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RESEARCH ARTICLE

Discounted densities of overshoot and undershoot for Lévy processes with applications in finance

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Abstract
This paper considers the first passage times to constant boundaries and the two-sided exit problem for Lévy process with a characteristic exponent in which at least one of the two jumps having rational Laplace transforms. The joint distribution of the first passage times and undershoot/overshoot are obtained. The processes recover many models that have appeared in the literature such as the compound Poisson risk models, the perturbed compound Poisson risk models, and their dual ones. As applications, we obtain the solutions for popular path-dependent options such as lookback and barrier options in terms of Laplace transforms.

1. Introduction

In this paper, we suppose that $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t : t \geq 0\}, P)$ is a filtered probability space with $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$ satisfies the usual conditions. On this space, we define a real valued Lévy process $X = (X_t)_{t\geq 0}$. For $x \in \mathbb{R}$, let us denote by $\mathbb{P}_x$ the law of X when it starts at x and for simplicity we write $\{P_x : x \in \mathbb{R}\}$ probabilities such that under P_x, $X(0) = x$ with probability one and for a special case $P = P_0$. Furthermore, we shall denote the expectation operator associated to P_x and P by E_x and E. For $u \in \mathbb{R}$, the Lévy-Khintchine formula states that $E(e^{iuX_t}) = e^{t\Psi(u)}$, where the characteristic exponent of the process is given as:

$$\Psi(u) = i\mu u - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} (e^{iux} - 1 - iux\mathbf{1}_{\{|x|\leq 1\}})\Pi(dx), \tag{1.1}$$

here $\mu \in \mathbb{R}$, $\sigma > 0$, and Π is called the Lévy measure concentrated on $\mathbb{R} \setminus \{0\}$ satisfying:

$$\int_{-\infty}^{\infty} (1 \wedge x^2)\Pi(dx) < \infty.$$

The triple (μ, σ, Π) is known as the generating triplet of X. It will be useful to refer to Bertoin [8], Sato [42], and Kyprianou [32, 33] for more information account on Lévy processes. It is usually to assume that the process does not degenerate, i.e. $\sigma \neq 0$ or $\Pi \neq 0$.

Given two constant barriers h and H ($h < H$), let τ_H^+ and τ_h^- denote the first upwards passage time over H and the first downwards passage time under h, respectively, as follows:

$$\tau_H^+ := \inf\{t \geq 0 : X(t) \geq H\}, \quad \tau_h^- := \inf\{t \geq 0 : X(t) \leq h\}.$$

The overshoot and undershoot through the boundaries H and h, are given by $T^H = X(\tau_H^+) - H$ and $T_h = h - X(\tau_h^-)$, respectively.

We introduce the first exit time τ of $\{X(t)\}$ to the interval (h, H), i.e.

$$\tau = \inf\{t \geq 0 : X(t) \notin (h, H)\}.$$

Clearly, $\tau = \tau_h^- \wedge \tau_H^+$. Note that it may either hit h (H) or jump over h (H) when crossing h (H) from above (below) depending on the components of the process. Further, we use

$$Y = (X(\tau) - H)\mathbf{1}_{A^H} + (-X(\tau) + h)\mathbf{1}_{A_h}$$

that is a nonnegative random variable, to model the size of the jump of the process cross the boundary at the first-exit time from the interval (h, H), where

$$A^H = \{X(\tau) \geq H\}, \quad A_h = \{X(\tau) \leq h\}.$$

Throughout this paper, we define an exponential random variable by $e(q)$ with parameter $q > 0$, which is independent with X, and for the special case of $q = 0$, denote $e(0) = \infty$. Moreover, for $q > 0$, the running supremum and running infimum of X killed at rate q is expressed by:

$$\overline{X}_{e(q)} := \sup_{0 \leq u \leq e(q)} X_u, \quad \underline{X}_{e(q)} := \inf_{0 \leq u \leq e(q)} X_u.$$

When $q \to 0$, the random variables (possibly degenerated) $\overline{X}_\infty$ and $\underline{X}_\infty$ can arrive at the supremum and infimum of the processes, respectively. In regard to the case in which $q = 0$, it is always assumed that $E(X_1) < 0$, therefore, the process tends to $-\infty$. In other words, when $P(\underline{X}_\infty = -\infty) = 1$, the random variable $\overline{X}_\infty$ is suitable. Their characteristic functions are given by:

$$\Psi^+(u) = E(e^{iu\overline{X}_{e(q)}}), \quad \Psi^-(u) = E(e^{iu\underline{X}_{e(q)}}).$$

It follows from Tauberian theorem that $\Psi^+(-\infty) = P(\overline{X}_{e(q)} = 0)$ and $\Psi^-(\infty) = P(\underline{X}_{e(q)} = 0)$. The Wiener-Hopf factorization states that (see Lewis and Mordecki [37]) $\overline{X}_{e(q)} - X_{e(q)}$ is independent of $\overline{X}_{e(q)}$ and $X_{e(q)} - \underline{X}_{e(q)}$ is independent of $\underline{X}_{e(q)}$. Moreover,

$$\overline{X}_{e(q)} - X_{e(q)} =^d -\underline{X}_{e(q)}, \quad X_{e(q)} - \underline{X}_{e(q)} =^d \overline{X}_{e(q)}.$$

As a consequence, for $u \in \mathbb{R}$,

$$E(\exp(iuX_{e(q)})) = q(q - \Psi(u))^{-1} = \Psi^+(u)\Psi^-(u).$$

The explicit integral expressions for the Wiener–Hopf factors Ψ^+ and Ψ^- are rather complex, if needed, please refer to Lewis and Mordecki [37]. It is worth noting that the Wiener–Hopf factors can be determined clearly in some cases. For example, when $\Psi(u)$ is a rational function. In this case, Kuznetsov [29] pointed out $q(q - \Psi(u))^{-1}$ is also a rational function, and thus it has a finite number of zeros/poles in the complex plane $\mathbb{C}$. Since $\overline{X}_{e(q)}$ $(\underline{X}_{e(q)})$ is positive (negative) and infinitely divisible, Ψ^+ (respectively, Ψ^-) admits the analytic continuation into the upper half-plane $\mathbb{C}^+ = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ (respectively, lower half-plane $\mathbb{C}^- = \{z \in \mathbb{C} : \mathrm{Im}(z) < 0\}$) and does not vanish there. Therefore, one can uniquely identify both Ψ^+ and Ψ^- as a rational function, which has values one at $z = 0$ and whose poles/zeros coincide with poles/zeros of $q(q - \Psi(u))^{-1}$ in $\mathbb{C}^+$ and $\mathbb{C}^-$, respectively.

The integral transforms of the joint distribution of $\{\tau_H^+, T^H\}$ and $\{\tau_h^-, T_h\}$ were obtained by Pehcerskii and Rogozin [41], one can find a simple proof in Kadankov and Kadankova [22] and Alili and

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Kyprianou [3]. According to those joint distributions of one-boundary functionals, Kadankov and Kadankova [22] determined the Laplace transform of the joint distribution of $\{\tau, Y\}$. Closed form expressions are obtained for a particular class of Lévy processes, see for example, Kou and Wang [27] considered a double exponential jump-diffusion process; Kuznetsov, Kyprianou and Pardo [30] for Meromorphic Lévy Process; Cai [9] for a hyper-exponential jump diffusion process; Kadankova and Veraverbeke [23], Chi [13] and Chi and Lin [14] for the two-sided jump-diffusion process with an exponential component; Villarroel and Vega [44] for a compound renewal process. In the applied probability, the passage time problems for Lévy processes have also been studied, related to theory theories of insurance risks, queues, mathematical finance, dams, etc. For instance, in the theory of actuarial mathematics, the problem of first exit from a half-line concerning the classical ruin problem, the expected discounted penalty function, and the expected total discounted dividends until ruin, have attracted much attention and many research results appeared. See e.g., Avram, Palmowski & Pistorius [7], Klüppelberg et al. [25], Mordecki [40], Xing et al. [46], Cai et al. [10], Zhang et al. [49], Chi [13], Chi and Lin [14], Yin et al. [47], Hu et al. [20]. In the context of mathematical finance, the first passage time is crucial for the pricing of many path-dependent options, American-type and Russian-type options, such as Kou [26], Kou and Wang ([27], [28]), Asmussen et al. [6], Alili and Kyprianou [3], Cai et al. [10], Cai and Kou [11], Jeannin and Pistorius [21], Kim et al. [24] and together with certain credit risk models, see, for example, Hilberink and Rogers [19], Le Courtois and Quittard-Pinon [34], Dong et al. [16], and Leippold and Vasiljevic [35]. Cai and Sun [12] investigate pricing problems of both infinite- and finite-maturity stock loans under a hyper-exponential jump diffusion model. Many optimal stopping strategies have also been demonstrated to come down to first passage problems for jump diffusion processes, see e.g. Mordecki [39], and Alvarez et al. [4]. Many of these have been established for some specific subclasses of Lévy processes, but very little is known in the general case. For the passage problem, we generalize the results of Kou and Wang [27], where the double exponential jump-diffusion process was considered. In a recent paper by Ai et al. [1] further results were obtained, in particular for a refracted jump diffusion process with hyper-exponential jumps. It can only be solved for certain kinds of jump distributions. In this paper, we study a more complex Lévy process with a characteristic exponent of the form (1.1) in which at least one of the two jumps having rational Laplace or Fourier transforms.

The aim of this paper is two-fold. First, motivated by Lewis and Mordecki [36, 37], we study explicit discounted densities of overshoot and undershoot to the one-sided and two-sided first-exit problems for the Lévy process with a characteristic exponent of the form (1.1) in which at least one of the two jumps having rational Laplace or Fourier transforms. Second, we obtain analytical solutions to the pricing problem of one barrier options and lookback options.

The rest of the paper is organized as follows. In Sections 2 and 3, we consider one-sided passage problems from below or above and the two-sided exit problems from a finite interval for the Lévy process with a characteristic exponent of the form (1.1). Section 4 gives the analytical solutions for the pricing problem of one barrier options and lookback options.

2. The overshoot and undershoot at first passage

In this section, we study the overshoot and undershoot of a Lévy process at first passage time, based on the assumption in Lewis and Mordecki [37]. Let X be a Lévy process and its jump measure is given by:

$$\Pi(dx) = \begin{cases} \pi^+(dx) = \lambda p(x)dx, \text{ if } x > 0, \\ \pi^-(dx), \text{ if } x < 0, \end{cases} \tag{2.1}$$

where π^- is an arbitrary Lévy measure supported on $(-\infty, 0)$ characterizing the behavior of negative jumps of the process. The positive jumps of the process have finite intensity $\lambda > 0$ and magnitude

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distributed on the basis of the probability density given by:

$$p(x) = \sum_{k=1}^{\nu}\sum_{j=1}^{n_k} c_{kj} \frac{\alpha_k{}^j x^{j-1}}{(j-1)!} e^{-\alpha_k x},\ x > 0, \tag{2.2}$$

where the parameters c_{kj} and $0 < \alpha_1 \leq \Re(\alpha_2) \leq \cdots \leq \Re(\alpha_\nu)$ are constants, $\Re(\alpha_k)$ is the real part of $\alpha_k, k = 2, \cdots, \nu$. This is the general form of the density of a random variable whose Fourier transform is a rational function:

$$\hat{p}(u) = \int_0^\infty e^{iux} p(x)dx = \sum_{k=1}^{\nu}\sum_{j=1}^{n_k} c_{kj} \left(\frac{i\alpha_k}{u + i\alpha_k}\right)^j.$$

We denote the characteristic exponent of X by:

$$\psi_1(u) := \log E[\exp(iuX_1)] = \psi_1^-(u) + \lambda(\hat{p}(u) - 1),$$

where

$$\psi_1^-(u) = i\mu u - \frac{1}{2}\sigma^2 u^2 + \int_{-\infty}^0 (e^{iux} - 1 - iux\mathbf{1}_{\{|x|\leq 1\}})\pi^-(dx),$$

is the characteristic exponent of a Lévy process X^- that has no positive jumps.

As pointed out by Lewis and Mordecki [37], the characteristic exponent $\psi_1(u)$ extend analytically to the complex atrip $-\alpha < \mathrm{Im}(z) < 0$, and continuous in $-\alpha < \mathrm{Im}(z) \leq 0$, under the formula:

$$\psi_1(z) = \psi_1^-(z) + \lambda(\hat{p}(z) - 1),$$

that satisfying $\log E[\exp(izX_1)] = \psi_1(z)$. Moreover, $\psi_1(z)$ is a meromorphic function in the set $\mathrm{Im}(u) < 0$, with poles $-i\alpha_1, \cdots, -i\alpha_\nu$ and respective multiplicities $n_1, \cdots, n_\nu$. The total pole count is $n := n_1 + \cdots + n_\nu$.

