

THE FUNDAMENTAL UNIT AND BOUNDS FOR CLASS NUMBERS OF REAL QUADRATIC FIELDS

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Introduction

Although class number one problem for imaginary quadratic fields was solved in 1966 by A. Baker [3] and by H. M. Stark [25] independently, the problem for real quadratic fields remains still unsettled. However, since papers by Ankeny-Chowla-Hasse [2] and H. Hasse [9], many papers concerning this problem or giving estimate for class numbers of real quadratic fields from below have appeared. There are three methods used there, namely the first is related with quadratic diophantine equations ([2], [9], [27, 28, 29, 31], [17]), and the second is related with continued fraction expansions ([8], [4], [16], [14], [18]). The third is related with Dirichlet's classical class number formula

$$h_D = (2 \log \varepsilon_D)^{-1} \sqrt{D} L(1, \chi_D),$$

where $L(1, \chi_D)$ is the value at $s = 1$ of the L -function

$$L(s, \chi_D) = \sum_{n=1}^{\infty} \chi_D(n) n^{-s}$$

with Kronecker character χ_D belonging to the real quadratic field $\mathbf{Q}(\sqrt{D})$ ([12], [30], [20]). There, T. Tatzuza's lower bound for $L(1, \chi_D)$:

$$L(1, \chi_D) > 0.655(c/D^c) \quad (\text{with one possible exception of } D)$$

plays very important role (cf. [26], [10]).

On the other hand, regarding estimate for the class number of real quadratic fields from above, there are two methods. One of them uses L. K. Hua's upper bound (cf. [11]) for $L(1, \chi_D)$:

$$L(1, \chi_D) < 2^{-1} \log D + 1$$

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in Dirichlet's formula ([23], [33]). Another uses D. A. Burgess' upper bound (cf. [5]) for $L(1, \chi_p)$:

$$L(1, \chi_p) < 0.2456 \log p \quad (p > p_0)$$

([15], [6], [33]).

Among all real quadratic fields, especially real quadratic fields of R-D type whose fundamental unit was well-known by Richaud [24] and Degert [7] were first studied ([2], [9], [28, 29], [17, 19], [14]), and later by H. Yokoi [27], T. Nakahara [22] and R. A. Mollin [20] real quadratic fields of non R-D type were studied.

In recent papers [31, 32, 33, 34], we defined some new p -invariants for any rational prime p congruent to 1 mod 4, and studied relationships among these new invariants and already known invariants. Above all, the new p -invariant n_p defined by

$$|t_p/u_p^2 - n_p| < 1/2$$

through the fundamental unit

$$\varepsilon_p = (t_p + u_p\sqrt{p})/2 \quad (> 1)$$

of real quadratic field $\mathbf{Q}(\sqrt{p})$ with prime discriminant was fundamental as far as $n_p \neq 0$ (i.e. $t_p/u_p^2 > 1/2$). On the other hand, it became clear that the case $n_p = 0$ (i.e. $0 < t_p/u_p^2 < 1/2$) is more important than the case $n_p \neq 0$ for the purpose of solving the class number one problem or Artin's conjecture for real quadratic fields (cf. [1], [32], [34]).

Therefore, one of our purposes in this paper is to define and to study similarly new invariants valuable in case of $n_p = 0$, and another is to generalize p -invariants to D -invariants for any positive square-free integer D .

All results obtained in this paper are valid for any positive square-free integer D provided we add a few unessential modifications, but for the sake of simplicity, we shall restrict integer D to a positive square-free integer satisfying $N\varepsilon_D = -1$ i.e. $t_D^2 - Du_D^2 = -4$.

In order to deal with the case $n_D = 0$ in the same way as the case $n_D \neq 0$, in §1 we shall slightly reform the definition of n_D . Namely, we shall newly define n_D by $n_D = [t_D/u_D^2]$ (or $n_D = [D/t_D]$ if $u_D > 2$), and first express uniquely D (resp. t_D) as a quadratic (resp. linear) polynomial of n_D with D -invariant coefficients, where $[x]$ means the greatest integer less than or equal to x . Using the Dirichlet's class number formula, we shall

next provide bounds for the class number h_D of real quadratic field $\mathbf{Q}(\sqrt{D})$ in terms of D and n_D .

