ASYMPTOTIC PROPERTIES OF KOLMOGOROV WIDTHS

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Abstract

We consider two problems concerning Kolmogorov widths of compacts in Banach spaces. The first problem is devoted to relations between the asymptotic behavior of the sequence of n-widths of a compact and of its projections onto a subspace of codimension one. The second problem is devoted to comparison of the sequence of n-widths of a compact in a Banach space $\mathcal{Y}$ and of the sequence of n-widths of the same compact in other Banach spaces containing $\mathcal{Y}$ as a subspace.

1. Introduction

Our Banach space theory terminology and notation follow [3]. We denote the closed unit ball of a Banach space $\mathcal{Y}$ by $B_\mathcal{Y}$, the unit ball of ℓ_p^n by B_p^n, and the norm closure of a set $M \subset \mathcal{Y}$ by $\overline{M}$. Let $\mathcal{Z}$ be a subset of a Banach space $\mathcal{X}$ and $x \in \mathcal{X}$. The *distance from x to $\mathcal{Z}$* is defined as $\mathrm{dist}(x, \mathcal{Z}) = \inf\{\|x - z\| : z \in \mathcal{Z}\}$.

DEFINITION 1.1. Let K be a subset of a Banach space $\mathcal{X}$, $n \in \mathbb{N} \cup \{0\}$. The *Kolmogorov n-width* of K is given by

$$d_n(K, \mathcal{X}) = \inf_{\mathcal{X}_n} \sup_{x \in K} \mathrm{dist}(x, \mathcal{X}_n),$$

where the infimum is over all n-dimensional subspaces. We use $d_n(K)$ instead of $d_n(K, \mathcal{X})$ if $\mathcal{X}$ is clear from context.

This notion was introduced by Kolmogorov [6] in 1936. It has been the subject of extensive study and has found many applications. See [7, 11, 14] for information on the Kolmogorov n-width. In [9] it was discovered that some general asymptotic properties of Kolmogorov widths are useful in the study of closures of sets of operators in the weak operator topology. The purpose of this paper is to continue the analysis of asymptotic properties of widths.

One of the results on asymptotic properties of widths proved in [9, Lemma 3.3] is the following lemma.

LEMMA 1.2. *Let K be a bounded subset in a Banach space $\mathcal{X}$. If $K_0 = K \cap L$, where L is a closed linear subspace in $\mathcal{X}$ which does not contain K, then there exists a constant $0 < C < \infty$ such that $d_n(K_0) \leq C d_{n+1}(K)$ for all $n \in \mathbb{N} \cup \{0\}$.*

In view of possible applications in the spirit of [9] it is important to find out whether one can prove a version of Lemma 1.2 for projections. In this connection we consider the following problem. *Let K be a compact in a Banach space $\mathcal{X}$ such that $\mathcal{X} = \overline{\mathrm{lin}(K)}$. Let $K_0 = P(K)$, where P is a bounded linear projection of $\mathcal{X}$ onto its subspace of codimension one. Does there exist $0 < C < \infty$ such that $d_n(K_0, \mathcal{X}) \leq C d_{n+1}(K, \mathcal{X})$ for all $n \in \mathbb{N} \cup \{0\}$?* The first purpose of this paper is to answer this problem in the negative. Our example is an infinite-dimensional ellipsoid in a Hilbert space.

DEFINITION 1.3. A set K of the form $A(B_{\mathcal{H}_0})$, where A is an infinite-dimensional bounded compact operator from a Hilbert space $\mathcal{H}_0$ to a Hilbert space $\mathcal{H}$, is called an *ellipsoid*. An ellipsoid K is called *lacunary* if

$$\liminf_{n \to \infty} (d_{n+1}(K, \mathcal{H})/d_n(K, \mathcal{H})) = 0.$$

THEOREM 1.4. *There exists a lacunary ellipsoid K with dense linear span in a Hilbert space $\mathcal{H}$ and an orthogonal projection $P : \mathcal{H} \to \mathcal{H}$ with one-dimensional kernel such that there exists $0 < C < \infty$ for which $d_n(K, \mathcal{H}) \leq C d_n(P(K), \mathcal{H})$.*

Since for a lacunary ellipsoid K there is no $0 < C < \infty$ such that $d_n(K, \mathcal{H}) \leq C d_{n+1}(K, \mathcal{H})$, Theorem 1.4 answers the above-mentioned problem in the negative. Theorem 1.4 is proved in Section 2.

