THE SIZE OF THE UNIT SPHERE

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Banach (**1**, pp. 242–243) defines, for two Banach spaces X and Y, a number $(X, Y) = \inf (\log (||L|| \, ||L^{-1}||))$, where the infimum is taken over all isomorphisms L of X onto Y. He says that the spaces X and Y are *nearly isometric* if $(X, Y) = 0$ and asks whether the concepts of near isometry and isometry are the same; in particular, whether the spaces c and c_0, which are not isometric, are nearly isometric. In a recent paper (**2**) Michael Cambern shows not only that c and c_0 are not nearly isometric but obtains the elegant result that for the class of Banach spaces of continuous functions vanishing at infinity on a first countable locally compact Hausdorff space, the notions of isometry and near isometry coincide.

We introduce two numbers for a normed linear space X, which give some measure of the size of the unit sphere in X. We show that if either of these numbers differs for two spaces, then these spaces cannot be nearly isometric. The first number, the thickness of X, is related to F. Riesz's theorem: a normed linear space is finite dimensional if (and only if) its closed unit sphere is compact. The second number, the thinness of X, gives us a geometric way of showing that c and c_0 are not nearly isometric, since one is thinner than the other.

Recall that for a metric space S with distance d, *an ϵ-net for S* is a set F of points of S with the property that for each s in S there is a t in F with $d(s, t) \leqslant \epsilon$ and also recall that a complete metric space is compact if and only if it has a finite ϵ-net for each $\epsilon > 0$. For a normed linear space X we denote *the surface of the unit sphere of* X by $S(X) = \{x \text{ in } X: ||x|| = 1\}$. If $\{x_1, \ldots, x_n\}$ is an ϵ-net for $S(X)$ and $\{a_1, \ldots, a_m\}$ is an ϵ'-net for the unit sphere $\{a: |a| \leqslant 1\}$ in the scalars (either the reals or the complexes) then $\{a_i x_j: 1 \leqslant j \leqslant n, 1 \leqslant i \leqslant m\}$ is an $(\epsilon + \epsilon')$-net for the unit sphere $\{x \text{ in } X: ||x|| \leqslant 1\}$. Consequently, for an infinite-dimensional normed linear space X we see from Riesz's theorem (**3**, Theorem IV.3.5, p. 245) that $S(X)$ fails to have a finite ϵ-net for some $\epsilon > 0$. (For X not complete, this follows from Lemma 2 below or from considering the completion of X.) The possible size of this ϵ should give an indication of the size of the unit sphere in X and to this end we define $A = \{a \geqslant 0: \text{for each } \epsilon > a, S(X) \text{ has a finite } \epsilon\text{-net}\}$. A simple argument shows that $T(X) = \inf A$ belongs to A.

1. *Definition.* For a normed linear space X we define $T(X)$, *the thickness of* X, to be the number described in the above paragraph, i.e. it is the largest

Received October 28, 1966. The author was supported by National Science Foundation grant GP 5424.

non-negative number with the property that each ϵ-net of $S(X)$ must be infinite if $0 < \epsilon < T(X)$.

2. LEMMA. *Let X be a normed linear space.*

(1) *If X is finite dimensional, then* $T(X) = 0$.

(2) *If X is infinite dimensional, then* $1 \leqslant T(X) \leqslant 2$.

(3) *For each T in* $[1, 2]$ *there is a Banach space X with* $T(X) = T$.

Proof. If X is finite dimensional, its unit sphere is compact (**3**, Corollary IV.3.3, p. 245), and so $T(X) = 0$.

Now suppose that $\{x_1, \ldots, x_n\}$ is an ϵ-net for $S(X)$ with $0 < \epsilon < 1$. By the Hahn–Banach theorem there are continuous linear functionals $x_1^*, \ldots, x_n^*$ of norm one with $x_i^*(x_i) = 1$. For each x in $S(X)$ and the appropriate index i, $1 > \epsilon \geqslant \|x - x_i\| \geqslant |x_i^*(x) - 1|$, so $x_i^*(x) \neq 0$. It follows that the map L of X into E^n defined by $Lx = (x_1^*(x), \ldots, x_n^*(x))$ is one-to-one, which forces X to be finite dimensional. Any single point $\{x\}$ of norm one is a 2-net for $S(X)$, which bounds $T(X)$ above by 2.