In §2, we shall first define a new invariant m_D by $m_D = [t_D/D]$ (or $m_D = [u_D^2/t_D]$ if $D > 5$), and express uniquely u_D^2 (resp. t_D) as a quadratic (resp. linear) polynomial of m_D with D -invariant coefficients. Next, in terms of new D -invariants, we shall give a necessary and sufficient condition for Artin's conjecture on the fundamental unit of real quadratic fields with prime discriminant to be true. Finally, using Dirichlet's formula, we shall estimate the class number h_D of $\mathbf{Q}(\sqrt{D})$ in terms of D and m_D .

Throughout this paper, we denote by $\mathbf{N}_0 = \{0, 1, 2, \dots\}$ the set of all non-negative rational integers and by $[x]$ the greatest integer less than or equal to x . Moreover, for any positive square-free integer D , we denote by h_D and $\varepsilon_D = (t_D + u_D\sqrt{D})/2 (> 1)$ the class number and the fundamental unit of the real quadratic field $\mathbf{Q}(\sqrt{D})$ respectively. Furthermore, we denote by \mathbf{D}_- the set of all positive square free integers such that the norm of the fundamental unit $(t_D + u_D\sqrt{D})/2$ of $\mathbf{Q}(\sqrt{D})$ is equal to -1 .

§ 1.

For any D in \mathbf{D}_- , set

$$\mathbf{A}_D = \{a: 0 \leq a < u_D^2, a^2 \equiv -4 \pmod{u_D^2}\}$$

and

$$(A, B)_D = \{(a, b): a \in \mathbf{A}_D, a^2 + 4 = bu_D^2\}.$$

Then, we can first prove the following theorem:

THEOREM 1.1. *For any D in \mathbf{D}_- , there are an uniquely determined integer n_D in \mathbf{N}_0 and an element (a_D, b_D) in $(A, B)_D$ such that*

$$\begin{cases} t_D = u_D^2 \cdot n_D + a_D, \\ D = u_D^2 \cdot n_D^2 + 2a_D \cdot n_D + b_D. \end{cases}$$

Additionally, if $u_D > 2$, then $0 < b_D < a_D < u_D^2$ and $[D/t_D] = n_D$.

Proof. For any D in \mathbf{D}_- , put

$$[t_D/u_D^2] = n_D \quad \text{and} \quad t_D = u_D^2 \cdot n_D + a_D,$$

then $n_D (\in \mathbf{N}_0)$ and $a_D (\geq 0)$ are uniquely determined and

$$0 \leq a_D < u_D^2$$

holds. Moreover, since

$$Du_D^2 = t_D^2 + 4 = u_D^4 \cdot n_D^2 + 2a_D \cdot u_D^2 \cdot n_D + (a_D^2 + 4),$$

we obtain

$$a_D^2 + 4 \equiv 0 \pmod{u_D^2}$$

and so a_D is in \mathbf{A}_D .

Now, put

$$a_D^2 + 4 = b_D \cdot u_D^2$$

then $b_D (> 0)$ is also uniquely determined and (a_D, b_D) is in $(A, B)_D$, and moreover

$$D = u_D^2 \cdot n_D^2 + 2a_D \cdot n_D + b_D$$

holds.

Especially, in the case $u_D > 2$, it follows from $0 \leq a_D < u_D^2$ that

$$b_D \cdot u_D^2 = a_D^2 + 4 < u_D^4 + 4,$$

and so $b_D \leq u_D^2$. Here, if we assume $b_D = u_D^2$, then $a_D^2 + 4 = u_D^4$ implies $(u_D^2 - a_D)(u_D^2 + a_D) = 4$, which contradicts $u_D > 2$. Hence, we have first $0 < b_D < u_D^2$.

