Canad. Math. Bull. Vol. **32** (1), 1989

BIREFLECTIONALITY IN ABSOLUTE GEOMETRY

BY
DRAGOSLAV LJUBIĆ

ABSTRACT. If G is any group then $g \in G$ is called an involution if $g \neq 1$ and $g \circ g = 1$. A group G is called bireflectional if every element in G is a product of two involutions. It is known that 2- dimensional, 3-dimensional, and some types of n-dimensional ($n > 3$) absolute geometries (in the sense of H. Kinder) are bireflectional. In this article the author proves the general result that every n-dimensional absolute geometry is bireflectional.

1. **Introduction.** If G is any group, then $g \in G$ is called an *involution* if $g \neq 1$ and $g \circ g = 1$. A group G is called *bireflectional* if every element in G is a product of two involutions. For example, symmetries in k-planes of the n-dimensional real euclidean space are involutions and the group of isometries is bireflectional ([3], p. 3). Bireflectionality is thoroughly explored in [2], and in particular it is proved that the isometry group of 2-dimensional absolute geometry is bireflectional. J. Ahrens ([1], p. 165) extended this result to dimension 3. The main goal of the present note is to prove that the same result holds in the n-dimensional case for all $n \geqq 2$. The axiomatic system used is adopted from [10] (see also [2], §20.9).

In order to put our work in perspective, we first list some closely related results. Starting with an isometry group (given axiomatically) one defines ideal points, lines, ..., orthogonality, and imbeds the resulting incidence-orthogonality structure into a projective metric space of the corresponding dimension ([2], [1], [10]). Via coordinization of the projective metric space the original isometry group becomes a subgroup of projective orthogonal group $PO_{n+1}(k, f)$ ([2]. §20. 8 – 10). The coordinate field k is a commutative of characteristic $\neq 2$ and the symmetric bilinear form f is of rank $n+1$ and index 0 or 1, or of rank n and index 1. Unfortunately, only in the elliptic case (f or rank $n+1$ and index 0) does one obtain the whole group PO_{n+1}, so a nice result of M. J. Wonenburger does not solve our problem completely. She proved in [13] that a linear transformation A of a finite dimensional vector space over a commutative field of characteristic $\neq 2$ is a product of two involutions if and only if A is invertible and similar to its inverse. In particular, she proved that if $A \in O_{n+1}(k,f)$, where f is of

Received by the editors February 2, 1987 and, in revised form, February 9, 1988.
AMS Subject Classification (1980): 51F10, 20G15.

rank $n+1$, then involutions of the decomposition can be chosen from $O_{n+1}(k,f)$; hence the orthogonal group $O_{n+1}(k,f)$ is bireflectional. This trivially implies bireflectionality of the elliptic isometry group. Surprisingly, the last fact seems to have been as yet unnoticed.

E. W. Ellers and W. Nolte in [6] extended the results of M. J. Wonenburger for fields of characteristic 2 (for simplectic groups also). E. W. Ellers generalized the concept of bireflectionality to unitary groups ([5]).

In section 2 we give the definitions and results that we use in the proof of the main theorem. We begin with the axiomatic system of H. Kinder. This nice theory includes the classical Euclidean, hyperbolic, and elliptic geometry. In section 3 we prove the main theorem and in section 4 we post a problem concerning one important implication of it.

2. Preliminaries. We start with a brief survey of the notation, definitions and results we are going to use later. In the definitions and results quoted we follow [10].