We shall show in Lemma 4 that for $1 \leqslant p < \infty$, $T(l^p) = 2^{1/p}$ and in Lemma 3 that $T(c) = 1$, which will establish (3).

The bound $T(X) \geqslant 1$, for X infinite dimensional, can be obtained alternatively by using Riesz's lemma (**5**, Theorem 3.12 E, p. 96). Note that the demonstration of this given in Lemma 2 yields a proof of Riesz's theorem which differs from the usual proofs.

For a topological space Q, $C(Q)$ is the space of bounded continuous scalar-valued functions on Q with the supremum norm. For Q locally compact, $C_0(Q)$ is the subspace of $C(Q)$ of functions which vanish at infinity.

3. LEMMA. *Let Q be a completely regular Hausdorff space which contains an infinite number of points. Then $T(C(Q)) = 1$ if Q contains an isolated point and $T(C(Q)) = 2$ otherwise. If Q is also locally compact, then*

$$T(C_0(Q)) = T(C(Q)).$$

Proof. If Q has an isolated point s, let f be the characteristic function of the set $\{s\}$. The set $\{f, -f\}$, for the case of real scalars, or the set $\{f, -f, if, -if\}$, for the case of complex scalars, is a 1-net for the surface of the unit sphere in $C(Q)$. Thus, $T(C(Q)) = 1$.

Suppose that Q has no isolated points and that $\{f_1, \ldots, f_n\}$ is an ϵ-net for the surface of the unit sphere in $C(Q)$. Let $\epsilon' > 0$ be given. There are points s_i with $|f_i(s_i)| \geqslant 1 - \epsilon'$ and neighbourhoods U_i of s_i with $|f_i(s_i) - f_i(t)| < \epsilon'$ for t in U_i. Using the complete regularity of Q we can construct a function g of norm one in $C(Q)$ so that for each i the scalars in $g(U_i)$ form an ϵ'-net for the set of scalars $\{a\colon |a| \leqslant 1\}$. If this holds, then for each i there is a point t_i in U_i with $|g(t_i) + f_i(s_i)| < \epsilon'$, so that

$$\begin{aligned}\|f_i - g\| \geqslant |g(t_i) - f_i(t_i)| &\geqslant 2|f_i(s_i)| - |g(t_i) + f_i(s_i)| - |f_i(s_i) - f_i(t_i)| \\ &\geqslant 2 - 4\epsilon',\end{aligned}$$

for each i. Thus ϵ must be at least 2 and so $T(C(Q)) = 2$.

https://doi.org/10.4153/CJM-1968-041-1 Published online by Cambridge University Press

The argument of the last paragraph works for $C_0(Q)$, since in that case each neighbourhood U_i is contained in a compact set and we can thus choose g to have compact support. The argument of the first paragraph is the same.

4. LEMMA. *For* $1 \leqslant p < \infty$, $T(l^p) = 2^{1/p}$.

Proof. Suppose that $\{x_1, \ldots, x_n\}$ is an ϵ-net for $S(l^p)$ with

$$x_i = \{x_i(1), x_i(2), \ldots\}.$$

Let $1 > \epsilon' > 0$ be given. There is an index N with

$$\sum_{j=N}^{\infty} |x_i(j)|^p < (\epsilon')^p \qquad \text{for } 1 \leqslant i \leqslant n.$$

The vector e_N in l^p has one in the Nth coordinate and zeros elsewhere. For this vector of norm one and some index i,

$$\epsilon^p \geqslant ||x_i - e_N||^p = \sum\nolimits_{j \neq N} |x_i(j)|^p + |1 - x_i(N)|^p \geqslant 1 - (\epsilon')^p + (1 - \epsilon')^p.$$

Since ϵ' is an arbitrary number in $(0, 1)$ we see that $\epsilon \geqslant 2^{1/p}$. Thus $T(l^p) \geqslant 2^{1/p}$.