Next, if we put

$$g(x) = -x^2 + u_D^2 x - 4,$$

then $g(1) = g(u_D^2 - 1) = u_D^2 - 5 > 0$, and so $g(b_D) = a_D^2 - b_D^2 > 0$. Therefore, we obtain

$$0 < b_D < a_D < u_D^2.$$

Finally, in the expression

$$D = n_D \cdot t_D + (a_D \cdot n_D + b_D),$$

we have $a_D \cdot n_D + b_D > 0$ and

$$t_D - (a_D \cdot n_D + b_D) = (u_D^2 - a_D)n_D + (a_D - b_D) > 0.$$

Hence we obtain $[D/t_D] = n_D$.

COROLLARY 1.1. *For any $D (> 2)$ in \mathbf{D}_- ,*

(1) *The following are equivalent:*

(i) $n_D = 0$, (ii) $D < t_D$, (iii) $t_D < u_D^2$ (in this case $u_D > 2$ holds).

(2) $a_D = 0$ holds only if $u_D = 1$ or 2 .

In the special case where $D \not\equiv 2 \pmod{4}$ i.e. $D \equiv 1 \pmod{4}$,

$a_D = 0$ if and only if $u_D = 1$ or 2 .

Proof. (1) It is clear from $[t_D/u_D^2] = n_D$ that

$n_D = 0$ if and only if $0 < t_D < u_D^2$.

Suppose $n_D = 0$. Then the above implies $u_D > 1$, and if $u_D = 2$, then $t_D \equiv u_D \pmod{2}$ yields

$$t_D = 2 \text{ i.e. } D = 2.$$

Therefore, $n_D = 0$ implies $u_D > 2$ and by the theorem $D < t_D$. Finally $D < t_D$ implies

$$t_D - D = (u_D^2 n_D + a_D)(1 - n_D) - a_D n_D - b_D > 0$$

and hence $n_D = 0$ holds except for $D = 2$.

(2) It is clear from Theorem 1.1 that

$a_D = 0$ if and only if $t_D \equiv 0 \pmod{u_D^2}$,

which implies easily $u_D = 1$ or $u_D = 2$.

On the other hand, in the case $u_D = 1$, we have $n_D = t_D$, $a_D = 0$, $b_D = 4$, and in the case $u_D = 2$, we have

$$a_D = 0, \quad b_D = 1, \quad D = 4 \cdot n_D^2 + 1 \equiv 1 \pmod{4}$$

or

$$a_D = 2, \quad b_D = 2, \quad D = 4 \cdot n_D^2 + 4n_D + 2 \equiv 2 \pmod{4}.$$

Hence, in the special case $D \equiv 2 \pmod{4}$ or $D \equiv 1 \pmod{4}$,

$a_D = 0$ if and only if $u_D = 1$ or 2 .

Next, we provide upper bounds for the class number h_D of $\mathbf{Q}(\sqrt{D})$.

THEOREM 1.2. For any D in \mathbf{D}_- ,

(1) $\varepsilon_D > (D - 4)/(n_D + 1)$,

(2) $h_D < \{(1/4)\sqrt{D} \cdot (2 + \log D)\} / \{\log((D - 4)/(n_D + 1))\}$.

Additionally, in the special case where $D = p$ is prime, there exists a constant p_0 such that if a prime p in \mathbf{D}_- satisfies $p > p_0$, then

$$(3) \quad h_p < \{(1/8)\sqrt{p} \cdot \log p\} / \{\log((p - 4)/(n_p + 1))\}.$$

In order to prove this theorem, we need the following two lemmas:

LEMMA 1.1 (L.K. Hau [11]). *For the value $L(1, \chi_d)$ at $s = 1$ of L -function $L(s, \chi_d)$ belonging to the real quadratic field $\mathbf{Q}(\sqrt{d})$,*

$$L(1, \chi_d) < (1/2) \log d + 1$$

holds.

LEMMA 1.2 (D.A. Burgess [5]). *For any prime p in \mathbf{D}_- , let $L(1, \chi_p)$ be the value at $s = 1$ of L -function $L(s, \chi_p)$ belonging to $\mathbf{Q}(\sqrt{p})$. Then there exists a prime constant p_0 such that*

$$L(1, \chi_p) < 0.2456 \log p$$

holds provided $p > p_0$.