Moreover, based on the assumption in Lewis and Mordecki [36], we introduce a Lévy process X' whose jump measure is given by:

$$\Pi'(dx) = \begin{cases} \pi'^+(dx), \text{ if } x > 0, \\ \pi'^-(dx) = \lambda' q(x)dx, \text{ if } x < 0, \end{cases} \tag{2.3}$$

where π'^+ is an arbitrary Lévy measure supported on $(0, \infty)$ characterizing the behavior of positive jumps of the process, the negative jumps of the process have finite intensity $\lambda' > 0$ and magnitude distributed on the basis of the probability density given by:

$$q(x) = \sum_{k=1}^{\nu}\sum_{j=1}^{n_k} c_{kj} \frac{\alpha_k{}^j (-x)^{j-1}}{(j-1)!} e^{\alpha_k x},\ x < 0, \tag{2.4}$$

where c_{kj} and $0 < \alpha_1 \leq \Re(\alpha_2) \leq \cdots \leq \Re(\alpha_\nu)$ are constants, $\Re(\alpha_k)$ is the real part of $\alpha_k, k = 2, \cdots, \nu$. This is the general form of the density of a random variable whose Fourier transform is a

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rational function:

$$\hat{q}(u) = \int_{-\infty}^{0} e^{iux} q(x)dx = \sum_{k=1}^{\nu} \sum_{j=1}^{n_k} c_{kj} \left(\frac{-i\alpha_k}{u - i\alpha_k} \right)^j .$$

We denote the characteristic exponent of X by:

$$\psi_2(u) := \log E[\exp(iuX_1)] = \psi_2^+(u) + \lambda(\hat{q}(u) - 1),$$

where

$$\psi_2^+(u) = -i\mu u - \frac{1}{2}\sigma^2 u^2 + \int_0^\infty (e^{iux} - 1 - iux\mathbf{1}_{\{|x|\le 1\}})\pi^+(dx),$$

is the characteristic exponent of a Lévy process X^+ that has no negative jumps. Observe that $\psi_2(u)$ can analytic continuation into the strip $0 \le \mathrm{Im}(u) < \alpha_1$, and more generally, it can be continued to a meromorphic function $\psi_2(z)$ defined in the set $\mathrm{Im}(z) > 0$, with the poles $i\alpha_1, \cdots, i\alpha_\nu$ and respective multiplicities $n_1, \cdots, n_\nu$.

In this section, we obtain a fluctuation identity for overshoot up a given level and undershoot below a given level of a Lévy process. To a fixed level $a > 0$, we define the first strict passage time τ_a^+ up a by:

$$\tau_a^+ = \inf\{t > 0 : X_t \ge a\},$$

and the overshoot $X_{\tau_a^+} - a$. The first passage time τ_{-a}^- below $-a$ by

$$\tau_{-a}^- = \inf\{t > 0 : X_t < -a\},$$

and define the undershoot by: $-X_{\tau_{-a}^+} - a$. We shall only provide proof about the fluctuation identity for overshoot up a given level of a Lévy process because the fluctuation identity for undershoot can be obtained by using $-X$ in place of X.

Theorem 2.1. *(i) For $q, s > 0$, we have:*

$$E\left(e^{-q\tau_a^+ - s(X_{\tau_a^+} - a)} \boldsymbol{1}_{\{\tau_a^+ < \infty\}}\right) = \sum_{k=1}^{N} \sum_{j=1}^{m_k} \frac{d_{j,k} e^{-\beta_k a}}{(j-1)!} \sum_{l=1}^{j} (a\beta_k)^{l-1} \Delta_{k,j,l}(s); \tag{2.5}$$

(ii) For $y \ge 0$, we have:

$$\begin{aligned} E\left(e^{-q\tau_a^+} \boldsymbol{1}_{\{X_{\tau_a^+} - a \in dy\}}\right) &= \delta_0(z) \sum_{k=1}^{N} \sum_{j=1}^{m_k} f_{k0} \frac{d_{j,k} e^{-\beta_k a}}{(j-1)!} (a\beta_k)^{j-1} dy \\ &+ \sum_{k=1}^{N} \sum_{j=1}^{m_k} \frac{d_{j,k} e^{-\beta_k a}}{(j-1)!} \sum_{l=1}^{j} (a\beta_k)^{l-1} f_{(k,j,l)} dy, \end{aligned} \tag{2.6}$$

where $\delta_0(x)$ is the Dirac delta at $x = 0$, $\beta_1, \cdots, \beta_N$ are the roots of $\psi_1(z) = q$ (see Lewis and Mordecki [37] (Lemma 1.1)) and the coefficients $d_0, d_{1,1}$ and $d_{j,k}$ are given by:

$$d_0 = \begin{cases} 0, & \text{if } -X^- \text{ is a not subordinator,} \\ \prod_{j=1}^{N} \beta_j^{m_j} \prod_{k=1}^{\nu} \alpha_k^{-n_k}, & \text{if } -X^- \text{ is a subordinator,} \end{cases} \tag{2.7}$$

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$$d_{1,1} = \prod_{j=1}^{\nu} \left(\frac{\alpha_j - \beta_1}{\alpha_j} \right)^{n_j} \prod_{k=2}^{N} \left(\frac{\beta_k}{\beta_k - \beta_1} \right)^{m_k}, \tag{2.8}$$

and for $k = 2, \cdots, N$ *and* $j = 0, 1, \cdots, m_k - 1$,

$$d_{k,m_k-j} = \frac{1}{j!\beta_k^{m_k-j}} \left[\frac{\partial^j}{\partial u^j} \left(\phi^+(-u)(u+\beta_k)^{m_k} \right) \right] |_{u=-\beta_k} \tag{2.9}$$

where

$$\phi^+(-u) := E[e^{-u\overline{X}_{e(q)}}] = \prod_{k=1}^{\nu} \left(\frac{u+\alpha_k}{\alpha_k} \right)^{n^k} \prod_{j=1}^{N} \left(\frac{\beta_j}{u+\beta_j} \right)^{m_j}. \tag{2.10}$$

$$\Delta_{k,j,l}(s) = \frac{\Pi_{i=1,i\neq k}^{N}(1+\frac{s}{\beta_i})^{m_i}(1+\frac{s}{\beta_k})^{m_k-j+l-1}}{\Pi_{r=1}^{\nu}(1+\frac{s}{\alpha_r})^{n_r}}, \tag{2.11}$$

$$f_{k0} = \begin{cases} 0, & \text{if } -X^- \text{ is a subordinator,} \\ \beta_k \prod_{i=1}^{\nu} \alpha_i^{n_i} / \prod_{j=1}^{N} \beta_j^{m_j}, & \text{if } -X^- \text{ is not a subordinator,} \end{cases} \tag{2.12}$$

$$f_{(k,j,l)} = \sum_{k_1=1}^{\nu} \sum_{j_1=1}^{n_{k_1}} f_{k_1,j_1}^{(k,j,l)} \frac{y^{j_1-1}e^{-\alpha_{k_1}y}}{(j_1-1)!}.$$

Here

$$f_{k_1,n_{k_1}-j_1}^{(k,j,l)} = \frac{1}{j_1!\alpha_{k_1}^{n_{k_1}-j_1}} \left[\frac{\partial^{j_1}}{\partial s^{j_1}} \frac{(1+\frac{s}{\alpha_{k_1}})^{n_{k_1}}}{(1+\frac{s}{\beta_k})^{j-l+1}\phi^+(-s)} \right] |_{s=-\alpha_{k_1}}, \tag{2.13}$$

$k_1 = 1, \cdots, \nu, j_1 = 0, 1, \cdots, n_{k_1}$.

Proof. (i). Using (2.6) in Lewis and Mordecki [37], we get:

$$\begin{aligned} E(e^{-s(\overline{X}_{e(q)}-a)}\mathbf{1}_{\{\overline{X}_{e(q)}>a\}}) &= \int_a^\infty e^{-s(y-a)} f_{\overline{X}_{e(q)}}(y)dy \\ &= \sum_{k=1}^{N} \sum_{j=1}^{m_k} \frac{d_{j,k}\beta_k^j e^{-\beta_k a}}{(j-1)!} \sum_{l=1}^{j} \frac{a^{l-1}}{(s+\beta_k)^{j-l+1}}, \end{aligned}$$

this, together with (2.2) in Lewis and Mordecki [37], yields:

$$\frac{E(e^{-s(\overline{X}_{e(q)}-a)}\mathbf{1}_{\{\overline{X}_{e(q)}>a\}})}{E(e^{-s\overline{X}_{e(q)}})} = \sum_{k=1}^{N} \sum_{j=1}^{m_k} \frac{d_{j,k}e^{-\beta_k a}}{(j-1)!} \sum_{l=1}^{j} (a\beta_k)^{l-1} \Delta_{k,j,l}(s), \tag{2.14}$$

where $\Delta_{k,j,l}(s)$ is given by (2.11). The result (2.5) follows from the following well-known formula:

$$E\left(e^{-q\tau_a^+ - s(X_{\tau_a^+}-a)}\mathbf{1}_{\{\tau_a^+<\infty\}}\right) = \frac{E(e^{-s(\overline{X}_{e(q)}-a)}\mathbf{1}_{\{\overline{X}_{e(q)}\geq a\}})}{E(e^{-s\overline{X}_{e(q)}})}, \tag{2.15}$$

which is due to Pecherskii and Rogozin [41], see also Kadankov and Kadankova [22] and Alili and Kyprianou [3] for a simple proof.

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(*ii*). By the fractional expansion,

$$\Delta_{k,j,l}(s) = \sum_{k_1=1}^{\nu} \sum_{j_1=0}^{n_{k_1}} f_{k_1,j_1}^{(k,j,l)} \left(1 + \frac{s}{\alpha_{k_1}}\right)^{-j_1}, \tag{2.16}$$

where

$$\sum_{k_1=1}^{\nu} f_{k_1,0}^{(k,j,l)} = \begin{cases} 0, & m = n, \\ \beta_k \prod_{i=1}^{\nu} \alpha_i^{n_i} / \prod_{j=1}^{N} \beta_j^{m_j}, & m = n+1, l = j, \\ 0, & m = n+1, l \neq j, \end{cases} \tag{2.17}$$

and

$$f_{k_1,n_{k_1}-j_1}^{(k,j,l)} = \frac{1}{j_1! \alpha_{k_1}^{n_{k_1}-j_1}} \left[\frac{\partial^{j_1}}{\partial s^{j_1}} \left(\Delta_{k,j,l}(s)(1 + \frac{s}{\alpha_{k_1}})^{n_{k_1}} \right) \right] \Big|_{s=-\alpha_{k_1}}, \tag{2.18}$$

$k_1 = 1, \cdots, \nu, j_1 = 0, 1, \cdots, n_{k_1}$.

Substituting (2.16)-(2.18) into (2.5), we can obtain (2.6) immediately by inverting it on s. This ends the proof of Theorem 2.1. □

Remark 2.2. The class of rational type densities of form (2.2) are a wide class of probability densities, including exponential, combinations and mixtures of exponentials, Erlang and Cox distributions and phase type distributions; See Asmussen and Albrecher [5], Yin et al. [47], and Yin et al. [48].