Let e_1 be the vector in l^p which is 1 in the first coordinate and zero in the other coordinates. The set $\{e_1, -e_1\}$, for the case of real scalars, or the set $\{e_1, -e_1, ie_1, -ie_1\}$ for the case of complex scalars, is a $2^{1/p}$-net for $S(l^p)$. Hence $T(l^p) \leqslant 2^{1/p}$.

5. THEOREM. *Let X and Y be normed linear spaces and $L\colon X \rightarrow Y$ be an isomorphism of X onto Y. Then $T(Y) \leqslant ||L||\, ||L^{-1}||\, T(X)$. In particular, if X and Y are of different thickness, then they cannot be nearly isometric.*

Proof. We may assume that X and Y are infinite dimensional, for the inequality holds in the finite-dimensional case by Lemma 2. Let $\epsilon > 0$ be given. From the definition of the thickness of X there are points $x_1, \ldots, x_n$ of norm one with $\min ||x - x_i|| < T(X) + \epsilon$ for each point x in $S(X)$. Suppose that z is an element in X with $0 < ||z|| < 1$. For any number a in $(0, 1)$,

$$\begin{aligned}\min||a(z/||z||) - x_i|| &= \min||a(z/||z|| - x_i) + (1 - a)(-x_i)|| \\ &< a(T(X) + \epsilon) + (1 - a) \leqslant T(X) + \epsilon,\end{aligned}$$

since $T(X) \geqslant 1$. In particular, for $a = ||z||$, we get $\min ||z - x_i|| < T(X) + \epsilon$. Thus $\min ||x - x_i|| < T(X) + \epsilon$ for all x with $||x|| \leqslant 1$.

Let y be in $S(Y)$. For $x = L^{-1}y$, $||x|| \leqslant ||L^{-1}||$. By what we have just shown, for some x_i we have $||x/||L^{-1}|| - x_i|| < T(X) + \epsilon$. Applying L, we get $||y - ||L^{-1}||Lx_i|| < ||L||\, ||L^{-1}||\, (T(X) + \epsilon) = C$; we call the right-hand side C for convenience. Notice that $||\, ||L^{-1}||\, Lx_i|| \geqslant 1$. Thus we have found points $y_1, \ldots, y_n$ in Y with $||y_i|| \geqslant 1$, which have the property that $\min||y_i - y|| < C$ for each point y in Y with $||y|| = 1$.

Let y be of norm one in Y. For some index i, $||y - y_i|| < C$. For any scalar b in $(0, 1]$, $||y - by_i|| \leqslant b\, ||y - y_i|| + (1 - b) < C$, since $C \geqslant 1$.

Taking b to be $1/||y_i||$, $||y - y_i/||y_i|| \, || < C$. That is to say that for all y of norm one, $\min ||y - y_i/||y_i|| \, || < ||L|| \, ||L^{-1}|| \, (T(X) + \epsilon)$. Hence,

$$T(Y) \leqslant ||L|| \, ||L^{-1}|| \, T(X).$$

For example, the space m (of bounded sequences with the supremum norm) cannot be nearly isometric to a space $C(Q)$ where Q is a completely regular Hausdorff space without isolated points; in fact we have $(m, C(Q)) \geqslant \log 2$. This does not follow from the results of (**2**), since in representing m as $C(\beta N)$, where βN is the Stone-Čech compactification of the integers, the topological space βN is not first countable (**4**, Corollary 9.6, p. 132).

We now consider another estimate of the size of the unit sphere. Our idea is that the sphere in X is rather small and cramped if for any finite set $\{x_1, \ldots, x_n\}$ of points in $S(X)$ we can find another point in $S(X)$ which is close to every one of the x_i. To make this quantitative we introduce, for a normed linear space X, the set $B = \{b \geqslant 0$: for $x_1, \ldots, x_n$ a finite set in $S(X)$ and $\epsilon > 0$, there is a point x in $S(X)$ with $||x - x_i|| < b + \epsilon$ for $1 \leqslant i \leqslant n\}$. We see that the number $t(X) = \inf B$ belongs to B.