Proof of Theorem 1.2. (1) It follows from $[t_D/u_D^2] = n_D$ that

$$D - 4u_D^{-2} = t_D^2 \cdot u_D^{-2} < t_D(n_D + 1),$$

and so

$$t_D > (D - 4u_D^{-2}) / (n_D + 1) \geq (D - 4) / (n_D + 1).$$

Hence, we get

$$-4 = t_D^2 - Du_D^2 > ((D - 4)/(n_D + 1))^2 - Du_D^2,$$

and so

$$u_D^2 > D^{-1}((D - 4)/(n_D + 1))^2 + 4/D > D^{-1}((D - 4)/(n_D + 1))^2,$$

which implies

$$u_D > ((D - 4)/(n_D + 1)) \cdot (1/\sqrt{D}).$$

Therefore, we obtain first

$$\varepsilon_D = (t_D + u_D\sqrt{D})/2 > (D - 4)/(n_D + 1).$$

(2) By applying the above assertion (1) and Lemma 1.1 to the Dirichlet's class number formula, we can get immediately the upper bound for h_p in the Theorem.

(3) Using Lemma 1.2 instead of Lemma 1.1, we can prove assertion (3) by the same way as assertion (2).

COROLLARY 1.2. *For any positive ε and for an arbitrarily fixed invariant $n = n_D (\geq 0)$, there is an effectively determined constant $D_0 = D_0(\varepsilon, n)$, depending on both ε and n , such that if D in \mathbf{D}_- satisfies $D > D_0$ and $n_D = n$, then*

$$h_D < ((1 + \varepsilon)/4)\sqrt{D}.$$

Proof. For any given and fixed $n = n_D (\geq 0)$, it holds

$$\lim_{D \rightarrow \infty} (2 + \log D) / \{\log ((D - 4)/(n + 1))\} = 1.$$

Hence, we can get Corollary 1.2 by Theorem 1.2, (2).

COROLLARY 1.3. *For an arbitrarily fixed $n = n_p (\geq 0)$, there is a constant $p_0 = p_0(n)$ depending only on n , such that if a prime p in \mathbf{D}_- satisfies $p > p_0$, then for the class number h_p of $\mathbf{Q}(\sqrt{p})$*

$$h_p < (1/8)\sqrt{p}$$

holds.

Proof. For an arbitrarily fixed $n = n_p (\geq 0)$, we get

$$\lim_{p \rightarrow \infty} (\log p) / \{\log ((p - 4)/(n + 1))\} = 1.$$

On the other hand, since $(1/8) - (0.2456/2) > 0$, we can choose a positive number ε satisfying $0.2456/2 = (1/8)(1 - \varepsilon)$. Hence, for this ε , we can choose a prime number $p'_0(n)$ such that

$$(\log p) / \{\log ((p - 4)/(n + 1))\} < 1 + \varepsilon$$

holds provided $p > p'_0(n)$.

Therefore, using Lemma 1.2 for any prime p in \mathbf{D}_- satisfying $p > p_0(n) := \max(p'_0(n), p_0)$ for p_0 in Lemma 1.2, we have

$$\begin{aligned} h_p &< \{(0.2456/2)\sqrt{p} \log p\} / \{\log ((p - 4)/(n + 1))\} \\ &< (1/8)(1 - \varepsilon)\sqrt{p}(1 + \varepsilon) \\ &< (1/8)\sqrt{p}. \end{aligned}$$

Finally, we provide lower bounds for the class number h_D of $\mathbf{Q}(\sqrt{D})$.

THEOREM 1.3. *For any D in \mathbf{D}_- , $n_D \neq 0$ implies the following:*

- (1) $\varepsilon_D < (D/n_D) + 1$,
- (2) $h_D > (0.3275/s) \cdot D^{(s-2)/2s} / \{\log ((D/n_D) + 1)\}$

for any $s \geq 11.2$ with one exception of D , and without any exception under the Generalized Riemann Hypothesis.