Another problem considered in this paper is related to the well-known observation that there exist a Banach space $\mathcal{X}$, its closed subspace $\mathcal{Y}$, and a subset $K \subset \mathcal{Y}$ such that for some n the strict inequality

$$d_n(K, \mathcal{X}) < d_n(K, \mathcal{Y}) \tag{1.1}$$

holds. Examples of this type can be found in [1], [7, p. 446, Problem 10.3], [11, pp. 10 and 35], and [14]. Observe that the nonstrict inequality in (1.1) follows immediately from the definition and holds for all triples $K \subset \mathcal{Y} \subset \mathcal{X}$.

Let K be a compact in a Banach space $\mathcal{X}$ and let $\mathcal{Y} = \overline{\mathrm{lin}\, K}$. We consider the following problem: *how small in comparison with $d_n(K, \mathcal{Y})$ can $d_n(K, \mathcal{X})$ be?* Using Kashin's decomposition (see [4, 13]) we obtain the following result (proved in Section 3).

THEOREM 1.5. *For each n the Banach space ℓ_1^{3n} contains a $2n$-dimensional subspace $\mathcal{Y}_{2n}$ and a compact $K_n \subset \mathcal{Y}_{2n}$ such that $d_n(K_n, \ell_1^{3n}) \leq 1$, but $d_n(K_n, \mathcal{Y}_{2n}) \geq c\sqrt{n}$ for some absolute constant $c > 0$.*

Theorem 1.5 shows that the sequence $\{d_n(K, \mathcal{X})\}_n$ depends heavily on the ambient Banach space. In this connection it is natural to recall a notion introduced in [2, Section 2].

DEFINITION 1.6. Let K be a compact in a Banach space $\mathcal{Y}$ and $n \in \mathbb{N}$. The nth *absolute width* $d_n^a(K)$ of K is defined by $d_n^a(K) = \inf_{\mathcal{X}} d_n(K, \mathcal{X})$, where the inf is over all Banach spaces $\mathcal{X}$ containing $\mathcal{Y}$ as a subspace.

In Section 4 we use Theorem 1.5 to establish one asymptotic property of d_n^a.

It is worth mentioning that some aspects of the natural problem of evaluating or estimating numbers $\{d_n^a(K)\}$, given a compact K in a Banach space $\mathcal{Y}$, were considered in [5, 8].

2. Asymptotic properties of widths of quotients

PROOF OF THEOREM 1.4. Let $\{e_i\}$ be the unit vector basis in $\mathcal{H}$, $\{\beta_i\}$ be a sequence satisfying $\beta_i \geq \beta_{i+1} > 0$, and let

$$K = \left\{ \sum_{i=1}^{\infty} x_i e_i : \sum_{i=1}^{\infty} \left(\frac{x_i}{\beta_i} \right)^2 \leq 1 \right\}.$$

Let $v = \sum_{i=1}^{\infty} a_i e_i$ be a unit vector ($\|v\| = 1$) and let P be an orthogonal projection in $\mathcal{H}$ whose kernel is the linear span of v, so $Px = x - \langle x, v \rangle v$. Desired properties of sequences $\{\beta_i\}$ and $\{a_i\}$ will be described later. It is well known that $d_n(K) = \beta_{n+1}$ (see [7, p. 401]).