6. *Definition.* For a normed linear space X we define $t(X)$, *the thinness of* X, to be the number which is described in the paragraph above, i.e. it is the smallest non-negative number with the property that for $x_1, \ldots, x_n$ in $S(X)$ and $\epsilon > 0$, there is a point x in $S(X)$ with $\max||x - x_i|| < t(X) + \epsilon$.

7. LEMMA. *Let X be a normed linear space.*

(1) *If X is finite dimensional, then* $t(X) = 2$.

(2) *For all X,* $1 \leqslant t(X) \leqslant 2$.

(3) *For each t in* $[1, 2]$ *there is a Banach space X with* $t(X) = t$.

Proof. Suppose that X is finite dimensional and that $\{x_1, \ldots, x_n\}$ is an ϵ-net for $S(X)$. Then $\max||x - x_i|| \geqslant 2 - \epsilon$ for each x in $S(X)$ and result (1) follows since it is clear that $t(X) \leqslant 2$.

Choose any x in $S(X)$ and let y be in $S(X)$. From

$$1 = ||y|| \leqslant \tfrac{1}{2}(||y - x|| + ||y + x||) \leqslant \max(||y - x||, ||y + x||),$$

we see that we cannot have $t(X) < 1$ for any X.

We show in Lemmas 8 and 9 that $t(l^p) = 2^{1/p}$, $1 \leqslant p < \infty$, and that $t(c_0) = 1$, from which (3) follows.

8. LEMMA. *Let Q be a completely regular Hausdorff space containing an infinite number of points. Then $t(C(Q)) = 2$. Let s_0 be a point of Q which is not isolated. Then the maximal ideal* $I = \{f$ in $C(Q)$: $f(s_0) = 0\}$ *in $C(Q)$ has $t(I) = 1$. In particular, if Q is locally compact but not compact, then the space $C_0(Q)$ of continuous functions which vanish at infinity has $t(C_0(Q)) = 1$.*

Proof. Let $\epsilon > 0$ be given and let $\{a_1, \ldots, a_m\}$ be an ϵ-net for the scalars $\{a: |a| = 1\}$. We consider the set of functions $\{a_1 1, \ldots, a_m 1\}$ and see that for any f of norm one in $C(Q)$, $\max||f - a_i 1|| \geqslant 2 - \epsilon$. Hence, $t(C(Q)) = 2$.

https://doi.org/10.4153/CJM-1968-041-1 Published online by Cambridge University Press

Suppose that Q contains a point s_0 which is not isolated and let

$$I = \{f \text{ in } C(Q): f(s_0) = 0\}.$$

Let a finite set $\{f_1, \ldots, f_n\}$ of functions of norm one in I and $\epsilon > 0$ be given. There is a neighbourhood U of s_0 with $|f_i(s)| < \epsilon$ for s in U and $1 \leqslant i \leqslant n$. Since Q is completely regular, there is a g of norm one in I which vanishes on $Q - U$ and for this g, $\max||g - f_i|| < 1 + \epsilon$. Thus $t(I) = 1$.

For the sequence spaces c and c_0, $t(c) = 2$ and $t(c_0) = 1$. The only maximal ideal in c whose thinness differs from that of c is c_0.

9. LEMMA. *For* $1 \leqslant p < \infty$, $t(l^p) = 2^{1/p}$.

Proof. Let $\{x_1, \ldots, x_n\}$ be an ϵ-net for $S(l^p)$ and let $\epsilon' > 0$ be given. As in the proof of Lemma 4, $||x_i - e_N||^p \leqslant 1 + (1 + \epsilon')^p$ for large enough N, and have $t(l^p) \leqslant 2^{1/p}$.