Corollary 2.3. *Assume the process $X = \{X_t\}_{t\geq 0}$ has the Lévy triplet (μ, σ^2, Π) with Laplace exponent ϕ_1, where Π is given by (2.1) with p is a combination of ν exponential distributions*

$$p(x) = \sum_{k=1}^{\nu} b_k \beta_k e^{-\beta_k x}, x > 0,$$

for some $0 < \beta_1 < \cdots \beta_\nu < \infty$ and the $b_1 + \cdots + b_\nu = 1$. This is different from a mixture because some of the b_k values can be negative so long as $p(x) \geq 0$. Then

(i) for $q, s > 0$,

$$E\left(e^{-q\tau_a^+ - s(X_{\tau_a^+} - a)} \mathbf{1}_{\{\tau_a^+ < \infty\}})\right) = \sum_{k=1}^{J} B_k \frac{\prod_{i=1, i\neq k}^{J} (1 + \frac{s}{r_i})}{\prod_{i=1}^{\nu} (1 + \frac{s}{\beta_i})} e^{-r_k a}, \tag{2.19}$$

(ii) for $q > 0, y \geq 0$,

$$E\left(e^{-q\tau_a^+} \mathbf{1}_{\{X_{\tau_a^+} - a \in dy\}}\right) = \sum_{k=1}^{J} B_k \left(A_{k0} \delta_0(y) + \sum_{l=1}^{\nu} A_{kl} \beta_l e^{-\beta_l y} \right) e^{-r_k a} dy,$$

(iii) for $q > 0$,

$$E\left(e^{-q\tau_a^+} \mathbf{1}_{\{X_{\tau_a^+} = a\}}\right) = \sum_{k=1}^{J} B_k A_{k0} e^{-r_k a},$$

(iv) for $q > 0$,

$$E\left(e^{-q\tau_a^+} \mathbf{1}_{\{X_{\tau_a^+} > a\}}\right) = \sum_{k=1}^{J} B_k \left(\sum_{l=1}^{\nu} A_{kl} \right) e^{-r_k a} = \sum_{k=1}^{J} B_k (1 - A_{k0}) e^{-r_k a},$$

where $r_1, \cdots, r_J$ are the positive roots of the equation $\phi_1(r) = q$, and

$$J = \begin{cases} \nu + 1, & \sigma > 0, \text{or } \sigma = 0 \text{ and } \mu > 0, \\ \nu, & \sigma = 0 \text{ and } \mu \leq 0, \end{cases}$$

$$B_j = \frac{\prod_{k=1}^{\nu}(1-\frac{r_j}{\beta_k})}{\prod_{k=1,k\neq j}^{J}(1-\frac{r_j}{r_k})}, j=1,\cdots,J,$$

$$A_{k0} = \begin{cases} \frac{\prod_{i=1}^{\nu}\beta_i}{\prod_{i=1,i\neq k}^{J} r_i}, & \sigma>0, \text{or } \sigma=0 \text{ and } \mu>0,\\ 0, & \sigma=0 \text{ and } \mu\le 0,\end{cases}$$

$$A_{kl} = \frac{\prod_{i=1,i\neq k}^{J}(1-\beta_l/r_i)}{\prod_{i=1,i\neq l}^{\nu}(1-\beta_l/\beta_i)}, \ l=1,2,\cdots,\nu.$$

Remark 2.4. The result (2.19) extended the result of Theorem 3.3 in Cai and Kou [11] in which only the system of linear equations is obtained for a mixed-exponential jump diffusion.

Example 2.5. Letting $\nu=1$ in Corollary 2.3, when $\sigma>0$ or $\sigma=0$ and $\mu>0$, we recover the result (3.1)-(3.3) and the result of Corollary 3.3 in Kou and Wang [27], and (3.3) in Chi and Lin [13] (see also Yin et al. [48]):

$$E(e^{-\delta\tau_a^+}\mathbf{1}_{\{X_{\tau_a^+}=a\}}) = \frac{\beta_1-r_1}{r_2-r_1}e^{-r_1a} + \frac{r_2-\beta_1}{r_2-r_1}e^{-r_2a},$$

$$E\left(e^{-\delta\tau_a^+}\mathbf{1}_{\{X(\tau_a^+)-a\in dy\}}\right) = e^{-\beta_1y}\frac{(\beta_1-r_1)(r_2-\beta_1)}{(r_2-r_1)}(e^{-r_1a}-e^{-r_2a})dy, \ y>0,$$

$$E(e^{-\delta\tau_a^+}) = \frac{r_2(\beta_1-r_1)}{\beta_1(r_2-r_1)}e^{-r_1a} + \frac{r_1(r_2-\beta_1)}{\beta_1(r_2-r_1)}e^{-r_2a},$$

$$E\left(e^{-\delta\tau_a^+-s(X(\tau_a^+)-a)}\mathbf{1}_{\{\tau_a^+<\infty\}}\right) = \frac{(r_2+s)(\beta_1-r_1)}{(\beta_1+s)(r_2-r_1)}e^{-r_1a} + \frac{(r_1+s)(r_2-\beta_1)}{(\beta_1+s)(r_2-r_1)}e^{-r_2a},$$

where $0<r_1<\beta_1<r_2<\infty$.

When $\sigma=0$ and $\mu\le 0$, we have:

$$E(e^{-\delta\tau_a^+}) = E\left(e^{-\delta\tau_a^+}\mathbf{1}_{\{X(\tau_a^+)-a>0\}}\right) = \frac{\beta_1-r_1}{\beta_1}e^{-r_1a},$$

$$E\left(e^{-\delta\tau_a^+-s(X(\tau_a^+)-a)}\mathbf{1}_{\{\tau_a^+<\infty\}}\right) = \frac{\beta_1-r_1}{\beta_1+s}e^{-r_1a},$$

$$E\left(e^{-\delta\tau_a^+}\mathbf{1}_{\{X(\tau_a^+)-a>l\}}\right) = e^{-\beta_1l}\frac{\beta_1-r_1}{\beta_1}e^{-r_1a}, \ l\ge 0,$$

where $0<r_1<\beta_1<\infty$.

Theorem 2.6. *The (generalized) probability density function of* $\underline{X}_{e(q)}$ *is given by:*

$$f_{\underline{X}_{e(q)}}(y) = d_0\delta_0(y) + d_{1,1}\beta_1e^{\beta_1y} + \sum_{k=2}^{N}\sum_{j=1}^{m_k} d_{j,k}(\beta_k)^j\frac{(-y)^{j-1}}{(j-1)!}\exp(\beta_ky), \ y\le 0, \tag{2.20}$$

where $\delta_0(y)$ *is the Dirac delta at* $y=0$, $\beta_1,\cdots,\beta_N$ *are the roots of* $\psi_2(z)-q=0$ *(see Lewis and Mordecki [36] (Lemma 1.2)) and* $d_0, d_{1,1},\cdots,d_{j,k}$ *are given in Theorem 2.1.*

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Theorem 2.7. *(i) For $q, s > 0$, we have:*

$$E\left(e^{-q\tau_{-a}^{-}-s(-X_{\tau_{-a}^{-}}-a)}\mathbf{1}_{\{\tau_{-a}^{-}<\infty\}}\right) = \sum_{k=1}^{N}\sum_{j=1}^{m_k}\frac{d_{j,k}e^{-\beta_k a}}{(j-1)!}\sum_{l=1}^{j}(a\beta_k)^{l-1}\Delta_{k,j,l}(s); \tag{2.21}$$

(ii) For $q > 0, y \geq 0$, we have:

$$E\left(e^{-q\tau_{-a}^{-}}\mathbf{1}_{\{-X_{\tau_{-a}^{-}}-a\in dy\}}\right) = \delta_0(y)\sum_{k=1}^{N}\sum_{j=1}^{m_k}f_{k0}\frac{d_{j,k}e^{-\beta_k a}}{(j-1)!}(a\beta_k)^{j-1}dy + \sum_{k=1}^{N}\sum_{j=1}^{m_k}\frac{d_{j,k}e^{-\beta_k a}}{(j-1)!}\sum_{l=1}^{j}(a\beta_k)^{l-1}f_{(k,j,l)}dy, \tag{2.22}$$

where

$$\Delta_{k,j,l}(s) = \frac{\prod_{i=1,i\neq k}^{N}(1+\frac{s}{\beta_i})^{m_i}(1+\frac{s}{\beta_k})^{m_k-j+l-1}}{\prod_{r=1}^{\nu}(1+\frac{s}{\alpha_r})^{n_r}}, \tag{2.23}$$

$$f_{k0} = \begin{cases} 0, & \text{if } X^+ \text{ is a subordinator,} \\ \beta_k\prod_{i=1}^{\nu}\alpha_i^{n_i}/\prod_{j=1}^{N}\beta_j^{m_j}, & \text{if } X^+ \text{ is not a subordinator,} \end{cases} \tag{2.24}$$

$$f_{(k,j,l)} = \sum_{k_1=1}^{\nu}\sum_{j_1=1}^{n_{k_1}}f_{k_1,j_1}^{(k,j,l)}\frac{y^{j_1-1}e^{-\alpha_{k_1}y}}{(j_1-1)!}.$$

Here

$$f_{k_1,n_{k_1}-j_1}^{(k,j,l)} = \frac{1}{j_1!\alpha_{k_1}^{n_{k_1}-j_1}}\left[\frac{\partial^{j_1}}{\partial s^{j_1}}\frac{(1+\frac{s}{\alpha_{k_1}})^{n_{k_1}}}{(1+\frac{s}{\beta_k})^{j-l+1}\phi^+(-s)}\right]\Big|_{s=-\alpha_{k_1}}, \tag{2.25}$$

$k_1 = 1,\cdots,\nu, j_1 = 0, 1,\cdots,n_{k_1}$.

Remark 2.8. The similar results as (2.20) and (2.21) for a general phase-type Lévy process are also obtained in Asmussen et al. [6] and Alili and Kyprianou [3].

The following results are generalized the corresponding results in Yin et al. [48].

Corollary 2.9. *Assume the process $X = \{X_t\}_{t\geq 0}$ has the Lévy triplet $(-\mu, \sigma^2, \Pi)$ with Laplace exponent ϕ_2, where Π is given by (1.1) with q is a combination of ν exponential distributions:*

$$q(x) = \sum_{k=1}^{\nu}b_k\beta_k e^{\beta_k x}, x < 0,$$

for some $0 < \beta_1 < \cdots\beta_\nu < \infty$ and $b_1 + \cdots + b_\nu = 1$. Then

(i) for $q, s > 0$,

$$E\left(e^{-q\tau_{-a}^{-}-s(-X_{\tau_{-a}^{-}}-a)}\mathbf{1}_{\{\tau_{-a}^{-}<\infty\}}\right) = \sum_{k=1}^{J}B_k\frac{\prod_{i=1,i\neq k}^{J}(1+\frac{s}{r_i})}{\prod_{i=1}^{\nu}(1+\frac{s}{\beta_i})}e^{-r_k a}, \tag{2.26}$$

(ii) for $q > 0, y \geq 0$,

$$E\left(e^{-q\tau_{-a}^{-}}\mathbf{1}_{\{-X_{\tau_{-a}^{-}}-a\in dy\}}\right) = \sum_{k=1}^{J}B_k\left(A_{k0}\delta_0(y) + \sum_{l=1}^{\nu}A_{kl}\beta_l e^{-\beta_l y}\right)e^{-r_k a}dy, \tag{2.27}$$

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(iii) for $q > 0$,

$$E\left(e^{-q\tau_{-a}^{-}}\boldsymbol{1}_{\{X_{\tau_{-a}^{-}}=-a\}}\right) = \sum_{k=1}^{J} B_k A_{k0} e^{-r_k a},$$

(iv) for $q > 0$,

$$E\left(e^{-q\tau_{-a}^{-}}\boldsymbol{1}_{\{X_{\tau_{-a}^{-}}<-a\}}\right) = \sum_{k=1}^{J} B_k \left(\sum_{l=1}^{\nu} A_{kl}\right) e^{-r_k a} = \sum_{k=1}^{J} B_k \left(1 - A_{k0}\right) e^{-r_k a},$$

where $-r_1, \cdots, -r_J$ are the negative roots of the equation $\phi_2(r) = q$, and

$$J = \begin{cases} \nu + 1, & \sigma > 0, \text{or } \sigma = 0 \text{ and } \mu > 0, \\ \nu, & \sigma = 0 \text{ and } \mu \le 0, \end{cases}$$

$$B_j = \frac{\prod_{k=1}^{\nu}(1 - \frac{r_j}{\beta_k})}{\prod_{k=1,k\neq j}^{J}(1 - \frac{r_j}{r_k})}, \; j = 1, \cdots, J,$$

$$A_{k0} = \begin{cases} \frac{\prod_{i=1}^{\nu}\beta_i}{\prod_{i=1,i\neq k}^{J} r_i}, & \sigma > 0, \text{or } \sigma = 0 \text{ and } \mu > 0, \\ 0, & \sigma = 0 \text{ and } \mu \le 0, \end{cases}$$

$$A_{kl} = \frac{\prod_{i=1,i\neq k}^{J}(1 - \beta_l/r_i)}{\prod_{i=1,i\neq l}^{\nu}(1 - \beta_l/\beta_i)}, \; l = 1, 2, \cdots, \nu.$$

Remark 2.10. The result (2.27) extended the results (3.2) and (9.3) in Albrecher, Gerber and Yang [2] in which only for special π^+ is considered. By (iii) and (iv), we get:

$$E\left(e^{-q\tau_{-a}^{-}}\right) = \sum_{k=1}^{J} B_k e^{-r_k a},$$

which generalizes the main result of Mordecki [40] (Theorem 1.1), where only the case $\sigma > 0$ and $b_i \geq 0$ $(i = 1, \cdots, \nu)$ is considered.

