma 1.10. *In Proposition* 1.9, *$\pi(V_f^{\perp})$ is the "$V_f^{\perp}$" of G/Γ.*

Proof. First, if $\pi_a(\Gamma) = \Gamma_a$, we have $G/\Gamma = \lim G_a/\Gamma_a$. Indeed, since Γ is closed, $\Gamma = \lim \Gamma_a$. The Γ_a are discrete, and in $Z(G_a)$, so $G/\Gamma = \lim G_a/\Gamma_a$ ([2], Theorem 2.7).

Next, consider the diagram

$$\begin{array}{ccc} G & \xrightarrow{\ \pi\ } & G/\Gamma \\ {\scriptstyle \pi_a}\downarrow & & \downarrow{\scriptstyle \tilde{\pi}_a} \\ G_a & \xrightarrow{\ \tilde{\pi}\ } & G_a/\Gamma_a \end{array}$$

where $\tilde{\pi}_a$ is defined to make the diagram commute, and the other maps are the obvious projections. We have $\pi_a(V_f^{\perp}) = V_{af}^{\perp}$ by definition of $V_f^{\perp}$. The group $\tilde{\pi}(V_{af}^{\perp})$ is the "$V_{af}^{\perp}$" of G_a/Γ_a since these groups are Lie and have the same Lie algebras, from which the $V_{af}^{\perp}$ are selected uniquely. Consequently, the "$V_f^{\perp}$" of G/Γ is $\lim \tilde{\pi}(V_{af}^{\perp}) = \lim V_{af}^{\perp}$. Now $\tilde{\pi}_a(\pi(V_f^{\perp})) = \tilde{\pi}(\pi_a(V_f^{\perp})) = V_{af}^{\perp}$, and $\pi(V_f^{\perp})$ is closed in G/Γ, so $\pi(V_f^{\perp}) = \lim V_{af}^{\perp} =$ the "$V_f^{\perp}$" of G/Γ.

Finally, $V_f^{\perp}\, S$ is closed by the same arguments as in Case 1.

Theorem 1.11. *If G is a connected locally compact group with abelian radical, $n(G) = V_f^{\perp}\, S$.*

PROOF. Since $\pi_a(V_f^\perp S)$ is contained in $n(G_a)$ for every a, $V_f^\perp S$ belongs to $n(G)$. Furthermore, if x belongs to $n(G)$ then $\pi_a(x) = v_{af}^\perp s_a$ for every a, and $f_{ba}(\pi_a(x)) = \pi_b(x)$, so x belongs to $\lim V_{af}^\perp S_a = V_f^\perp S$.

COROLLARY 1.12. *If G is as in the theorem*, $n(G) = \lim n(G_a)$.

PROOF. Since $V_f^\perp = \lim V_{af}^\perp$, $S = \lim S_a$, and $V_f^\perp S$ are all closed subgroups of G, Lemma 1.7 and Theorem 1.11 imply that $n(G) = V_f^\perp S = \lim V_{af}^\perp \cdot \lim S_a = \lim V_{af}^\perp S_a$. But each G_a is a connected Lie group and therefore $n(G_a) = V_{af}^\perp S_a$ ([5], Theorem 1.2). The result follows.

COROLLARY 1.13. *If G is as in the theorem, $n(G)$ is a Lie group.*

THEOREM 1.14. *If G is a connected locally compact group, and π and $V_f^\perp$ are defined as above, then $n(G) = \pi^{-1}(V_f^\perp \cdot \pi(S))$.*

PROOF. The proof is the same as in [5]. See the discussion following our Proposition 1.2.

2. Extensions and consequences

LEMMA 2.1. *If $G = \lim G_a$, and G^0 (respectively G_a^0) is the identity component of G (respectively G_a), then $G^0 = \lim G_a^0$.*

We omit the proof, which is well known.

LEMMA 2.2. *If G is an arbitrary locally compact group and G/G^0 is compact, then $n(G) \subset G^0$ and $n(G^0) \subset n(G)$.*

PROOF. The finite dimensional unitary representations of G/G^0 separate the points, so for every element of G which does not lie in G^0 there is a representation of G which does not send the element to the identity, so $n(G) \subset G^0$. The second statement follows easily also since the restrictions of representations of G are representations of G^0.

LEMMA 2.3. *If G has a finite number of components*, $n(G) = n(G^0)$.