The well-known results on widths (see [7, Ch. 13, Section 5]) imply that in order to prove the theorem it suffices to show that for suitable $\{\beta_n\}$, $\{a_n\}$, and $c > 0$ we have $\inf\{\|Px\| : x \in \mathcal{S}_n\} \geq c\beta_n$, where

$$\mathcal{S}_n = \left\{ \sum_{i=1}^{\infty} x_i e_i : \sum_{i=1}^{n} \left(\frac{x_i}{\beta_i} \right)^2 = 1 \text{ and } 0 = x_{n+1} = x_{n+2} = \cdots \right\}.$$

Since $\beta_{n+1} = d_n(K)$ and we are looking for a lacunary ellipsoid, we require that $\{\beta_n\}$ satisfies $\liminf_{n \to \infty} (\beta_{n+1}/\beta_n) = 0$.

To simplify the computation, we prove that $\inf\{\|Px\| : x \in \mathcal{R}_n\} \geq c\beta_n$ for

$$\mathcal{R}_n = \left\{ \sum_{i=1}^{\infty} x_i e_i : \frac{\sum_{i=1}^{n-1} x_i^2}{\beta_{n-1}^2} + \left(\frac{x_n}{\beta_n} \right)^2 = 1 \text{ and } 0 = x_{n+1} = x_{n+2} = \cdots \right\}.$$

Let $\{x_i\}_{i=1}^{\infty} \in \mathcal{R}_n$. Denote by $v(n)$ the orthogonal projection of v onto the subspace E_n spanned by the first n vectors of $\{e_i\}$. Then

$$\|Px\|^2 = \|x\|^2 - \langle x, v \rangle^2.$$

Let $x = w + ze_n \in \mathcal{R}_n$, where w is the orthogonal projection of x to E_{n-1}. Then $\|x\|^2 = \|w\|^2 + z^2$. Also

$$\left(\frac{\|w\|}{\beta_{n-1}}\right)^2 + \left(\frac{z}{\beta_n}\right)^2 = 1 \tag{2.1}$$

and $\langle x, v \rangle = \langle w, v(n-1) \rangle + za_n$. Therefore

$$\begin{aligned}
\|Px\|^2 &= \|w\|^2 + z^2 - (\langle w, v(n-1) \rangle + za_n)^2 \\
&\geq \|w\|^2 + z^2 - (\|w\|\|v(n-1)\| + za_n)^2 \\
&= \|w\|^2(1 - \|v(n-1)\|^2) + z^2(1 - a_n^2) - 2\|w\|\|v(n-1)\|za_n.
\end{aligned}$$

We denote the first two summands in the last line by B^2 and C^2, respectively. We show that for a suitable choice of the sequence $\{a_n\}$ there is a constant $0 < d < 1$ which does not depend on n and is such that

$$2\|w\|\|v(n-1)\|za_n \leq 2dBC. \tag{2.2}$$

Assume that we have shown (2.2). Then we get

$$\|Px\|^2 \geq B^2 + C^2 - 2dBC \geq (1 - d^2)\max\{B^2, C^2\}.$$

Observe that (2.1) implies that either $\|w\|^2 \geq \beta_{n-1}^2/2$ or $z^2 \geq \beta_n^2/2$.

In the former case we get

$$\|Px\|^2 \geq (1 - d^2)\|w\|^2(1 - \|v(n-1)\|^2) \geq \frac{1-d^2}{2}\beta_{n-1}^2(1 - \|v(n-1)\|^2),$$

and we are done if we assume that the sequences $\{\beta_n\}$ and $\{a_n\}$ are selected in such a way that

$$\beta_{n-1}^2(1 - \|v(n-1)\|^2) \geq \beta_n^2. \tag{2.3}$$

In the latter case we have

$$\|Px\|^2 \geq (1 - d^2)z^2(1 - a_n^2) \geq \frac{1-d^2}{2}\beta_n^2(1 - a_n^2) \geq \frac{1-d^2}{4}\beta_n^2,$$

if we assume that $a_n^2 \leq \frac{1}{2}$ for all n.