Example 2.11. Letting $\nu = 1$ in Corollary 2.9, when $\sigma > 0$ or $\sigma = 0$ and $\mu > 0$, we get:

$$E(e^{-q\tau_{-a}^{-}}\mathbf{1}_{\{X_{\tau_{-a}^{-}}=-a\}}) = \frac{\beta_1 - r_1}{r_2 - r_1} e^{-r_1 a} + \frac{r_2 - \beta_1}{r_2 - r_1} e^{-r_2 a},$$

$$E\left(e^{-q\tau_{-a}^{-}}\mathbf{1}_{\{-X(\tau_{-a}^{-})-a\in dy\}}\right) = e^{-\beta_1 y}\frac{(\beta_1 - r_1)(r_2 - \beta_1)}{(r_2 - r_1)}(e^{-r_1 a} - e^{-r_2 a})dy, \; y > 0,$$

$$E(e^{-q\tau_{-a}^{-}}) = \frac{r_2(\beta_1 - r_1)}{\beta_1(r_2 - r_1)} e^{-r_1 a} + \frac{r_1(r_2 - \beta_1)}{\beta_1(r_2 - r_1)} e^{-r_2 a},$$

$$E\left(e^{-q\tau_{-a}^{-}+s(X(\tau_{-a}^{-})+a)}\mathbf{1}_{\{\tau_{-a}^{-}<\infty\}}\right) = \frac{(r_2 + s)(\beta_1 - r_1)}{(\beta_1 + s)(r_2 - r_1)} e^{-r_1 a} + \frac{(r_1 + s)(r_2 - \beta_1)}{(\beta_1 + s)(r_2 - r_1)} e^{-r_2 a},$$

where $0 < r_1 < \beta_1 < r_2 < \infty$.

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When $\sigma = 0$ and $\mu \leq 0$, we have:

$$E(e^{-q\tau_{-a}^-}) = E\left(e^{-\delta\tau_{-a}^-}\mathbf{1}_{\{X(\tau_{-a}^-)<-a\}}\right) = \frac{\beta_1 - r_1}{\beta_1}e^{-r_1 a},$$

$$E\left(e^{-q\tau_{-a}^- + s(X(\tau_{-a}^-)+a)}\mathbf{1}_{\{\tau_{-a}^-<\infty\}}\right) = \frac{\beta_1 - r_1}{\beta_1 + s}e^{-r_1 a},$$

$$E\left(e^{-q\tau_{-a}^-}\mathbf{1}_{\{-X(\tau_{-a}^-)-a>l\}}\right) = e^{-\beta_1 l}\frac{\beta_1 - r_1}{\beta_1}e^{-r_1 a},\ l \geq 0,$$

where $0 < r_1 < \beta_1 < \infty$.

3. The overshoot and undershoot at first exit

The two sided exit problem has been an interesting problem in Lévy process theory for a long time and it is rare to extract explicit identities for the overshoot and/or undershoot at first exit from an interval (see for example [23, 30, 47]). In this section, we derive the joint distributions of the first exit time and the values of the overshoot and undershoot at the first exit from a finite interval for a Lévy process X with linear drift $\mu \in \mathbb{R}$ and diffusion coefficients $\sigma \geq 0$, the intensity and the density of jumps are, respectively, $\lambda > 0$, and

$$p(x) = \sum_{j=1}^{\nu}\sum_{i=1}^{n_j} p_{ij}\frac{\rho_j^i x^{i-1}}{(i-1)!}e^{-\rho_j x}I_{\{x>0\}} + \sum_{j=1}^{\hat{\nu}}\sum_{i=1}^{\hat{n}_j} \hat{p}_{ij}\frac{\hat{\rho}_j^i |x|^{i-1}}{(i-1)!}e^{\hat{\rho}_j x}I_{\{x<0\}}, \tag{3.1}$$

where $\nu, \hat{\nu}, p_{ij}, \hat{p}_{ij} \in \mathbb{R}^+$, $\Re(\rho_j) > 0$, $\Re(\hat{\rho}_j) > 0$, $\Re(\rho_j)$ is the real part of ρ_j. and that $\rho_i \neq \rho_j$, $\hat{\rho}_i \neq \hat{\rho}_j$ for all $i \neq j$. Moreover,

$$\sum_{j=1}^{\nu}\sum_{i=1}^{n_j} p_{ij} + \sum_{j=1}^{\hat{\nu}}\sum_{i=1}^{\hat{n}_j} \hat{p}_{ij} = 1.$$

Clearly the process X can be identified as:

$$X_t = \sigma B_t + \mu t + \sum_{i=1}^{N_t} Y_i, \tag{3.2}$$

where $\{N_t\}$ is a Poisson process with intensity λ and $\{Y_i\}$ are iid random variables with density p, $\{B_t\}$ is the Brownian motion with $B_0 = 0$. Additionally, it is supposed that $\{N_t\}$, $\{B_t\}$ and $\{Y_i\}$ are independent. A similar model with $\sigma = 0$ was studied by Wen and Yin [45].

It is fairly easy to find out that the infinitesimal generator of $\{X_t\}_{t\geq 0}$ is given by:

$$(Lu)(x) = \frac{1}{2}\sigma^2 u''(x) + \mu u'(x) + \lambda\int_{-\infty}^{\infty}[u(x+y) - u(x)]p(y)dy,$$

for any twice continuously differentiable function $u(x)$ and the Lévy exponent $g(z) = \ln E[\exp(zX_1)]$ is given by:

$$g(z) = \frac{1}{2}z^2\sigma^2 + z\mu + \lambda\left(\sum_{j=1}^{\nu}\sum_{i=1}^{n_j}\frac{p_{ij}(\rho_j)^i}{(\rho_j - z)^i} + \sum_{j=1}^{\hat{\nu}}\sum_{i=1}^{\hat{n}_j}\frac{\hat{p}_{ij}(\hat{\rho}_j)^i}{(\hat{\rho}_j + z)^i} - 1\right).$$

By analytic continuation, the function $g(z)$ can be extended to the complex plane except at finitely many poles. In the following, we consider the resulting extension $G(z)$ of $g(z)$, i.e.,

$$G(z) = \frac{1}{2}z^2\sigma^2 + z\mu + \lambda\left(\sum_{j=1}^{\nu}\sum_{i=1}^{n_j}\frac{p_{ij}(\rho_j)^i}{(\rho_j - z)^i} + \sum_{j=1}^{\hat{\nu}}\sum_{i=1}^{\hat{n}_j}\frac{\hat{p}_{ij}(\hat{\rho}_j)^i}{(\hat{\rho}_j + z)^i} - 1\right),\ z \in \mathbb{C}\setminus\{\rho_j, \hat{\rho}_j\}.$$

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Let us denote $M = \sum_{j=1}^{\nu} n_j$ and $\hat{M} = \sum_{j=1}^{\hat{\nu}} \hat{n}_j$.

Kuznetsov and Pardo [31] have studied the roots of the equation $G(z) = \theta$. However, for this particular Lévy process X, we will give a simple proof for the roots of this equation.

Lemma 3.1. *Assume that* $\theta > 0$.

(i). If $\sigma > 0$, *then the equation:* $G(z) = \theta$ *has* $M + \hat{M} + 2$ *complex roots* $\gamma_1(\theta), \gamma_2(\theta), \ldots, \gamma_{M+1}(\theta)$ *with* $\Re(\gamma_i(\theta)) > 0$ *for* $i = 1, 2, \ldots, M+1$ *and* $-\hat{\gamma}_1(\theta), -\hat{\gamma}_2(\theta), \ldots, -\hat{\gamma}_{\hat{M}+1}(\theta)$ *with* $\Re(\hat{\gamma}_i(\theta)) > 0$ *for* $i = 1, 2, \ldots, \hat{M}+1$.

(ii). If $\sigma = 0$ *and* $\mu > 0$, *then the equation* $G(z) = \theta$ *has* $M + \hat{M} + 1$ *complex roots* $\gamma_1(\theta), \gamma_2(\theta), \ldots, \gamma_{M+1}(\theta)$ *with* $\Re(\gamma_i(\theta)) > 0$ *for* $i = 1, 2, \ldots, M+1$ *and* $-\hat{\gamma}_1(\theta), -\hat{\gamma}_2(\theta), \ldots, -\hat{\gamma}_{\hat{M}}(\theta)$ *with* $\Re(\hat{\gamma}_i(\theta)) > 0$ *for* $i = 1, 2, \ldots, \hat{M}$.

(iii). If $\sigma = 0$ *and* $\mu < 0$, *then the equation* $G(z) = \theta$ *has* $M + \hat{M} + 1$ *complex roots* $\gamma_1(\theta), \gamma_2(\theta), \ldots, \gamma_M(\theta)$ *with* $\Re(\gamma_i(\theta)) > 0$ *for* $i = 1, 2, \ldots, M$ *and* $-\hat{\gamma}_1(\theta), -\hat{\gamma}_2(\theta), \ldots, -\hat{\gamma}_{\hat{M}+1}(\theta)$ *with* $\Re(\hat{\gamma}_i(\theta)) > 0$ *for* $i = 1, 2, \ldots, \hat{M}+1$.

(iv). If $\sigma = 0$ *and* $\mu = 0$, *then the equation* $G(z) = \theta$ *has* $M + \hat{M}$ *complex roots* $\gamma_1(\theta), \gamma_2(\theta), \ldots, \gamma_M(\theta)$ *with* $\Re(\gamma_i(\theta)) > 0$ *for* $i = 1, 2, \ldots, M$ *and* $-\hat{\gamma}_1(\theta), -\hat{\gamma}_2(\theta), \ldots, -\hat{\gamma}_{\hat{M}}(\theta)$ *with* $\Re(\hat{\gamma}_i(\theta)) > 0$ *for* $i = 1, 2, \ldots, \hat{M}$.

Proof. We prove (i) only, since the rest can be proved similarly. We denote

$$G_1(z) = \frac{1}{2} z^2 \sigma^2 + zc - \lambda - \theta, \; z \in \mathbb{C},$$

$$G_2(z) = \lambda \left(\sum_{j=1}^{\nu} \sum_{i=1}^{n_j} \frac{p_{ij}(\rho_j)^i}{(\rho_j - z)^i} + \sum_{j=1}^{\hat{\nu}} \sum_{i=1}^{\hat{n}_j} \frac{\hat{p}_{ij}(\hat{\rho}_j)^i}{(\hat{\rho}_j + z)^i} \right), \; z \in \mathbb{C}.$$

Firstly, we prove that for given $\theta > 0$, $G(z) = \theta$ has $\hat{M} + 1$ roots with negative real parts. Set $C_r^- = \{z : |z| = r, z \in C^-\}$ with $r > \varepsilon + \max_{1 \le j \le \hat{\nu}}\{|\hat{\rho}_j|\}$, where ε is an arbitrary positive constant. Applying Rouchés theorem on the semi-circle C_r^-, consisting of the imaginary axis running from $-ir$ to ir and with radius r running clockwise from ir to $-ir$. We let $r \to \infty$ and denote by C^- the limiting semi-circle. It is known that both $\left(\prod_{j=1}^{\hat{\nu}} (\hat{\rho}_j + z)^{\hat{n}_j}\right) G_1(z)$ and $\left(\prod_{j=1}^{\hat{\nu}} (\hat{\rho}_j + z)^{\hat{n}_j}\right) G_2(z)$ are analytic in C^-. We want to show that:

$$\left| \left(\prod_{j=1}^{\hat{\nu}} (\hat{\rho}_j + z)^{\hat{n}_j} \right) G_1(z) \right| > \left| \left(\prod_{j=1}^{\hat{\nu}} (\hat{\rho}_j + z)^{\hat{n}_j} \right) G_2(z) \right|, z \in C^-.$$