Let e_1 be the vector which has 1 in the first coordinate and zero elsewhere and let $\{a_1, \ldots, a_m\}$ be an ϵ-net for the set $\{a: |a| = 1\}$ of scalars. We consider the set of functions $\{a_1 e_1, \ldots, a_m e_1\}$. Let x have norm one in l^p, $x = \{x(1), x(2), \ldots\}$. If $x(1) = 0$, then $||x - a_1 e_1|| = 2^{1/p}$. If $x(1) \neq 0$, let $a = x(1)/|x(1)|$. Then

$$||x + ae_1||^p = |1 + |x(1)||^p + \sum_{i=2} |x(i)|^p \geqslant 1 + |x(1)|^p + \sum_{i=2} |x(i)|^p = 2.$$

There is some a_i with $|a + a_i| < \epsilon$ and for this scalar,

$$||x - a_i e_1|| \geqslant ||x + ae_1|| - ||ae_1 + a_i e_1|| \geqslant 2^{1/p} - \epsilon.$$

So $t(l^p) \geqslant 2^{1/p}$ and, from the first paragraph, $t(l^p) = 2^{1/p}$.

10. THEOREM. *Let X and Y be normed linear spaces with $L: X \to Y$ an isomorphism of X onto Y. Then $t(Y) \leqslant ||L|| \, ||L^{-1}|| \, t(X)$. In particular, if X and Y have different thinness, then they cannot be nearly isometric.*

Proof. Let $y_1, \ldots, y_n$ be points of norm one in Y and let $\epsilon > 0$ be given. Let $x_i = L^{-1}y_i$ and note that $||x_i|| \leqslant ||L^{-1}||$. Consider the set

$$\{x_1/||x_1||, -x_1/||x_1||, \ldots, x_n/||x_n||, -x_n/||x_n||\}$$

in $S(X)$. From the definition of $t(X)$, there is a point x in $S(X)$ with

$$\max||\pm x_i/||x_i|| - x|| < t(X) + \epsilon.$$

Any real number b in $[-1, 1]$ can be written in the form $a(-1) + (1 - a)(1)$ for some a in $[0, 1]$. Then

$$||bx_i/||x_i|| - x|| \leqslant a\, ||x_i/||x_i|| + x|| + (1 - a)\, ||x_i/||x_i|| - x|| < t(X) + \epsilon.$$

For the choices $b = \pm||x_i||/||L^{-1}||$, $||\pm x_i - ||L^{-1}||\, x|| < ||L^{-1}||\, (t(X) + \epsilon)$. Then it follows that $\max||\pm y_i - y|| < ||L||\, ||L^{-1}||\, (t(X) + \epsilon) = C$, where $y = ||L^{-1}||\, Lx$ has $||y|| \geqslant 1$.

Once more we write any b in $[-1, 1]$ as $b = a(1) + (1 - a)(-1)$ for some a in $[0, 1]$. And then $||y_i - by|| \leqslant a\,||y_i - y|| + (1 - a)\,||y_i + y|| < C$. Choosing $b = 1/||y||$, $\max||y_i - y/||y||\,|| < C$. Hence $t(Y) \leqslant ||L||\,||L^{-1}||\,t(X)$ and the proof is complete.

If Q is a completely regular Hausdorff space containing an infinite number of points, then its Stone–Čech compactification βQ contains a point which is not isolated and so from Lemma 8 we see that $C(Q) = C(\beta Q)$ contains a maximal ideal I which is not nearly isometric to $C(Q)$; in fact any isomorphism L of $C(Q)$ onto I must have $||L||\,||L^{-1}|| \geqslant 2$. For an isomorphism L of c onto c_0 we obtain the same lower bound for Banach's constant (c, c_0) as was obtained in **(2)**: $(c, c_0) \geqslant \log 2$.

Added December 28, 1966. The example given after Theorem 5 will follow from D. Amir's paper, *On isomorphisms of continuous function spaces*, Israel J. Math., 3 (1965), 205–210, which extends Cambern's result to arbitrary compact spaces. In connection with Riesz's theorem, we mention an interesting proof due to A. Wilansky: If $D_1, \ldots, D_n$ are closed disks of radius less than one and $\cup D_i$ contains the surface of the unit sphere of X, then $\cup D_i$ is weakly closed and does not contain 0 and from Problem 11 on page 245 of Wilansky, *Functional Analysis* (Blaisdell, 1964), it follows that X is finite dimensional.

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