It remains to establish (2.2). Analysis of this inequality shows that we need to prove the inequality

$$\|v(n-1)\|a_n \leq d\sqrt{(1 - \|v(n-1)\|^2)(1 - a_n^2)},$$

but this follows from the obvious inequality $\|v(n-1)\|^2 \leq (1 - a_n^2)$ and the observation that we can select $\{a_n\}$ in such a way that

$$a_n^2 \leq d^2(1 - \|v(n-1)\|^2).$$

This can be done because $1 - \|v(n-1)\|^2 = a_n^2 + a_{n+1}^2 + \cdots$, and we may select the sequence $\{a_n\}$ in such a way that, for example, $a_{n+1}^2 + \cdots \geq a_n^2$.

For clarity we describe the way in which $\{a_i\}$ and $\{b_i\}$ can be chosen. First we choose $\{a_i\}$ in such a way that $\sum_n a_n^2 = 1$, $a_n^2 \leq a_{n+1}^2 + a_{n+2}^2 + \cdots$, and $a_n^2 \leq \frac{1}{2}$ for all n; one of the possible choices is $a_n = (\sqrt{2})^{-n}$.

Now we choose $\{\beta_i\}$ satisfying $\beta_i \geq \beta_{i+1} > 0$, $\liminf_{n \to \infty}(\beta_{n+1}/\beta_n) = 0$, and (2.3). It is clear that such choice is possible and that with these choices all steps of our argument work. $\square$

3. Dependence of widths on the ambient Banach space

PROOF OF THEOREM 1.5. It is well known and is easy to see that the unit ball B_1^{3n} of ℓ_1^{3n} contains $(1/\sqrt{3n})B_2^{3n}$, where B_2^{3n} is the unit ball of ℓ_2^{3n}. When we use the term 'orthogonal projection' we mean a projection orthogonal with respect to the inner product corresponding to ℓ_2^{3n}.

The celebrated result of Kashin [4] (see also [13] and [12, Ch. 6]) states that there is an absolute constant A and a $(2n)$-dimensional subspace $\mathcal{Y}_{2n}$ of ℓ_1^{3n} such that $\mathcal{Y}_{2n} \cap B_1^{3n} \subset (A/\sqrt{3n})B_2^{3n}$, so that $\mathcal{Y}_{2n}$ is A-isomorphic to a Hilbert space.

We let P_n be an orthogonal projection onto $\mathcal{Y}_{2n}$ and $K_n = P_n(B_1^{3n})$. Since the kernel of P_n is n-dimensional, it follows that $d_n(K_n, \ell_1^{3n}) \leq 1$.

It remains to show that $d_n(K_n, \mathcal{Y}_{2n}) \geq c\sqrt{n}$. Since the norm of $\mathcal{Y}_{2n}$ is A-equivalent to the $\sqrt{3n}$-multiple of the norm of ℓ_2^{3n} restricted to $\mathcal{Y}_{2n}$, it suffices to show the inequality $d_n(K_n, \mathcal{Y}_{2n}) \geq c_1$ with respect to the norm of ℓ_2^{3n} on $\mathcal{Y}_{2n}$.

Let E_n be any n-dimensional subspace of $\mathcal{Y}_{2n}$. Let $Q : \mathcal{Y}_{2n} \to E_n^\perp$ be an orthogonal projection onto the orthogonal complement of E_n in $\mathcal{Y}_{2n}$. It suffices to show that the diameter of $Q(K_n)$ with respect to the norm of ℓ_2^{3n} is greater than or equal to c_1.

Let $\{e_j\}_{j=1}^{3n}$ be the unit vector basis of ℓ_1^{3n}. We need to show that $\|QP_ne_j\|_{\ell_2^{3n}} \geq c_1$ for some $j \in \{1, \ldots, 3n\}$.