Notice that $|G_1(z)| \to \infty$ for $\Re(z) \to -\infty$, and

$$|G_2(z)| \le \lambda \sum_{j=1}^{\nu} \sum_{i=1}^{n_j} \frac{|p_{ij}||\rho_j|^i}{\varepsilon^i} + \lambda \sum_{j=1}^{\hat{\nu}} \sum_{i=1}^{\hat{n}_j} \frac{|\hat{p}_{ij}||\hat{\rho}_j|^i}{\varepsilon^i},$$

is bounded for $\Re(z) \to -\infty$. Hence, for $\Re(z) \to -\infty$,

$$\left| \left(\prod_{j=1}^{\hat{\nu}} (\hat{\rho}_j + z)^{\hat{n}_j} \right) G_1(z) \right| > \left| \left(\prod_{j=1}^{\hat{\nu}} (\hat{\rho}_j + z)^{\hat{n}_j} \right) G_2(z) \right|$$

on the boundary of the half circle in C^-. For $a \in \mathbb{R}$, we have $|G_2(ia)| < \lambda$ (see Lewis and Mordecki [37]). On the other hand,

$$|G_1(ia)| \ge -\Re(G_1(ia)) = \frac{1}{2} \sigma^2 a^2 + \lambda + \theta > \lambda.$$

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Thus, we have $|G_1(ia)| > |G_2(ia)|$. Since $\left(\prod_{j=1}^{\hat{\nu}}(\hat{\rho}_j + z)^{\hat{n}_j}\right) G_1(z)$ has $\hat{M}+1$ roots with negative real parts, so equation $G(z) = \theta$ has $\hat{M} + 1$ roots with negative real parts. Similarly, we can prove $G(z) = \theta, \theta > 0$ has $M + 1$ roots with positive real parts. This ends the proof of Lemma 3.1. □

We assume that N of $\gamma_i(\theta)$'s are distinct and denote by $\gamma_i(i = 1, 2, \ldots, N)$ with respective multiplicities $1, m_2, \cdots, m_N$, ordered such that $0 < \gamma_1 \le \Re(\gamma_2) \le \cdots \le \Re(\gamma_N)$. Furthermore,

$$M = \begin{cases} \sum_{i=2}^{N} m_i, & \sigma > 0, \text{or } \sigma = 0 \text{ and } \mu > 0, \\ 1 + \sum_{i=2}^{N} m_i, & \sigma = 0 \text{ and } \mu \le 0. \end{cases}$$

Likewise, we assume that $\hat{N}$ of $\hat{\gamma}_i(\theta)$'s are distinct and denote by $\hat{\gamma}_i(i = 1, 2, \ldots, \hat{N})$ with respective multiplicities $1, \hat{m}_2, \cdots, \hat{m}_N$, ordered such that $0 < \hat{\gamma}_1 \le \Re(\hat{\gamma}_2) \le \cdots \le \Re(\hat{\gamma}_{\hat{N}})$. Furthermore,

$$\hat{M} = \begin{cases} \sum_{i=2}^{\hat{N}} \hat{m}_i, & \sigma > 0, \text{or } \sigma = 0 \text{ and } \mu < 0, \\ 1 + \sum_{i=2}^{\hat{N}} \hat{m}_i, & \sigma = 0 \text{ and } \mu \ge 0. \end{cases}$$

For $0 < u < b$ ($b > 0$ is a constant), define

$$\tau = \inf\{t \ge 0 : X_t \notin (-u, b - u)\},$$

to be the first exit time of X from the interval $(-u, b-u)$. We consider the random events $A^{b-u} = \{X_\tau \ge b - u\}$ and $A_u = \{X_\tau \le -u\}$ meaning that the process exits the interval through the upper and lower boundary, respectively. Further, define the overshoot and undershoot of the process over the boundaries at the first exit time by:

$$\chi = (X_\tau - b + u)\mathbf{1}_{A^{b-u}} + (-X_\tau - u)\mathbf{1}_{A_u}.$$

For $y \ge 0$, define

$$T^y = X_{\tau_y^+} - y, \quad T_y = -X_{\tau_{-y}^-} - y,$$

where as before:

$$\tau_y^+ = \inf\{t \ge 0 : X_t \ge y\}, \quad \tau_y^- = \inf\{t \ge 0 : X_t \le -y\}.$$

Using a general result on Lévy processes (see Kadankov and Kadankova [22] (Theorem 1), or Kadankova and Veraverbeke [23] (Lemma 1)), the joint distribution of random variables $\{\tau, \chi\}$ can be expressed in terms of distributions of the one-boundary functionals $\{\tau^x, T^x\}$ and $\{\tau_x, T_x\}$, we give the following Theorem 3.2. For convenience, we now introduce some notions. Denote

$$\Phi(-u) = \prod_{k=1}^{\nu} \left(\frac{u + \rho_k}{\rho_k}\right)^{n_k} \prod_{j=1}^{N} \left(\frac{\gamma_j}{u + \gamma_j}\right)^{m_j},$$

$$\hat{\Phi}(u) = \prod_{k=1}^{\hat{\nu}} \left(\frac{-u + \hat{\rho}_k}{\hat{\rho}_k}\right)^{\hat{n}_k} \prod_{j=1}^{\hat{N}} \left(\frac{\hat{\gamma}_j}{-u + \hat{\gamma}_j}\right)^{\hat{m}_j},$$

$$K_+(v, dy, \theta) = \int_0^\infty E[e^{-\theta\tau_{-v-b}^-}; T_{v+b} \in dl]E[e^{-\theta\tau_{l+b}^+}; T^{l+b} \in dy],$$

$$K_-(v, dy, \theta) = \int_0^\infty E[e^{-\theta\tau_{v+b}^+}; T^{v+b} \in dl]E[e^{-\theta\tau_{-l-b}^-}; T_{l+b} \in dy].$$

We introduce the sequences $K_\pm^{(n)}(v, dy, \theta), n \in \mathbb{N}$ by means of recurrence relation:

$$K_\pm^{(1)}(v, dy, \theta) = K_\pm(v, dy, \theta),$$

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$$K_{\pm}^{(n+1)}(v,dy,\theta)=\int_0^{\infty}K_{\pm}^{(n)}(v,dl,\theta)K_{\pm}(l,dy,\theta),\ n\in\mathbb{N},$$

Theorem 3.2. *Let $\{X_t\}_{t\geq 0}$ be a jump-diffusion process defined by (3.2). For $\theta>0$ and $y\geq 0$, the Laplace transforms of the joint distribution of random variables $\{\tau,\chi\}$ satisfy the equations*

$$E[e^{-\theta\tau};\chi\in dy,A^{b-u}]=f_+^{\theta}(b-u,dy)+\int_0^{\infty}f_+^{\theta}(b-u,dv)K_+^{\theta}(v,dy),$$

$$E[e^{-\theta\tau};\chi\in dy,A_u]=f_-^{\theta}(u,dy)+\int_0^{\infty}f_-^{\theta}(u,dv)K_-^{\theta}(v,dy),$$

where

$$f_+^{\theta}(b-u,dy)=E[e^{-\theta\tau_{b-u}^+};T^{b-u}\in dy]-\int_0^{\infty}E[e^{-\theta\tau_{-u}^-};T_u\in dv]E[e^{-\theta\tau_{v+b}^+};T^{v+b}\in dy],$$

$$f_-^{\theta}(u,dy)=E[e^{-\theta\tau_{-u}^-};T_u\in dy]-\int_0^{\infty}E[e^{-\theta\tau_{b-u}^+};T^{b-u}\in dv]E[e^{-\theta\tau_{-v-b}^-};T_{v+b}\in dy],$$

and

$$K_{\pm}^{\theta}(v,dy)=\sum_{n=1}^{\infty}K_{\pm}^{(n)}(v,dy,\theta),\ v\geq 0.$$

Here

$$\begin{aligned}E\left(e^{-q\tau_a^+};T^a\in dy\right)&=\delta_0(z)\sum_{k=1}^{N}\sum_{j=1}^{m_k}f_{k0}\frac{d_{j,k}e^{-\gamma_k a}}{(j-1)!}(a\gamma_k)^{j-1}dy\\&\quad+\sum_{k=1}^{N}\sum_{j=1}^{m_k}\frac{d_{j,k}e^{-\gamma_k a}}{(j-1)!}\sum_{l=1}^{j}(a\gamma_k)^{l-1}f_{(k,j,l)}dy,\end{aligned}$$

where

$$d_{1,1}=\prod_{j=1}^{\nu}\left(\frac{\rho_j-\gamma_1}{\rho_j}\right)^{n_j}\prod_{k=2}^{N}\left(\frac{\gamma_k}{\gamma_k-\gamma_1}\right)^{m_k},$$

$$d_{k,m_k-j}=\frac{1}{j!\gamma_k^{m_k-j}}\left[\frac{\partial^j}{\partial u^j}\left(\Phi(-u)(u+\gamma_k)^{m_k}\right)\right]|_{u=-\gamma_k}$$

for $k=2,\cdots,N$ and $j=0,1,\cdots,m_k-1$,

$$f_{k0}=\begin{cases}0, & \text{if } \sigma=0 \text{ and } \mu\leq 0,\\ \gamma_k\prod_{i=1}^{\nu}\rho_i^{n_i}/\prod_{j=1}^{N}\gamma_j^{m_j}, & \text{if } \sigma>0, \text{or } \sigma=0 \text{ and } \mu>0,\end{cases}$$

$$f_{(k,j,l)}=\sum_{k_1=1}^{\nu}\sum_{j_1=1}^{n_{k_1}}f_{k_1,j_1}^{(k,j,l)}\frac{y^{j_1-1}e^{-\rho_{k_1}y}}{(j_1-1)!},$$

$$f_{k_1,n_{k_1}-j_1}^{(k,j,l)}=\frac{1}{j_1!\rho_{k_1}^{n_{k_1}-j_1}}\left[\frac{\partial^{j_1}}{\partial s^{j_1}}\frac{(1+\frac{s}{\alpha_{k_1}})^{n_{k_1}}}{(1+\frac{s}{\gamma_k})^{j-l+1}\Phi^+(-s)}\right]|_{s=-\rho_{k_1}},$$

$k_1=1,\cdots,\nu,j_1=0,1,\cdots,n_{k_1},$

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and

$$E\left(e^{-q\tau_{-a}^{-}}; T_a \in dy\right) = \delta_0(y)\sum_{k=1}^{\hat{N}}\sum_{j=1}^{\hat{m}_k}\hat{f}_{k0}\frac{\hat{d}_{j,k}e^{-\hat{\gamma}_k a}}{(j-1)!}(a\hat{\gamma}_k)^{j-1}dy + \sum_{k=1}^{\hat{N}}\sum_{j=1}^{\hat{m}_k}\frac{\hat{d}_{j,k}e^{-\hat{\gamma}_k a}}{(j-1)!}\sum_{l=1}^{j}(a\hat{\gamma}_k)^{l-1}\hat{f}_{(k,j,l)}dy,$$

where

$$\hat{d}_{1,1} = \prod_{j=1}^{\hat{\nu}}\left(\frac{\hat{\rho}_j - \hat{\gamma}_1}{\hat{\rho}_j}\right)^{\hat{n}_j}\prod_{k=2}^{\hat{N}}\left(\frac{\hat{\gamma}_k}{\hat{\gamma}_k - \hat{\gamma}_1}\right)^{\hat{m}_k},$$

$$\hat{d}_{k,m_k-j} = \frac{1}{j!\hat{\gamma}_k^{\hat{m}_k-j}}\left[\frac{\partial^j}{\partial u^j}\left(\hat{\Phi}(u)(u+\gamma_k)^{\hat{m}_k}\right)\right]|_{u=\gamma_k}$$

for $k = 2, \cdots, \hat{N}$ *and* $j = 0, 1, \cdots, \hat{m}_k - 1$,

$$\hat{f}_{k0} = \begin{cases} 0, & \text{if } \sigma = 0 \text{ and } \mu \geq 0, \\ \hat{\gamma}_k \prod_{i=1}^{\hat{\nu}}\hat{\rho}_i^{\hat{n}_i}/\prod_{j=1}^{\hat{N}}\hat{\gamma}_j^{\hat{m}_j}, & \text{if } \sigma > 0, \text{or } \sigma = 0 \text{ and } \mu < 0, \end{cases}$$

$$\hat{f}_{(k,j,l)} = \sum_{k_1=1}^{\hat{\nu}}\sum_{j_1=1}^{\hat{n}_{k_1}}\hat{f}_{k_1,j_1}^{(k,j,l)}\frac{y^{j_1-1}e^{-\hat{\rho}_{k_1}y}}{(j_1-1)!}.$$

Here

$$\hat{f}_{k_1,\hat{n}_{k_1}-j_1}^{(k,j,l)} = \frac{1}{j_1!\hat{\rho}_{k_1}^{\hat{n}_{k_1}-j_1}}\left[\frac{\partial^{j_1}}{\partial s^{j_1}}\frac{(1+\frac{s}{\hat{\rho}_{k_1}})^{\hat{n}_{k_1}}}{(1+\frac{s}{\hat{\gamma}_k})^{j-l+1}\hat{\Phi}(s)}\right]|_{s=\hat{\rho}_{k_1}},$$

$k_1 = 1, \cdots, \nu, j_1 = 0, 1, \cdots, \hat{n}_{k_1}$.