PROOF. Let x belong to $n(G)$. Then x belongs to $n(G^0)$. Let r be a finite dimensional unitary representation of G^0, and consider U^r, the representation induced from r. U^r is also unitary and finite dimensional, so $U^r(x)$ is trivial. Calculating,

$$U^r(x)f(g) = f(gx) = f(gxg^{-1}g) = r(gxg^{-1})f(g)$$

so that $r(gxg^{-1})$ is trivial for every g, and in particular, when $g = e$. Therefore, $r(x) = e$ for every finite dimensional unitary representation r, so x belongs to $n(G^0)$. Together with Lemma 2.2, this implies that $n(G) = n(G^0)$.

PROPOSITION 2.4. *Let G be a locally compact group with abelian radical and G/G^0 compact. Then $n(G) = n(G^0)$.*

PROOF. Since G/G^0 is compact, $G = \lim G_a$, where each G_a has a finite number of components and has abelian radical as well. From Lemma 2.3, $n(G_a) = n(G_a^0)$, so $\lim n(G_a) = \lim n(G_a^0) = n(G^0)$, with the last equality following from Lemma 2.1 and Corollary 1.12. Now $\pi_a(n(G)) \subset n(G_a)$, so $n(G) \subset \lim n(G_a) = n(G^0)$. Since $n(G^0) \subset n(G)$, the proposition follows.

LEMMA 2.5. *Let G be a connected locally compact group. Then* $n(G) = \lim n(G_a)$.

PROOF. $n(G) = \pi^{-1}(V_f^\perp\, \pi(S)) = V_f^\perp\, S[R, R]^-$, so

$$\pi_a(n(G)) = \pi_a(V_f^\perp)\pi_a([R, R]^-)S_a = V_{af}^\perp\, [R_a, R_a]^- \cdot S_a = n(G_a).$$

Consequently $n(G) = \lim n(G_a)$.

THEOREM 2.6. *If G is locally compact and G/G^0 is compact then* $\lim n(G_a)$ *exists and $n(G) = n(G^0)$.*

PROOF. $\lim n(G_a^0)$ exists since G_a^0 is connected. Since G/G^0 is compact, G can be expressed as the projective limit of groups G_a with G_a/G_a^0 finite. Consequently, $n(G_a) = n(G_a^0)$ by Lemma 2.3, so $\lim n(G_a)$ exists. Now $n(G^0) \subset n(G) \subset \lim n(G_a) = \lim n(G_a^0) = \lim n(G_a^0) = n(G^0)$, so $n(G) = n(G^0)$.

COROLLARY 2.7. *If G/G^0 is compact*, $n(G) = \lim n(G_a)$.

Maximally almost periodic groups are those with $n(G) = (e)$.

COROLLARY 2.8 (see [1], Corollary 2.10). *If G/G^0 is compact, then G is maximally almost periodic if and only if G^0 is maximally almost periodic.*

We conclude with a theorem about representations of semisimple groups.

THEOREM 2.9. *Let π be a representation of a connected semisimple group L on a separable vector space V. Suppose that V has a subspace W which is L-fixed and of finite co-dimension. Then π is completely reducible.*

PROOF. Consider the collection $\mathfrak{W}$ of all subspaces of W of finite co-dimension. If W' belongs to $\mathfrak{W}$, $V' = V/W'$ is finite dimensional and the intersection of all such W' is trivial. Thus $V = \lim V'$. With the trivial bracket operation, V and all the V' become Lie algebras, and, since each W' is L-fixed, L acts on each V'. There is a locally compact group R with Lie algebra V. Indeed, for each V' the image W'' of W has a discrete subgroup D' such that W''/D' is compact. Take V'/D' for the choice of Lie group with Lie algebra V'. Then $R = \lim V'/D'$ exists and is a locally compact group with Lie algebra V. Furthermore, L acts on each V'/D', and so it acts on R. Form the group $G = R \times L$. G is a connected locally compact group with abelian radical. As above, R decomposes into $V_f^{\perp} \times H$, where H is L-fixed and $V_f^{\perp}$ is L-stable. Let V^* be the Lie algebra of $V_f^{\perp}$. Then V^* is L-stable, and $V = V^* \oplus W$. The representation π restricted to V^* is completely reducible, since $\pi(L)$ is a semisimple Lie group and Weyl's Theorem applies, and W decomposes into 1-dimensional L-fixed subspaces.

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