Let $\{f_i\}_{i=1}^n$ be an orthonormal basis of $E_n^\perp$. Then

$$f_i = \sum_{j=1}^{3n} \langle f_i, e_j \rangle e_j.$$

Therefore $\sum_{j=1}^{3n} \langle f_i, e_j \rangle^2 = 1$ and

$$\sum_{i=1}^n \sum_{j=1}^{3n} \langle f_i, e_j \rangle^2 = n.$$

This implies that there exists $j \in \{1, 2, \ldots, 3n\}$ such that

$$\sum_{i=1}^n \langle f_i, e_j \rangle^2 \geq \frac{1}{3},$$

and this inequality means that $\|QP_ne_j\| \geq 1/\sqrt{3}$. $\square$

REMARK 3.1. Later we shall need a slight generalization of the statement proved above: the set K_n contains $3n$ vectors $\{P_n e_i\}_{i=1}^{3n}$ such that for each orthogonal projection Q with rank at least αn ($0 < \alpha < 1$) on $\mathcal{Y}_{2n}$ endowed with the norm whose unit ball is $(1/\sqrt{3n})B_2^{3n} \cap \mathcal{Y}_{2n}$ there exists $j \in \{1, \ldots, 3n\}$ such that $\|QP_n e_j\| \geq \sqrt{\alpha n}$.

4. Absolute widths

We start by mentioning the fact observed in [2, Section 2] that there exists a wide class of compacts K for which their widths in $\overline{\text{lin}(K)}$ is the same as their absolute widths. Examples can be constructed in the following way. Let $\mathcal{Y}$ be an arbitrary infinite-dimensional Banach space. Let $\{\mathcal{Z}_n\}_{n=1}^{\infty}$ be a family of subspaces of $\mathcal{Y}$ satisfying $\dim \mathcal{Z}_n = n$ and $\mathcal{Z}_n \subset \mathcal{Z}_{n+1}$, let B_n be the unit ball of $\mathcal{Z}_n$ and let $\{t_n\}$ be a decreasing sequence of positive numbers with $\lim_{n \to \infty} t_n = 0$. Consider the compact

$$K = \overline{\text{conv}\left(\bigcup_{n=1}^{\infty} t_n B_n\right)}. \tag{4.1}$$

Then $d_n(K, \mathcal{X}) = t_{n+1}$ for each $n \in \mathbb{N}$ and each Banach space $\mathcal{X}$ containing $\mathcal{Y}$ as a subspace. The estimate from above follows from $K \subset \mathcal{Z}_n + t_{n+1}B_{\mathcal{X}}$. The estimate from below is an immediate consequence of [7, Theorem 5.1, p. 419].

The proof of Theorem 1.5 can be used to construct an infinite-dimensional compact K in a Hilbert space $\mathcal{H}$ for which $\liminf_{n \to \infty} d_n^a(K)/d_n(K, \mathcal{H}) = 0$.

See [3, p. 5] for definitions of direct sums used below.

THEOREM 4.1. *There exist a compact $K \subset \mathcal{H}$ and an isometric embedding I of $\mathcal{H}$ into a Banach space $\mathcal{X}$ isomorphic to $\mathcal{L} = (\sum_{n=1}^{\infty} \ell_1^{3n})_2$, such that*

$$\liminf_{n \to \infty}(d_n(IK, \mathcal{X})/d_n(K, \mathcal{H})) = 0.$$

PROOF. We let $\{n_i\}_{i=1}^{\infty}$ be an increasing sequence of positive integers satisfying

$$\sum_{i=1}^{k}(2n_i) \leq \frac{n_{k+1}}{2}. \tag{4.2}$$

We represent $\mathcal{H}$ as $(\sum_{i=1}^{\infty} \ell_2^{2n_i})_2$. We define $E_i : \ell_2^{2n_i} \to \ell_1^{3n_i}$ as a linear embedding which maps $B_2^{2n_i}$ onto the intersection $(1/\sqrt{3n_i})B_2^{3n_i} \cap \mathcal{Y}_{2n_i}$, where $\mathcal{Y}_{2n_i}$ are the same as in Theorem 1.5. Combining these embeddings we get an embedding E of $\mathcal{H}$ into $\mathcal{L}$. This embedding is not isometric but it satisfies $\|x\|/A \leq \|Ex\| \leq \|x\|$ for the absolute constant A introduced in the proof of Theorem 1.5.