Proof. The proof is similar to the one of Kadankov and Kadankova [22] (Theorem 1). Therefore, it is omitted here.

Corollary 3.3. *Consider the special case of (3.1) in which* $n_j = \hat{n}_j = 1$. *Then*

$$\begin{aligned}(i)\ f_+^{\theta}(b-u,dy) &= r_1(u)\delta_0(y)dy + \sum_{k=1}^{J}C_k\left(\sum_{l=1}^{\nu}D_{kl}\rho_l e^{-\rho_l y}\right)e^{-\gamma_k(b-u)}dy \\ &\quad -\left(\sum_{i=1}^{\hat{J}}\hat{C}_i\hat{D}_{i0}e^{-\hat{\gamma}_i u}\right)\left(\sum_{j=1}^{J}C_j e^{-\gamma_j b}\sum_{l=1}^{\nu}D_{jl}\rho_l e^{-\rho y}\right)dy \\ &\quad -\sum_{i=1}^{\hat{J}}\sum_{j=1}^{J}\hat{C}_i C_j e^{-\hat{\gamma}_i u}e^{-\gamma_j b}\sum_{l_1=1}^{\hat{\nu}}\sum_{l_2=1}^{\nu}\hat{D}_{i,l_1}D_{j,l_2}\frac{\hat{\rho}_{l_1}\rho_{l_2}e^{-\rho_{l_2}y}}{\gamma_j + \hat{\rho}_{l_1}}dy, \end{aligned} \tag{3.3}$$

where

$$J = \begin{cases} \nu + 1, & \sigma > 0, \text{or } \sigma = 0 \text{ and } \mu > 0, \\ \nu, & \sigma = 0 \text{ and } \mu \leq 0, \end{cases}$$

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$$C_j = \frac{\prod_{k=1}^{\nu}(1-\frac{\gamma_j}{\rho_k})}{\prod_{k=1,k\neq j}^{J}(1-\frac{\gamma_j}{\gamma_k})},\ j=1,\cdots,J,$$

$$D_{k0} = \begin{cases} \frac{\prod_{i=1}^{\nu}\rho_i}{\prod_{i=1,i\neq k}^{J}\gamma_i}, & \sigma>0, \text{or } \sigma=0 \text{ and } \mu>0,\\ 0, & \sigma=0 \text{ and } \mu\leq 0,\end{cases}$$

$$D_{kl} = \frac{\prod_{i=1,i\neq k}^{J}(1-\rho_l/\gamma_i)}{\prod_{i=1,i\neq l}^{\nu}(1-\rho_l/\rho_i)},\ l=1,2,\cdots,\nu,$$

$$\begin{aligned} r_1(u) &= \sum_{k=1}^{J} C_k D_{k0} e^{-\gamma_k(b-u)} - \sum_{i=1}^{\hat{J}}\sum_{j=1}^{J}\hat{C}_i\hat{D}_{i0}C_jD_{j0}e^{-\hat{\gamma}_i u}e^{-\gamma_j b} \\ &\quad - \sum_{i=1}^{\hat{J}}\sum_{j=1}^{J}\hat{C}_iC_jD_{j0}e^{-\hat{\gamma}_i u}e^{-\gamma_j b}\sum_{l=1}^{\hat{\nu}}\hat{D}_{il}\hat{\rho}_l\frac{1}{\hat{\rho}_l+\gamma_j}. \end{aligned} \tag{3.4}$$

$$\begin{aligned} (ii)\, f_-^{\theta}(u,dy) &= r_2(u)\delta_0(y)dy + \sum_{k=1}^{\hat{J}}\hat{C}_k\left(\sum_{l=1}^{\hat{\nu}}\hat{D}_{kl}\hat{\rho}_l e^{-\hat{\rho}_l y}\right)e^{-\hat{\gamma}_k u}dy \\ &\quad -\left(\sum_{j=1}^{J}C_jD_{j0}e^{-\gamma_j(b-u)}\right)\left(\sum_{i=1}^{\hat{J}}\hat{C}_i\sum_{l=1}^{\hat{\nu}}\hat{D}_{il}\hat{\rho}_l e^{-\hat{\rho}y}\right)e^{-\hat{\gamma}_i b}dy \\ &\quad -\sum_{i=1}^{\hat{J}}\sum_{j=1}^{J}\hat{C}_iC_je^{-\hat{\gamma}_i b}e^{-\gamma_j(b-u)}\sum_{l_1=1}^{\hat{\nu}}\sum_{l_2=1}^{\nu}\hat{D}_{i,l_1}D_{j,l_2}\frac{\rho_{l_2}\hat{\rho}_{l_1}e^{-\hat{\rho}_{l_1}y}}{\hat{\gamma}_i+\rho_{l_2}}dy, \end{aligned} \tag{3.5}$$

where

$$\hat{J} = \begin{cases} \hat{\nu}+1, & \sigma>0, \text{or } \sigma=0 \text{ and } \mu<0,\\ \hat{\nu}, & \sigma=0 \text{ and } \mu\geq 0,\end{cases}$$

$$\hat{C}_j = \frac{\prod_{k=1}^{\hat{\nu}}(1-\frac{\hat{\gamma}_j}{\hat{\rho}_k})}{\prod_{k=1,k\neq j}^{\hat{J}}(1-\frac{\hat{\gamma}_j}{\hat{\gamma}_k})},\ j=1,\cdots,\hat{J},$$

$$\hat{D}_{k0} = \begin{cases} \frac{\prod_{i=1}^{\hat{\nu}}\hat{\rho}_i}{\prod_{i=1,i\neq k}^{\hat{J}}\hat{\gamma}_i}, & \sigma>0, \text{or } \sigma=0 \text{ and } \mu<0,\\ 0, & \sigma=0 \text{ and } \mu\geq 0,\end{cases}$$

$$\hat{D}_{kl} = \frac{\prod_{i=1,i\neq k}^{\hat{J}}(1-\hat{\rho}_l/\hat{\gamma}_i)}{\prod_{i=1,i\neq l}^{\hat{\nu}}(1-\hat{\rho}_l/\hat{\rho}_i)},\ l=1,2,\cdots,\hat{\nu},$$

$$r_2(u) = \sum_{k=1}^{\hat{J}}\hat{C}_k\hat{D}_{k0}e^{-\hat{\gamma}_k u} - \sum_{i=1}^{\hat{J}}\sum_{j=1}^{J}\hat{C}_i\hat{D}_{i0}C_jD_{j0}e^{-\hat{\gamma}_i b}e^{-\gamma_j(b-u)}$$

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$$-\sum_{i=1}^{\hat{J}}\sum_{j=1}^{J}\hat{C}_iC_j\hat{D}_{i0}e^{-\hat{\gamma}_ib}e^{-\gamma_j(b-u)}\sum_{l=1}^{\nu}D_{jl}\rho_l\frac{1}{\rho_l+\hat{\gamma}_i}. \tag{3.6}$$

Proof. From Corollary 2.3 (*ii*), we get for $\theta > 0, y \geq 0$,

$$E\left(e^{-\theta\tau^+_{b-u}}, T^{b-u} \in dy\right) = \sum_{k=1}^{J} C_k\left(D_{k0}\delta_0(y) + \sum_{l=1}^{\nu} D_{kl}\rho_l e^{-\rho_l y}\right)e^{-\gamma_k(b-u)}dy. \tag{3.7}$$

From Corollary 2.9 (*ii*), we get for $\theta > 0, y \geq 0$,

$$E\left(e^{-\theta\tau^-_{-u}}, T_u \in dy\right) = \sum_{k=1}^{\hat{J}} \hat{C}_k\left(\hat{D}_{k0}\delta_0(y) + \sum_{l=1}^{\hat{\nu}} \hat{D}_{kl}\hat{\rho}_l e^{-\hat{\rho}_l y}\right)e^{-\hat{\gamma}_k u}dy. \tag{3.8}$$

It follows that:

$$\begin{aligned}
\int_0^\infty & E\left(e^{-\theta\tau^-_{-u}}, T_u \in dv\right)E\left(e^{-\theta\tau^+_{b+v}}, T^{b+v} \in dy\right)\\
= & \left(\sum_{i=1}^{\hat{J}}\sum_{j=1}^{J}\hat{C}_i\hat{D}_{i0}C_jD_{j0}e^{-\hat{\gamma}_iu}e^{-\gamma_jb}\right)\delta_0(y)dy\\
& +\left(\sum_{i=1}^{\hat{J}}\hat{C}_i\hat{D}_{i0}e^{-\hat{\gamma}_iu}\right)\left(\sum_{j=1}^{J}C_je^{-\gamma_jb}\sum_{l=1}^{\nu}D_{jl}\rho_le^{-\rho y}\right)dy\\
& +\delta_0(y)dy\sum_{i=1}^{\hat{J}}\sum_{j=1}^{J}\hat{C}_iC_jD_{j0}e^{-\hat{\gamma}_iu}e^{-\hat{\gamma}_jb}\sum_{l=1}^{\hat{\nu}}\hat{D}_{il}\hat{\rho}_l\frac{1}{\hat{\rho}_l+\gamma_j}\\
& +\sum_{i=1}^{\hat{J}}\sum_{j=1}^{J}\hat{C}_iC_je^{-\hat{\gamma}_iu}e^{-\gamma_jb}\sum_{l_1=1}^{\hat{\nu}}\sum_{l_2=1}^{\nu}\hat{D}_{i,l_1}D_{j,l_2}\frac{\hat{\rho}_{l_1}}{\gamma_j+\hat{\rho}_{l_1}}\rho_{l_2}e^{-\rho_{l_2}y}dy.
\end{aligned} \tag{3.9}$$

The relation (3.3) follows from (3.7) and (3.9).

Similarly,

$$\begin{aligned}
\int_0^\infty & E\left(e^{-\theta\tau^+_{b-u}}, T^{b-u} \in dv\right)E\left(e^{-\theta\tau^-_{-b-v}}, T_{b+v} \in dy\right)\\
= & \delta_0(y)dy\left(\sum_{i=1}^{\hat{J}}\sum_{j=1}^{J}\hat{C}_i\hat{D}_{i0}C_jD_{j0}e^{-\hat{\gamma}_ib}e^{-\gamma_j(b-u)}\right)\\
& +\left(\sum_{j=1}^{J}C_jD_{j0}e^{-\gamma_j(b-u)}\right)\left(\sum_{i=1}^{\hat{J}}\hat{C}_i\sum_{l=1}^{\hat{\nu}}\hat{D}_{il}\hat{\rho}_le^{-\hat{\rho}y}\right)e^{-\hat{\gamma}_ib}dy\\
& +\delta_0(y)dy\sum_{i=1}^{\hat{J}}\sum_{j=1}^{J}\hat{C}_iC_j\hat{D}_{i0}e^{-\hat{\gamma}_ib}e^{-\gamma_j(b-u)}\sum_{l=1}^{\nu}D_{jl}\rho_l\frac{1}{\rho_l+\hat{\gamma}_i}\\
& +\sum_{i=1}^{\hat{J}}\sum_{j=1}^{J}\hat{C}_iC_je^{-\hat{\gamma}_ib}e^{-\gamma_j(b-u)}\sum_{l_1=1}^{\hat{\nu}}\sum_{l_2=1}^{\nu}\hat{D}_{i,l_1}D_{j,l_2}\frac{\rho_{l_2}}{\hat{\gamma}_i+\rho_{l_2}}\hat{\rho}_{l_1}e^{-\hat{\rho}_{l_1}y}dy,
\end{aligned} \tag{3.10}$$

and thus the relation (3.5) follows from (3.8) and (3.10). This ends the proof of Corollary 3.3. □

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Corollary 3.4. *Consider the special case of (3.1) and (3.2) in which $\sigma = 0, \mu = 0$ and, when $x > 0$, $p(x) = p_{11}\rho_1 e^{-\rho_1 x}$. Then*

$$K_+^\theta(v, dy) = \frac{E(e^{-\theta\tau^-_{-v-b}-\gamma_1 T_{v+b}})E(e^{-\theta\tau_b^+})}{1 - E(e^{-\theta\tau_b^+})E(e^{-\theta\tau^-_{-\xi-b}-\gamma_1 T_{\xi+b}})}\rho_1 e^{-\rho_1 y}dy, \tag{3.11}$$

$$K_-^\theta(v, dy) = \frac{E(e^{-\theta\tau^-_{-\xi-b}}, T_{\xi+b} \in dy)E(e^{-\theta\tau_b^+})}{1 - E(e^{-\theta\tau_b^+})E(e^{-\theta\tau^-_{-\xi-b}-\gamma_1 T_{\xi+b}})}e^{-\gamma_1 v}, \tag{3.12}$$

where ξ is an exponential random variable with parameter ρ_1.