In order to prove this theorem, we need the following lemma:

LEMMA 1.3 (T. Tatzuzaawa [26], cf. [10]). *For any positive c satisfying $1/2 > c > 0$, let d be any positive integer such that $d \geq \max(e^{1/c}, e^{11.2})$, and let χ_d be any non-principal primitive character with modulus d .*

Then, it holds with one possible exception of d

$$L(1, \chi_d) > 0.655(c/d^c),$$

where $L(1, \chi_d)$ is the value at $s = 1$ of the L -function $L(s, \chi_d)$ corresponding to χ_d .

Proof of Theorem 1.3. (1) In the case $n_D \neq 0$, it follows from Theorem 1.1 that

$$D > t_D \cdot n_D \quad \text{i.e.} \quad t_D < D/n_D.$$

Hence we get

$$-4 = t_D^2 - Du_D^2 < (D/n_D)^2 - Du_D^2,$$

and so

$$u_D^2 < D^{-1}((D/n_D)^2 + 4) = Dn_D^{-2} + 4D^{-1},$$

which implies

$$u_D < (\sqrt{D}/n_D) + (2/\sqrt{D}).$$

Therefore, we obtain first

$$\varepsilon_D = (t_D + u_D\sqrt{D})/2 < (D/n_D) + 1.$$

(2) Putting $s = 1/c$ in Lemma 1.3, for any $s \geq 11.2$ we have

$$L(1, \chi_D) > (0.655/s)D^{-1/s}.$$

On the other hand, since under the Generalized Riemann Hypothesis Lemma 1.3 is true without any exception (cf. Kim [13]), we can prove Theorem 1.3, (2) by applying the above assertion (1) to the Dirichlet's class number formula.

From Theorem 1.3, we can obtain the following corollary:

COROLLARY 1.4. *There is an effectively determined D_0 such that if D in \mathbf{D}_- satisfies both $D > D_0$ and $n_D \neq 0$, then $h_D > 1$ holds with one more*

possible exception of D .

In other words, there exist only finitely many D in \mathbf{D}_- satisfying both $h_D = 1$ and $n_D \neq 0$.

Proof. If $n_D \neq 0$, then $\log((D/n_D) + 1) < \log 2D$ holds, and Theorem 1.3 yields

$$h_D > 0.3275 \cdot s^{-1} \cdot D^{(s-2)/2s} / \log 2D$$

for any fixed $s \geq 11.2$ with one possible exception of D .

On the other hand,

$$f_s(x) = x^{(s-2)/2s} / \log 2x$$

is monotone increasing on $[6, \infty)$, and $\lim_{x \rightarrow \infty} f_s(x) = \infty$ holds. Hence we can determine D_0 effectively such that

$$0.3275 \cdot s^{-1} \cdot D^{(s-2)/2s} / \log 2D > 1$$

holds for any $D \geq D_0$.

In fact, it is known that there are exactly 54 D 's satisfying both $h_D = 1$ and $n_D \neq 0$ (cf. Yokoi [34], Mollin and Williams [21]).

From Corollary 1.4, we can easily derive the following corollary closely related to Gauss' conjecture on class numbers of real quadratic fields:

COROLLARY 1.5. *There exist infinitely many primes p in \mathbf{D}_- satisfying $h_p = 1$ if and only if there exist infinitely many primes p in \mathbf{D}_- satisfying both $h_p = 1$ and $n_p = 0$.*

§ 2.

For any D in \mathbf{D}_- , set

$$\bar{\mathbf{A}}_D = \{a: 0 \leq a < D, a^2 \equiv -4 \pmod{D}\}$$

and

$$(\tilde{\mathbf{A}}, \tilde{\mathbf{B}})_D = \{(a, b): a \in \bar{\mathbf{A}}_D, a^2 + 4 = bD\}.$$

Then, we can first prove the following theorem:

THEOREM 2.1. *For any D in \mathbf{D}_- , there are uniquely determined m_D in \mathbf{N}_0 and (a_D, b_D) in $(\tilde{\mathbf{A}}, \tilde{\mathbf{B}})_D$ such that*

$$\begin{cases} t_D = D \cdot m_D + a_D, \\ u_D^2 = D \cdot m_D^2 + 2a_D \cdot m_D + b_D. \end{cases}$$