Let $\mathfrak{G}$ be a group generated by an invariant set $\mathfrak{S}$ of involutions. If for some involutions $x, y \in \mathfrak{G}\, x \circ y$ is an involution, we denote that by $x|y$. Also $x_{11}, \ldots, x_{1n_1}|x_{21}, \ldots, x_{2n_2}| \ldots |x_{k1}, \ldots x_{kn_k}$ abbreviates: $x_{ip}|x_{jq}$ whenever $i < j$. Elements of $\mathfrak{G}$ are called *hyperplane symmetries* and are denoted by lower case Greek letters. Let n be a fixed positive integer. Products $\alpha_1 \circ \ldots \circ \alpha_{n-k}$, such that $\alpha_1| \ldots |\alpha_{n-k}, 0 \leqq k < n$, are called *k-plane symmetries* (or just *plane symmetries* if k is obvious or not relevant) and are denoted by lower case indexed Greek letters α^k, β^k,.... For $k = 0, 1, 2$ the corresponding products are also called *point, line*, and *plane symmetries*, respectively. Point symmetries are also denoted by upper case Roman letters. We call a pair $(\mathfrak{G}, \mathfrak{S})$ (or just $\mathfrak{G}$) the *n-dimensional isometry group*, its elements *isometries*, and the theory thus obtained *n-dimensional absolute geometry*, if $n \geqq 2$ and the following axioms are satisfied: Axiom 1_n^*. Given $\alpha_1, \ldots, \alpha_{n-1}, A$ there is some α such that $\alpha|\alpha_1, \ldots, \alpha_{n-1}, A$. Axiom 1_n. Given $\alpha_1| \ldots |\alpha_{n-2}|A, B$ there is some α such that $\alpha|\alpha_1, \ldots . \alpha_{n-2}, A, B$. Axiom 2_n. If $\alpha_1| \ldots |\alpha_{n-2}|\alpha, \beta|A, B$, then $\alpha = \beta$ or $A = B$. Axiom 3_n. If $\alpha_1| \ldots |\alpha_{n-2}, A|\alpha, \beta, \gamma$ and $\alpha_{n-2} \neq A$, then $\alpha \circ \beta \circ \gamma \in \mathfrak{S}$. Axiom 4_n. If $\alpha_1| \ldots |\alpha_{n-1}|\alpha, \beta, \gamma$, then $\alpha \circ \beta \circ \gamma \in \mathfrak{S}$. Axiom X_n. There are $\alpha_1 \ldots, \alpha_n$ such that $\alpha_1| \ldots |\alpha_n$. Axiom D_n. Given $\alpha_1| \ldots |\alpha_n$ there is some α such that $\alpha \nmid \alpha_n, \alpha \neq \alpha_n$ and $\alpha|\alpha_1| \ldots |\alpha_{n-1}$.

An isometry group $\mathfrak{G}$ is called *elliptic* if it satisfies Axiom P_n. There are $\alpha_1, \ldots., \alpha_{n+1}$ such that $\alpha_1| \ldots |\alpha_{n+1}$.

In this paragraph we describe an incidence-orthogonality structure, called *(n-dimensional) group space*, which naturally corresponds to every isometry group. The *hyperplane* $\boldsymbol{\alpha}$ corresponding to a hyperplane symmetry α is defined as $\boldsymbol{\alpha} = \{A : A|\alpha\}$. If $\alpha^k = \alpha_1 \circ \ldots \circ \alpha_{n-k}$, with $\alpha_1| \ldots |\alpha_{n-k}$, then the *k-plane* $\boldsymbol{\alpha}^k, 0 \leqq k < n$, is defined as $\boldsymbol{\alpha}_1 \cap \boldsymbol{\alpha}_2 \cap \ldots \cap \boldsymbol{\alpha}_{n-k}$. 0, 1, 2-planes are also called *points, lines*, and *planes*, respectively and the generic term will be *plane*. Because no confusion is likely to arise we shall denote $\boldsymbol{\alpha}^k$ by α^k. As in an "ordinary geometry" we say that planes α^i and β^j

span the plane $(\alpha^i, \beta^j) = \cap\{x : x \supset \alpha^i, \beta^j, x \text{ a plane}\}$. If there is no such a plane (α^i, β^j) will denote (the set of all points of) the group space of $\mathfrak{G}$. Planes α^i and β^j are called *orthogonal* (symbolically $\alpha^1 \perp \beta^j$) if $\alpha^i = \gamma_1 \cap \ldots \cap \gamma_{n-k} \cap \delta_1 \cap \ldots \cap \delta_{k-i}$ and $\beta^j = \gamma_1 \cap \ldots \cap \gamma_{n-k} \cap \epsilon_1 \cap \ldots \cap \epsilon_{k-j}$ for some γ's, δ's and ϵ's such that $i, j < k \leqq i + j, k \leqq n$ and $\gamma_1 | \ldots | \gamma_{n-k} | \delta_1 | \ldots \delta_{k-i} | \epsilon_1 | \ldots | \epsilon_{k-j}$. The main feature of orthogonal planes we are going to use is that the corresponding plane symmetries commute. If $\alpha^k \cap \beta^l \ni O$, and $l \neq 0$, we shall denote by $(\alpha^k, (\beta^l)^{\perp})$ the span of α^k and γ^{n-1}, where $\gamma^{n-l} \cap \beta^l = O$ and $\gamma^{n-l} \perp \beta^l$.

