Appelons O le second foyer de la conique, et transformons par polaires réciproques en prenant le point O pour centre de la transformation. Nous obtenons ainsi ce théorème :—

Soient f l'axe radical d'un point O et d'un cercle C, et P le pôle de cet axe relativement au cercle C. Si les tangentes menées d'un point quelconque de f au cercle C coupent l'une des tangentes à ce cercle parallèles à f aux points I et I', et que les droites PI et PI' coupent la droite f aux points H et H', l'angle HOH' est droit.

Note on the Kinematics of a Quadrilateral.

By R. J. DALLAS.

I send a note on the following problem, a solution of which was requested of me by one of the tutors at King's College, Cambridge.

We are given a quadrilateral of four jointed bars ABCD (fig. 83). The bar CD being held fast, find the tangent to the locus of P, the intersection of DA, CB in any position ; and verify the following construction for the radius of curvature of the path of P :—

Let PQ be the third diagonal, draw through P a perpendicular to PQ meeting BA, CD in L and L'; through L and L' draw parallels to PQ meeting AD in M and M'; through M and M' draw perpendiculars to AD meeting the normal at P in O and O'; then will

$$-1/\rho = 1/\mathrm{OP} + 2/\mathrm{O'P}.$$

The first part of this is easily found. The important angles in the figure have been marked thus—

$$\mathrm{CPQ} = \alpha,\ \ \mathrm{DPQ} = \beta,\ \ \mathrm{AQP} = \gamma,\ \ \mathrm{DQP} = \epsilon.$$

Making PD rock through a small angle $\delta\theta$, we have, if δs is the resulting element of arc traced out by P,

$$\delta s = \mathrm{PD}\,\delta\theta \operatorname{cosec}\mathrm{TPD}, \text{ and so also, if } \phi = \angle\,\mathrm{PCD},$$

$$\delta s = \mathrm{PC}\,\delta\phi \operatorname{cosec}\mathrm{TPC}.$$

Now $\delta\phi/\delta\theta = \mathrm{QD}/\mathrm{QC}$ as is well known, (see Goodeve's *Elements of Mechanism* p. 110), and thus $\dfrac{\sin\mathrm{TPC}}{\sin\mathrm{TPD}} = \dfrac{\mathrm{QD}}{\mathrm{PD}}\dfrac{\mathrm{PC}}{\mathrm{QC}}$.

Now $$\frac{\mathrm{QD}}{\mathrm{PD}} = \frac{\sin\mathrm{DPQ}}{\sin\mathrm{PQD}} \text{ and } \frac{\mathrm{PC}}{\mathrm{QC}} = \frac{\sin\mathrm{PQC}}{\sin\mathrm{QPC}},$$

therefore $$\frac{\sin\mathrm{TPC}}{\sin\mathrm{TPD}} = \frac{\sin\beta}{\sin\alpha} \text{ and } \mathrm{TPC} - \mathrm{TPD} = \beta - \alpha\,;$$

therefore $$\mathrm{TPC} = \beta \text{ and } \mathrm{TPD} = \alpha.$$

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Thus TP makes an angle with DP equal to the angle QPC.

The construction in the figure is equivalent to asserting that

$$1/\rho = \sin\alpha\sin\beta(2\cot\epsilon + \cot\gamma)/PQ.$$

Let PT make an angle ψ with CD, then $\psi = \pi - \theta - \alpha$ and thus $\delta\psi = -(\delta\theta + \delta\alpha)$.

Also $\delta s = PD\delta\theta\mathrm{cosec}\alpha$, sensibly. We therefore must find $\delta\alpha$.

We have $\phi = \epsilon + \alpha$; therefore $\delta\alpha = \delta\phi - d\epsilon$.

To find $\delta\epsilon$ we displace PQ twice, first the end P into its new position* and then the end Q. Let δDQ be the increment of DQ.

Then $\delta\epsilon = (\delta s\sin(\alpha+\beta) - \delta DQ\sin\epsilon)/QP$.

To find δDQ, fix AD instead of CD, and rock DQ through an angle $\delta\theta$, then the tangent to the path of Q makes as before an angle equal to PQA with DQ (as shown by dotted line) and $\delta DQ = DQ\delta\theta\cot\gamma$. Thus

$$\delta\epsilon = \frac{\delta\theta}{PQ}\left[\frac{PD\sin(\alpha+\beta)}{\sin\alpha} - \frac{DQ\sin\epsilon}{\tan\gamma}\right]$$

and thus $\qquad \delta\psi = -(\delta\theta + \delta\alpha) = -(\delta\theta + \delta\phi - \delta\epsilon).$

From this, dividing by ds, we get

$$\frac{1}{\rho} = \frac{\sin\alpha}{PD}\left[1 + \frac{DQ}{CQ} + \frac{DQ}{PQ} - \frac{DP}{PQ}\frac{\sin(\alpha+\beta)}{\sin\alpha}\right]$$

$$= \frac{\sin\alpha\sin\beta}{PQ}\left[\frac{PQ}{PD\sin\beta} + \frac{DQ.PQ}{CQ.PD\sin\beta} + \frac{DQ}{DP}\frac{\cot\gamma\sin\epsilon}{\sin\beta} - \frac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta}\right]$$

$$= \frac{\sin\alpha\sin\beta}{PQ}\left[\frac{\sin\theta}{\sin\epsilon\sin\beta} + \frac{\sin\phi}{\sin\epsilon\sin\alpha} + \cot\gamma - \frac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta}\right]$$

$$= \frac{\sin\alpha\sin\beta}{PQ}\,\frac{\sin\theta\sin\alpha\sin\gamma + \sin\phi\sin\beta\sin\gamma + \cos\gamma\sin\alpha\sin\beta\sin\epsilon - \sin(\alpha+\beta)\sin\gamma\sin\epsilon}{\sin\alpha\sin\beta\sin\gamma\sin\epsilon}$$

Substituting for θ and ϕ in terms of the other angles, namely, $\theta = \beta + \epsilon$, $\phi = \alpha + \epsilon$, we readily get the result

$$-1/\rho = \sin\alpha\sin\beta(2\cot\epsilon + \cot\gamma)/PQ,$$

which agrees with the result already given.

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* Evidently for this purpose we may treat the element of arc as straight.