Additionally, if $D > 5$, then

$$0 < b_D < a_D < D \quad \text{and} \quad [u_D^2/t_D] = m_D.$$

Proof. For any D in \mathbf{D}_- , put

$$[t_D/D] = m_D \quad \text{and} \quad t_D = D \cdot m_D + a_D,$$

then m_D ($\in \mathbf{N}_0$) and a_D are uniquely determined and

$$0 \leq a_D < D.$$

Moreover, since

$$Du_D^2 = t_D^2 + 4 = D^2 \cdot m_D^2 + 2a_D \cdot Dm_D + (a_D^2 + 4),$$

we obtain

$$a_D^2 + 4 \equiv 0 \pmod{D},$$

and so a_D is in $\tilde{\mathbf{A}}_D$.

Now, put

$$a_D^2 + 4 = D \cdot b_D,$$

then b_D (> 0) is also uniquely determined and (a_D, b_D) is in $(\tilde{\mathbf{A}}, \tilde{\mathbf{B}})_D$, and moreover

$$u_D^2 = D \cdot m_D^2 + 2a_D \cdot m_D + b_D$$

holds.

Suppose $D > 5$. If we assume $b_D \geq D$, then

$$a_D^2 + 4 = D \cdot b_D \geq D^2,$$

and so

$$4 \geq D^2 - a_D^2 = (D - a_D)(D + a_D) \geq D + a_D \geq D,$$

which contradicts with $D > 5$. Hence we have first $0 < b_D < D$.

Next, if we put

$$g(x) = -x^2 + Dx - 4,$$

then $g(1) = g(D - 1) = D - 5 > 0$, and so $g(b_D) = a_D^2 - b_D^2 > 0$. Therefore we obtain

$$0 < b_D < a_D < D.$$

Finally, in the expression

$$u_D^2 = t_D \cdot m_D + (a_D \cdot m_D + b_D),$$

we have $a_D \cdot m_D + b_D > 0$ and

$$t_D - (a_D \cdot m_D + b_D) = (D - a_D) \cdot m_D + (a_D - b_D) > 0.$$

Hence, we have $[u_D^2/t_D] = m_D$.

COROLLARY 2.1. *For any D in \mathbf{D}_- ,*

(1) *if $D > 5$, then the following are equivalent:*

(i) $m_D = 0$ (ii) $t_D < D$ (iii) $u_D^2 < t_D$ (iv) $a_D = t_D, b_D = u_D^2$.

(2) $a_D = 0$ *if and only if $D = 2$*

(in this case $t_D = u_D = b_D = 2, m_D = 1$).

Proof. (1) In the case $D > 5$, it is clear from $[u_D^2/t_D] = m_D$ that

$$m_D = 0 \text{ if and only if } 0 < u_D^2 < t_D.$$

Moreover, it is also clear from $[t_D/D] = m_D$ that

$$m_D = 0 \text{ if and only if } 0 < t_D < D.$$

On the other hand, if $m_D = 0$, then by Theorem 2.1 we have

$$t_D = a_D \text{ and } u_D^2 = b_D.$$

Conversely, if $t_D = a_D$, then $0 < a_D < D$ implies $0 < t_D < D$, i.e.

$$m_D = [t_D/D] = 0.$$

(2) If we assume $a_D = 0$, then by Theorem 2.1 we have $D \leq 5$. On the other hand, in the case $D = 5$, we know $t_5 = u_5 = 1, m_5 = 0, a_5 = b_5 = 1$. In the case $D = 2$, we know also $t_2 = u_2 = 2, m_2 = 1, a_2 = 0, b_2 = 2$.

Hence $a_D = 0$ if and only if $D = 2$.

In the case where $D = p$ is prime congruent to 1 mod 4, the following conjecture is well-known as Artin's conjecture (cf. [1]):

CONJECTURE (Artin). *For the fundamental unit*

$$\varepsilon_p = (t_p + u_p \sqrt{p})/2 > 1$$

of the quadratic field $\mathbf{Q}(\sqrt{p})$,

$$u_p \not\equiv 0 \pmod{p}$$

holds.