Proof. From (3.7), we get for $\theta > 0, y \geq 0$,

$$E\left(e^{-\theta\tau^+_{b-u}}, T^{b-u} \in dy\right) = C_1 D_{11}\rho_1 e^{-\rho_1 y}e^{-\gamma_1(b-u)}dy,$$

this, together with the definition of the kernel we have:

$$\begin{aligned} K_+(v, dy, \theta) &= \int_0^\infty E[e^{-\theta\tau^-_{-v-b}}; T_{v+b} \in dl]E[e^{-\theta\tau^+_{l+b}}; T^{l+b} \in dy] \\ &= C_1 D_{11}\rho_1 E(e^{-\theta\tau^-_{-v-b}-\gamma_1 T_{v+b}})e^{-\rho_1 y}e^{-\gamma_1 b}dy. \end{aligned} \tag{3.13}$$

Similarly,

$$K_+^{(2)}(v, dy, \theta) = (C_1 D_{11}e^{-\gamma_1 b})^2\rho_1 e^{-\rho_1 y}E(e^{-\theta\tau^-_{-v-b}-\gamma_1 T_{v+b}})E(e^{-\theta\tau^-_{-\xi-b}-\gamma_1 T_{\xi+b}})dy.$$

The successive iterations $K_+^{(n)}(v, dy, \theta), n \in \mathbb{N}$, of kernel $K_+(v, dy, \theta)$ are found by induction:

$$K_+^{(n)}(v, dy, \theta) = (C_1 D_{11}e^{-\gamma_1 b})^n E(e^{-\theta\tau^-_{-v-b}-\gamma_1 T_{v+b}})(E(e^{-\theta\tau^-_{-\xi-b}-\gamma_1 T_{\xi+b}}))^{n-1}\rho_1 e^{-\rho_1 y}dy.$$

From which and note that $E(e^{-\theta\tau_b^+}) = C_1 D_{11}e^{-\gamma_1 b}$, we get (3.11) easily. Equation (3.12) can be proved similarly. The proof is complete.

The following theorem is a direct consequence of Theorem 3.1. □

Theorem 3.5. *Consider the special case of (3.1) and (3.2) in which $\sigma = 0, \mu = 0$ and, $p(x) = p_{11}\rho_1 e^{-\rho_1 x}\mathbf{1}_{\{x>0\}} + \sum_{j=1}^{\hat{v}}\hat{p}_{1j}\hat{\rho}_j e^{\hat{\rho}x}\mathbf{1}_{\{x<0\}}$. Then*

$$\begin{aligned} E[e^{-\theta\tau}; \chi \in dy, A^{b-u}] &= -\Delta(\theta)^{-1}(Ee^{-\theta\tau_b^+})\left(\sum_{i=1}^{\hat{J}}\hat{C}_i e^{-\hat{\gamma}_i u}\sum_{l=1}^{\hat{v}}\hat{D}_{i,l}\frac{\hat{\rho}_l}{\gamma_1 + \hat{\rho}_l}\right)\rho_1 e^{-\rho_1 y}dy \\ &\quad + \Delta(\theta)^{-1}(Ee^{-\theta\tau_b^+})\rho_1 e^{-\rho_1 y}e^{\gamma_1 u}dy, \end{aligned} \tag{3.14}$$

$$\begin{aligned} E[e^{-\theta\tau}; \mathcal{X} \in dy, A_u] &= -\Delta(\theta)^{-1}(Ee^{-\theta\tau_b^+})\left(\sum_{i=1}^{\hat{J}}\hat{C}_i e^{-\gamma_i u}\sum_{l=1}^{\hat{v}}\hat{D}_{il}\frac{\hat{\rho}_l}{\gamma_1 + \hat{\rho}_l}\rho_l e^{-\rho_l y}dy\right) \\ &\quad + \sum_{i=1}^{\hat{J}}\hat{C}_i e^{-\hat{\gamma}_i u}\sum_{l=1}^{\hat{v}}\hat{\rho}_l e^{-\hat{\rho}_l y}dy, \end{aligned}$$

where

$$\Delta(\theta) = 1 - Ee^{-\theta\tau_b^+}E(e^{-\theta\tau^-_{-\xi-b}-\gamma_1 T_{\xi+b}}).$$

Corollary 3.6. *Let $b = 0$ in Theorem 3.5, we have*

$$E[e^{-\theta\tau}] = -\Delta(\theta)^{-1}C_1 D_{11}\left(\sum_{i=1}^{\hat{J}}\hat{C}_i e^{-\hat{\gamma}_i u}\sum_{l=1}^{\hat{v}}\hat{D}_{i,l}\frac{\hat{\rho}_l}{\gamma_1 + \hat{\rho}_l} + \sum_{i=1}^{\hat{J}}\hat{C}_i e^{\gamma_i u}\sum_{l=1}^{\hat{v}}\hat{D}_{il}\frac{\rho_l}{\hat{\gamma}_1 + \rho_l} - 1\right)$$

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$$+\sum_{i=1}^{\hat{J}} \hat{C}_i e^{-\hat{\gamma}_i u} \sum_{l=1}^{\hat{\nu}} \hat{D}_{il},$$

when $\theta \to \infty$, $\Delta(\theta) = 1$, *we can get:*

$$P(\tau < \infty) = -C_1 D_{11} \left(\sum_{i=1}^{\hat{J}} \hat{C}_i e^{-\hat{\gamma}_i u} \sum_{l=1}^{\hat{\nu}} \hat{D}_{i,l} \frac{\hat{\rho}_l}{\gamma_1 + \hat{\rho}_l} + \sum_{i=1}^{\hat{J}} \hat{C}_i e^{\gamma_i u} \sum_{l=1}^{\hat{\nu}} \hat{D}_{il} \frac{\rho_l}{\hat{\gamma}_1 + \rho_l} - 1 \right) + \sum_{i=1}^{\hat{J}} \hat{C}_i e^{-\hat{\gamma}_i u} \sum_{l=1}^{\hat{\nu}} \hat{D}_{il}.$$

Example 3.7. Letting $\nu = \hat{\nu} = 1$ in Corollary 3.3, when $\sigma > 0$ or $\sigma = 0$ and $\mu > 0$, we get:

$$\begin{aligned} f_+^\theta(b-u, dy) =& \frac{(\rho_1-\gamma_1)(\gamma_2-\rho_1)}{\gamma_2-\gamma_1}(e^{-\gamma_1(b-u)} - e^{-\gamma_2(b-u)})e^{-\rho_1 y}dy \\ &- \frac{\hat{\rho}_1-\hat{\gamma}_1}{\hat{\gamma}_2-\hat{\gamma}_1}(e^{-\hat{\gamma}_1 u} - e^{-\hat{\gamma}_2 u})\left(\frac{(\rho_1-\gamma_1)(\gamma_2-\rho_1)}{\gamma_2-\gamma_1}(e^{-\gamma_1 b} - e^{-\gamma_2 b})\right)e^{-\rho_1 y}dy \\ &- \frac{(\hat{\rho}_1-\hat{\gamma}_1)(\hat{\gamma}_2-\hat{\rho}_1)}{\hat{\gamma}_2-\hat{\gamma}_1}(e^{-\hat{\gamma}_1 u} - e^{-\hat{\gamma}_2 u})\frac{(\rho_1-\gamma_1)(\gamma_2-\rho_1)}{\gamma_2-\gamma_1} \\ &\left(e^{-\gamma_1 b}\frac{e^{-\rho_1 y}}{\gamma_1+\hat{\rho}_1} - e^{-\gamma_2 b}\frac{e^{-\rho_1 y}}{\gamma_2+\hat{\rho}_1}\right)dy, \end{aligned}$$

$$\begin{aligned} f_-^\theta(b-u, dy) =& \frac{(\hat{\rho}_1-\hat{\gamma}_1)(\hat{\gamma}_2-\hat{\rho}_1)}{\hat{\gamma}_2-\hat{\gamma}_1}(e^{-\hat{\gamma}_1 u} - e^{-\hat{\gamma}_2 u})e^{-\hat{\rho}_1 y}dy \\ &- \frac{\rho_1-\gamma_1}{\gamma_2-\gamma_1}(e^{-\gamma_1(b-u)} - e^{-\gamma_2(b-u)})\left(\frac{(\hat{\rho}_1-\hat{\gamma}_1)(\hat{\gamma}_2-\hat{\rho}_1)}{\hat{\gamma}_2-\hat{\gamma}_1}(e^{-\hat{\gamma}_1 b} - e^{-\hat{\gamma}_2 b})\right)e^{-\hat{\rho}_1 y}dy \\ &- \frac{(\hat{\rho}_1-\hat{\gamma}_1)(\hat{\gamma}_2-\hat{\rho}_1)}{\hat{\gamma}_2-\hat{\gamma}_1}(e^{-\gamma_1(b-u)} - e^{-\gamma_2(b-u)})\frac{(\rho_1-\gamma_1)(\gamma_2-\rho_1)}{\gamma_2-\gamma_1} \\ &\left(e^{-\hat{\gamma}_1 b}\frac{e^{-\hat{\rho}_1 y}}{\hat{\gamma}_1+\rho_1} - e^{-\hat{\gamma}_2 b}\frac{e^{-\hat{\rho}_1 y}}{\hat{\gamma}_2+\rho_1}\right)dy. \end{aligned}$$

4. Pricing path-dependent options

In this section, we aim to study the price of barrier and lookback options. These options have a fixed maturity T and yield a payoff contingent upon the maximum (or minimum) of the asset price on [0, T]. Under a risk-neutral probability measure $\mathbb{P}$, the asset price process $\{S_t : t \geq 0\}$ is supposed as $S_t = S_0 e^{X_t}$, where X is given by (1.1), $S_0 = e^{X_0}$. Let the risk-free interest rate be $r > 0$. Further, to ensure that the asset price of S_t has finite expectation, we have occasion to assume $\rho_1 > 1$. We will use the results obtained in Section 2 to develop the pricing formulae for standard single barrier options and lookback options.

4.1. *Lookback options*

The value of a lookback option hinges on the maximum or minimum of the stock price during the entire life span of the option. With a strike price K and the maturity T, it is proverbial that (see e.g. Schoutens [43]) applying risk-neutral valuation and after selecting an equivalent martingale measure $\mathbb{P}$ the initial (i.e. $t = 0$) price of a fixed-strike lookback put option is given by:

$$L_{fix}^P(K, T) = e^{-rT}\mathbb{E}\left(\sup_{0 \leq t \leq T} S(t) - K\right)^+.$$

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The initial price of a fixed-strike lookback call option is given by

$$L_{fix}^{C}(K,T) = e^{-rT}\mathbb{E}\left(K - \inf_{0\le t\le T} S(t)\right)^{+};$$

The initial price of a floating-strike lookback put option is given by:

$$L_{floating}^{P}(T) = e^{-rT}\mathbb{E}\left(\sup_{0\le t\le T} S(t) - S_T\right)^{+};$$

The initial price of a floating-strike lookback call option is given by:

$$L_{floating}^{C}(T) = e^{-rT}\mathbb{E}\left(S_T - \inf_{0\le t\le T} S(t)\right)^{+}.$$

In the standard Black-Scholes setting, Merton [38] and Goldman et al. [18] deduced closed-form solutions for lookback options. Cai and Kou [11] obtained the Laplace transforms of the lookback put option price as a function of the maturity T for the double mixed-exponential jump diffusion model, however, the coefficients do not determinated explicitly. Yin et al. [48] presented the analytical solutions for lookback options in terms of Laplace transforms for mixed-exponential jump diffusion processes.

We shall only consider lookback put options by reason of the calculation for the lookback call option follows similarly.