The well-known argument [10, Proposition 1] implies that to complete the proof of the theorem it suffices to find a compact $K \subset \mathcal{H}$ such that

$$\liminf_{n \to \infty}(d_n(EK, \mathcal{L})/d_n(K, \mathcal{H})) = 0.$$

We let $K_{n_i} \subset \mathcal{Y}_{2n_i}$ be the compacts $P_{n_i}(B_1^{3n_i})$ defined in Theorem 1.5 and $H_i = E_i^{-1}K_{n_i}$ be their pre-images considered as compacts in $\ell_2^{2n_i}$. The desired compact K will be found in the form $K = \overline{\text{conv}(\bigcup_{i=1}^{\infty} \alpha_i H_i)}$, where H_i is considered as a subset of $\mathcal{H}$ and $\{\alpha_i\}$ is a converging to zero sequence of positive real numbers satisfying

$$\alpha_n \geq \sum_{i=n+1}^{\infty} (\alpha_i \, \text{diam}(H_i)).$$

We let $p = p_k = \sum_{i=1}^{k}(2n_i) + n_{k+1}$. To complete the proof of the theorem it suffices to show that:

(1) $d_p(EK, \mathcal{L}) \leq 2\alpha_{k+1}$;
(2) $d_p(K, \mathcal{H}) \geq \frac{1}{\sqrt{2}}\sqrt{n_{k+1}}\alpha_{k+1}$.

To prove (1) we consider the p-dimensional space $\mathcal{L}_p = \sum_{n=1}^{k} \mathcal{Y}_{2n_i} + \ker P_{n_{k+1}}$ in $\mathcal{L}$. Then

$$K \subset \mathcal{L}_p + \alpha_{k+1}B_1^{3n_{k+1}} + \overline{\text{conv}\left(\bigcup_{i=k+2}^{\infty} \alpha_i K_{n_i}\right)}.$$

Therefore

$$d_p(K, \mathcal{L}) \leq \alpha_{k+1} + \sum_{i=k+2}^{\infty} (\alpha_i \, \text{diam}(H_i)) \leq 2\alpha_{k+1}.$$

To prove (2) assume the contrary. Then for some k there is a p_k-dimensional subspace $\mathcal{A}_p \subset \mathcal{H}$ such that $K \subset \mathcal{A}_p + c\sqrt{n_{k+1}}\alpha_{k+1}B_{\mathcal{H}}$, where $c < \frac{1}{\sqrt{2}}$. Therefore

$$\alpha_{k+1}H_{k+1} \subset \mathcal{A}_p + c\sqrt{n_{k+1}}\alpha_{k+1}B_{\mathcal{H}}$$

and

$$\alpha_{k+1}H_{k+1} \subset \mathcal{Z}_p + c\sqrt{n_{k+1}}\alpha_{k+1}B_{\mathcal{H}},$$

where $\mathcal{Z}_p$ is a p-dimensional subspace in $\ell_2^{2n_{k+1}}$.

Let $Q : \ell_2^{2n_{k+1}} \to \mathcal{Z}_p^{\perp}$ be an orthogonal projection onto the orthogonal complement of $\mathcal{Z}_p$ in $\ell_2^{2n_{k+1}}$. The inequality (4.2) implies that $\dim \mathcal{Z}_p^{\perp} = 2n_{k+1} - p \geq \frac{1}{2}n_{k+1}$. By Remark 3.1 there is a vector $z \in \alpha_{k+1}H_{k+1}$ such that $\|Qz\| \geq \frac{1}{\sqrt{2}}\alpha_{k+1}\sqrt{n_{k+1}}$. We arrive at a contradiction. $\square$

In connection with the examples presented in this section we suggest the following problems.

PROBLEM 4.2. Does there exist an infinite-dimensional compact K such that

$$\lim_{n \to \infty} d_n^a(K)/d_n(K, \overline{\text{lin}(K)}) = 0?$$

PROBLEM 4.3. Characterize compacts K for which the absolute widths do not differ much from the widths of K in $\overline{\text{lin}(K)}$.

Acknowledgement

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