Concerning this conjecture, we are able to prove the following:

THEOREM 2.2. *Artin's conjecture is true if and only if*

$$a_p \cdot b_p \not\equiv 8m_p \pmod{p}.$$

In particular, if $m_p \equiv 0 \pmod{p}$, Artin's conjecture is true.

Proof. In the special case $p = 5$, ($t_5 = u_5 = 1$), we get $m_5 = 0$, $a_5 = b_5 = 1$, and hence both $u_5 \not\equiv 0 \pmod{5}$ and $a_5 \cdot b_5 \not\equiv 8m_5 \pmod{5}$ simultaneously hold. Therefore without loss of generality, we may assume $p > 5$, and so $(a_p, p) = 1$ by Theorem 2.1.

Moreover, it follows immediately from $u_p^2 = p \cdot m_p^2 + 2a_p \cdot m_p + b_p$ that

$$u_p \equiv 0 \pmod{p} \text{ if and only if } 2a_p^2 \cdot m_p + a_p b_p \equiv 0 \pmod{p}.$$

On the other hand, $a_p^2 + 4 = p \cdot b_p$ implies

$$2a_p^2 \cdot m_p + a_p b_p \equiv a_p b_p - 8m_p \pmod{p}.$$

Hence, it is clear that

$$u_p \not\equiv 0 \pmod{p} \text{ if and only if } a_p b_p \not\equiv 8m_p \pmod{p}.$$

Especially, in the case $p > 5$, we get $a_p b_p \not\equiv 0 \pmod{p}$ by Theorem 2.1, and hence if $m_p \equiv 0 \pmod{p}$, then $a_p b_p \not\equiv m_p \pmod{p}$ i.e. Artin's conjecture is true.

Finally, we provide a lower bound and an upper bound for the class number h_D of $\mathbb{Q}(\sqrt{D})$.

THEOREM 2.3. *For any $D > 5$ in \mathbf{D}_- ,*

- (1) $[\varepsilon_D/D] = m_D$,
- (2) *For any $s \geq 11.2$ and $D \geq e^s$,*

$$h_D > 0.3275 \cdot s^{-1} \cdot D^{(s-2)/2s} / \{\log((m_D + 1)D)\}$$

holds with one possible exception of D ,

- (3) $h_D < \{(1/4)\sqrt{D} \cdot (2 + \log D)\} / \{\log(D \cdot m_D)\}$.

Especially, if $m_D > e^2$, then

$$h_D < (1/4)\sqrt{D}.$$

Proof. (1) By Theorem 2.1, we have first $Dm_D \leq t_D < D(m_D + 1)$. Hence we get also

$$D^2 m_D^2 \leq t_D^2 = Du_D^2 - 4 < D^2(m_D + 1)^2,$$

i.e.

$$Dm_D^2 + (4/D) \leq u_D^2 < D(m_D + 1)^2 + (4/D),$$

which implies easily

$$m_D\sqrt{D} < u_D \leq (m_D + 1)\sqrt{D}$$

because of $D > 5$. Therefore we get

$$Dm_D < \varepsilon_D = (t_D + u_D\sqrt{D})/2 < D(m_D + 1),$$

from which implies $[\varepsilon_D/D] = m_D$.

Remark. We can prove also $[u_D/D] = m_D$.

(2) Since $\varepsilon_D < D(m_D + 1)$, we can prove assertion (2) by applying Lemma 1.3 to the Dirichlet's class number formula.

(3) Since $Dm_D < \varepsilon_D$, we can also obtain

$$h_D < \{(1/4)\sqrt{D} \cdot (2 + \log D)\} / \log(D \cdot m_D)$$

by applying Lemma 1.1 to the Dirichlet's class number formula.

Moreover, in the special case $m_D > e^2$,

$$0 < (2 + \log D) / \log(D \cdot m_D) < 1,$$

yields immediately $h_D < (1/4)\sqrt{D}$.

COROLLARY 2.2. *In the special case where $D = p$ is prime, if $m_D > e^2$ then*

$$h_p < (1/8)\sqrt{p}$$

holds except for at most finite number of p .