Theorem 4.1. *Suppose X is a Levy process with jump measure given by* (2.1)*. For all sufficiently large $\delta > 0$, then*

(i) for $K \ge S_0$,

$$\int_0^\infty e^{-\delta T} L_{fix}^{P}(K,T)dT = \frac{S_0}{r+\delta}\sum_{i=1}^{n+1}\sum_{j=1}^{m_k}\frac{d_{ji}}{(j-1)!}\sum_{l=1}^{j}\sum_{t=1}^{l}\frac{\beta_{i,r+\delta}^{l-1}(l-1)!}{(\beta_{i,r+\delta}-1)^t(l-t)!}\left(\ln\frac{K}{S_0}\right)^{l-t}\left(\frac{S_0}{K}\right)^{\beta_{i,r+\delta}-1},$$

$$(ii)\quad \int_0^\infty e^{-\delta T} L_{floating}^{P}(T)dT = \frac{S_0}{r+\delta}\sum_{i=1}^{n+1}\sum_{j=1}^{m_k}\frac{d_{ji}}{(j-1)!}\sum_{l=1}^{j}\frac{\beta_{i,r+\delta}^{l-1}(l-1)!}{(\beta_{i,r+\delta}-1)^l} + \frac{S_0}{r+\delta} - \frac{S_0}{\delta},$$

where $\beta_{1,r+\delta},\cdots,\beta_{n+1,r+\delta}$ are the $(n+1)$ positive roots of the equation $\psi_1(z) = r+\delta$ and d_{ij} is defined by (2.8) *and* (2.9)*.*

Proof. (i). Set $k = \ln\frac{K}{S_0} \ge 0$, then

$$L_{fix}^{P}(K,T) = S_0 e^{-rT}\int_k^\infty e^y \mathbb{P}\left(\sup_{0\le s\le T} X(s) \ge y\right)dy.$$

It follows that:

$$\begin{aligned}\int_0^\infty e^{-\delta T} L_{fix}^{P}(K,T)dT &= S_0\int_k^\infty e^y\left[\int_0^\infty e^{-(r+\delta)T}\mathbb{P}\left(\sup\nolimits_{0\le s\le T} X(s)\ge y\right)dT\right]dy \\ &= \frac{S_0}{r+\delta}\int_k^\infty e^y\mathbb{E}(e^{-(r+\delta)\tau_y^+})dy.\end{aligned} \tag{4.1}$$

From Theorem 2.1 and (4.1), we can get:

$$\int_k^\infty e^y\mathbb{E}(e^{-(r+\delta)\tau_y^+})dy = \sum_{i=1}^{n+1}\sum_{j=1}^{m_k}\frac{d_{ji}}{(j-1)!}\sum_{l=1}^{j}\int_k^\infty e^{(1-\beta_{i,r+\delta})y}(y\beta_{i,r+\delta})^{l-1}dy. \tag{4.2}$$

After a series of partial integrals of (4.2), the result can be proved.

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(*ii*). Since

$$L^P_{floating}(T) = S_0 e^{-rT}\mathbb{E}\left[\exp\left(\sup_{0\le t\le T} X(t)\right)\right] - S_0,$$

consequently, it can be deduced that:

$$\begin{aligned}
\int_0^\infty e^{-\delta T} L^P_{floating}(T)dT &= S_0\int_0^\infty e^{-(r+\delta)T}\mathbb{E}\left[\exp\left(\sup_{0\le t\le T} X(t)\right)\right]dT - \tfrac{S_0}{\delta} \\
&= \tfrac{S_0}{r+\delta}\mathbb{E}\left[\exp\left(\sup_{0\le t\le e(r+\delta)} X(t)\right)\right] - \tfrac{S_0}{\delta} \\
&= \tfrac{S_0}{r+\delta}\left[1+\int_0^\infty e^y\mathbb{P}\left(\sup_{0\le s\le e(r+\delta)} X(s)\ge y\right)dy\right] - \tfrac{S_0}{\delta} \\
&= \tfrac{S_0}{r+\delta}\left[1+\int_0^\infty e^y\mathbb{E}(e^{-\delta\tau_y^+})dy\right] - \tfrac{S_0}{\delta}.
\end{aligned} \tag{4.3}$$

The result follows from Theorem 2.1 and (4.3) by a similar calculation of (4.2). □

Example 4.2. Letting $\nu = \hat{\nu} = 1$, when $\sigma > 0$ or $\sigma = 0$ and $\mu > 0$, we get:

$$\begin{aligned}
&\int_0^\infty e^{-\delta T} L^P_{fix}(K,T)dT \\
&= \frac{S_0}{r+\delta}\frac{\beta_2(\alpha_1-\beta_1)}{\alpha_2(\beta_2-\beta_1)(\beta_1-1)}\left(\frac{S_0}{K}\right)^{\beta_1-1} + \frac{S_0}{r+\delta}\frac{\beta_1(\alpha_1-\beta_2-\alpha_2)}{\alpha_1(\beta_2-\beta_1)(\beta_2-1)}\left(\frac{S_0}{K}\right)^{\beta_2-1},
\end{aligned}$$

$$\int_0^\infty e^{-\delta T} L^P_{floating}(T)dT = \frac{S_0}{r+\delta}\left[1+\frac{\beta_2(\alpha_1-\beta_1)}{\alpha_2(\beta_{2,r+\delta}-\beta_1)(\beta_1-1)} + \frac{\beta_1(\alpha_1-\beta_2-\alpha_2)}{\alpha_1(\beta_2-\beta_1)(\beta_2-1)}\right] - \frac{S_0}{\delta},$$

where β_1, β_2 are the positive roots of the equation $\psi_1(z) = r+\delta$ and $1 < \beta_1 < \alpha_1 < \beta_2 < \infty$. By Laplace transform inversion, we can get:

$$L^P_{fix}(K,T)dT = \left[\frac{S_0^{\beta_1}\beta_2(\alpha_1-\beta_1)}{\alpha_2 K^{\beta_1-1}(\beta_2-\beta_1)(\beta_1-1)} + \frac{S_0^{\beta_2}\beta_1(\alpha_1-\beta_2-\alpha_2)}{\alpha_1 K^{\beta_2-1}(\beta_2-\beta_1)(\beta_2-1)}\right]e^{-rT},$$

$$L^P_{floating}(T) = S_0\left[1+\frac{\beta_2(\alpha_1-\beta_1)}{\alpha_2(\beta_2-\beta_1)(\beta_1-1)} + \frac{\beta_1(\alpha_1-\beta_2-\alpha_2)}{\alpha_1(\beta_2-\beta_1)(\beta_2-1)}\right]e^{-rT} - S_0.$$

When $\sigma = 0$ and $\mu \le 0$, we have:

$$\int_0^\infty e^{-\delta T} L^P_{fix}(K,T)dT = \frac{S_0}{r+\delta}\frac{\alpha_1-\beta_1}{\alpha_1(\beta_1-1)}\left(\frac{S_0}{K}\right)^{\beta_1-1},$$

$$\int_0^\infty e^{-\delta T} L^P_{floating}(T)dT = \frac{S_0}{r+\delta}\left[1+\frac{\alpha_1-\beta_1}{\alpha_1(\beta_1-1)}\right] - \frac{S_0}{\delta},$$

where $1 < \beta_1 < \alpha_1 < \infty$.

By Laplace transform inversion, we can easily get:

$$L^P_{fix}(K,T) = \frac{S_0^{\beta_{1,r+\delta}}(\alpha_1-\beta_1)}{\alpha_1 K^{\beta_1-1}(\beta_1-1)}e^{-rT},$$

$$L^P_{floating}(T) = S_0\left[1+\frac{\alpha_1-\beta_1}{\alpha_1(\beta_1-1)}\right]e^{-rT} - S_0.$$

4.2. Barrier options

The common term barrier options refers to the type of options whose payoff is determined by whether the maximum or minimum of the underlying asset hits the barrier level over a maturity T. There are eight classes of (one dimensional, single) barrier options: up (down)-and-in (out) call (put) options. For

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more comprehensive information on the barrier options, we suggest readers refer to Schoutens [43]. The closed-form price of up-and-in call barrier option for a double exponential jump diffusion model is shown in Kou and Wang [28]; the closed-form expressions of the up-and-in call barrier option under a double mixed-exponential jump diffusion model are obtained in Cai and Kou [11]. The analytical solutions for barrier options in terms of Laplace transforms for mixed-exponential jump diffusion processes are presented in Yin et al. [48]. Here, we only demonstrate how to handle the down-and-out call barrier option as the methodology is also valid for the other seven barrier options. Under the risk-neutral probability measure $\mathbb{P}$, the price of a down-and-out call option with a strike price K and a barrier level U is expressed as:

$$DOC = \exp(-rT)\mathbb{E}[(S_T - K)^+ \mathbf{1}_{(\inf_{0\le t\le T} S_t > U)}|S_0], \ U < S_0.$$

Let $h = \ln \frac{U}{S_0}$ and $k = -\ln K$. Then

$$DOC(k,T) := DOC = \exp(-rT)\mathbb{E}_x[(S_0 e^{X_T} - e^{-k})^+ \mathbf{1}_{(\tau_h^- > T)}].$$

Theorem 4.3. *Suppose X is a Levy process with jump measure given by (2.3). For any* $0 < \phi < \theta_1 - 1$ *and* $r + \varphi > \psi_1(\phi + 1)$*, then*

$$\int_0^\infty \int_{-\infty}^\infty e^{-\phi k - \varphi T} DOC(k,T)dkdT = \frac{S_0^{\phi+1}}{\phi(\phi+1)(\varphi + r - \psi_1(\phi+1))} \Big\{1 - \sum_{i=1}^{n+1} \sum_{j=1}^{m_k} \frac{d_{ji}}{(j-1)!} \left(\frac{S_0}{U}\right)^{\beta_{i,r+\phi}+\phi+1} \sum_{l=1}^{j} (h\beta_{i,r+\phi})^{l-1} \Delta_{i,j,l}(\phi+1)\Big\},$$

where $-\beta_{1,r+\phi}, \cdots, -\beta_{n+1,r+\phi}$ *are the* $(n+1)$ *negative roots of the equation* $\psi_2(z) = r + \phi$ *and* $\Delta_{i,j,l}(s)$ *is defined by (2.23).*

Proof. Using the same argument as that of the proof of Theorem 5.2 in Cai and Kou [11], we get:

$$\begin{aligned}\int_0^\infty \int_{-\infty}^\infty e^{-\phi k - \varphi T} DOC(k,T)dkdT &= \int_0^\infty \int_{-\infty}^\infty e^{-\phi k - (r+\varphi)T} \mathbb{E}_x[(S_0 e^{X_T} - e^{-k})^+ \mathbf{1}_{(\tau_h^- > T)}]dkdT \\ &= \frac{S_0^{\phi+1}}{\phi(\phi+1)} \frac{1}{\varphi + r - \psi_1(\phi+1)} \left(1 - \mathbb{E}_x[e^{-(r+\varphi)\tau_h^- + (\phi+1)X(\tau_h^-)}]\right),\end{aligned}$$

and the result is obtained from Theorem 2.7 (i). □

Example 4.4. Letting $\nu = \hat{\nu} = 1$, when $\sigma > 0$ or $\sigma = 0$ and $\mu > 0$, we get:

$$\begin{aligned}&\int_0^\infty \int_{-\infty}^\infty e^{-\phi k - \varphi T} DOC(k,T)dkdT \\ &= \frac{S_0^{\phi+1}}{\phi(\phi+1)} \frac{1}{\varphi + r - \psi_1(\phi+1)} \left(1 - \frac{(\beta_2 + \phi + 1)(\alpha_1 - \beta_1)}{(\alpha_1 + \phi + 1)(\beta_2 - \beta_1)} \left(\frac{S_0}{U}\right)^{\beta_1 + \phi + 1} \right. \\ &\left. + \frac{(\beta_1 + \phi + 1)(\beta_2 - \alpha_1)}{(\alpha_1 + \phi + 1)(\beta_2 - \beta_1)} \left(\frac{S_0}{U}\right)^{\beta_2 + \phi + 1}\right),\end{aligned}$$

where $-\beta_1, -\beta_2$ are the negative roots of the equation $\psi_2(z) = r + \phi$ and $1 < \beta_1 < \alpha_1 < \beta_2 < \infty$.

When $\sigma = 0$ and $\mu \le 0$, we have:

$$\int_0^\infty \int_{-\infty}^\infty e^{-\phi k - \varphi T} DOC(k,T)dkdT = \frac{S_0^{\phi+1}}{\phi(\phi+1)} \frac{1}{\varphi + r - \psi_1(\phi+1)} \left(1 - \frac{(\alpha_1 - \beta_1)}{\alpha_1 + \phi + 1} \left(\frac{S_0}{U}\right)^{\beta_1 + \phi + 1}\right),$$

where $1 < \beta_1 < \alpha_1 < \infty$.

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