Proof. Using Lemma 1.2 instead of Lemma 1.1, we can prove Corollary 2.2 in the same way as the proof of Theorem 2.3, (2).

$t_D = Dm_D + a_D$	•	$t_D = u_D^2 n_D + a_D$
$a_D^2 + 4 = b_D D$	•	$a_D^2 + 4 = b_D u_D^2$
$c_D = a_D m_D + b_D$	•	$c_D = a_D n_D + b_D$
$u_D^2 = m_D t_D + c_D$	•	$D = n_D t_D + c_D$

D	t_D	u_D	h_D	m_D	n_D	a_D	b_D	c_D
* 2	2	2	1	1		0	2	2
* 5	1	1	1		1	0	4	4
10	6	2	2		1	2	2	4
* 13	3	1	1		3	0	4	4
* 17	8	2	1		2	0	1	1
26	10	2	2		2	2	2	6
* 29	5	1	1		5	0	4	4
* 37	12	2	1		3	0	1	1
* 41	64	10	1	1		23	13	36
* 53	7	1	1		7	0	4	4
58	198	26	2	3		24	10	82
* 61	39	5	1		1	14	8	22
65	16	2	2		4	0	1	1
* 73	2136	250	1	29		19	5	556
74	86	10	2	1		12	2	14
82	18	2	4		4	2	2	10
85	9	1	2		9	0	4	4
* 89	1000	106	1	11		21	5	236
* 97	11208	1138	1	115		53	29	6124
* 101	20	2	1		5	0	1	1
106	8010	778	2	75		60	34	4534
* 109	261	25	1	2		43	17	103
* 113	1552	146	1	13		83	61	1140
122	22	2	2		5	2	2	12
130	114	10	4		1	14	2	16
* 137	3488	298	1	25		63	29	1604
145	24	2	4		6	0	1	1
* 149	61	5	1		2	11	5	27
* 157	213	17	1	1		56	20	76
170	26	2	4		6	2	2	14
* 173	13	1	1		13	0	4	4
* 181	1305	97	1	7		38	8	274

D	t_D	u_D	h_D	m_D	n_D	a_D	b_D	c_D
185	136	10	2		1	36	13	49
* 193	3528264	253970	1	18281		31	5	566716
* 197	28	2	1		7	0	1	1
202	6282	442	2	31		20	2	622
218	502	34	2	2		66	20	152
226	30	2	8		7	2	2	16
* 229	15	1	3		15	0	4	4
* 233	46312	3034	1	198		178	136	35380
* 241	142022136	9148450	1	589303		113	53	66591292
* 257	32	2	3		8	0	1	1
265	12144	746	2	45		219	181	10036
* 269	164	10	1		1	64	41	105
274	2814	170	4	10		74	20	760
* 277	2613	157	1	9		120	52	1132
* 281	2127064	126890	1	7569		175	109	1324684
290	34	2	4		8	2	2	18
* 293	17	1	1		17	0	4	4
298	819114	47450	2	2748		210	148	577228
* 313	253724736	14341370	1	810622		50	8	40531108
314	886	50	2	2		258	212	728
* 317	89	5	1		3	14	8	50
* 337			1					
346	186	10	6		1	86	74	160
* 349	18420	986	1	52		272	212	14356
* 353	142528	7586	1	403		269	205	108612
362	38	2	2		9	2	2	20
365	19	1	2		19	0	4	4
370	654	34	4	1		284	218	502
* 373	10236	530	1	27		165	73	4528
* 389	2564	130	1	6		230	136	1516
394	790046070	39801946	2					
* 397	3447	173	1	8		271	185	2353

D	t_D	u_D	h_D	m_D	n_D	a_D	b_D	c_D
* 401	40	2	5		10	0	1	1
* 409			1					
* 421	444939	21685	1	1056		363	313	383641
* 433			1					
442	42	2	8		10	2	2	22
445	21	1	4		21	0	4	4
* 449	378942664	17883410	1	843970		134	40	113092020

* indicates prime number

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