Inner automorphisms $X \mapsto YXY^{-1} \overset{def}{=} X^Y$ of $\mathfrak{G}$ induce bijections of the group space onto itself, which preserve incidence and orthogonality. The group $\mathfrak{G}^*$ obtained in this way is called *the group of isometries of the group space*. Its elements (*isometries*) inherit the names of the corresponding elements of $\mathfrak{G}$. Moreover we shall use the same notation for them in this paper. Groups $\mathfrak{G}$ and $\mathfrak{G}^*$ are isomorphic and group-theoretic statements about $\mathfrak{G}$ can be equivalently expressed in the (more "picturesque") language of the group space and its group of isometries.

We saw that in the elliptic case a point $A (= \alpha_2 \cap \ldots \cap \alpha_{n+1})$ and a hyperplane α_1 can generate the same isometry. If this is the case, we say that A is *conjugate* to every point $B \in \alpha_1$. A unique hyperplane α_1 corresponds to the given point A (so on every line $\lambda^1 \ni A$ there is exactly one point B conjugate to A) and points of α_1 are conjugate to A. Every line through A and some point of a plane $\alpha^k \subset \alpha_1$ is orthogonal to α^k. Conversely, if two different lines through A are orthogonal to some plane α^k, then $\alpha^k \subset \alpha_1$.

3. The main theorem. In this section we shall prove the main theorem (Theorem 3) using a few auxiliary propositions. Theorem 1 and Lemmas 1 and 2 will be established first; they make possible the induction in Lemma 4 and Theorem 3. Theorems 1 and 2 give a "construction" of invariant points and lines needed for the applications of Lemmas 1 and 2.

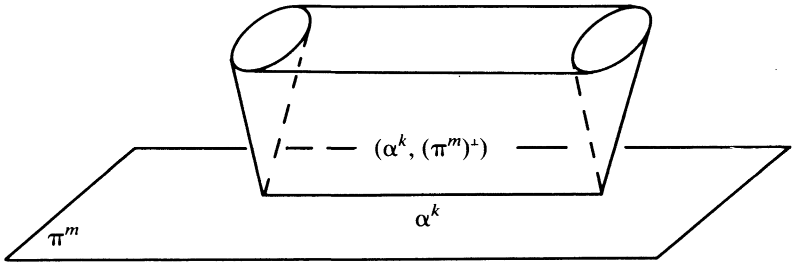
Let $\pi^m, 2 \leqq m < n$, be a fixed plane of the group space of $\mathfrak{G}$. The hyperplane symmetries $\alpha, \alpha \perp \pi^m$, generate a group which acts on (the set of all points of) π^m. We shall denote by $\mathfrak{G}^*/\pi^m$ the corresponding group of automorphisms of π^m. (Here we think of α as an element of $\mathfrak{G}^*$.) We easily check that $\mathfrak{G}^*/\pi^m$ is an m-dimensional isometry group with hyperplane symmetries $\alpha/\pi^m, \alpha \perp \pi^m$. The basic property which gives the proof is

(i) $\alpha/\pi^m | \beta/\pi^m$ iff $\alpha | \beta \, (\alpha, \beta \perp \pi^m)$.

The property (i) holds since $\alpha/\pi^m | \beta/\pi^m$ iff $\alpha/\pi^m = \alpha^\beta/\pi^m \neq \beta/\pi^m$ iff $\alpha/\pi^m = \alpha'/\pi^m \neq \beta/\pi^m$, where $\alpha' = \alpha^\beta$. Comparing fixed points of α, α' and β in π^m we conclude that $\alpha/\pi^m = \alpha'/\pi^m \neq \beta/\pi^m$iff $\alpha = \alpha' \neq \beta$, i.e., iff $\alpha | \beta$.

A straightforward consequence of (i) is

(ii) k-plane symmetries of $\mathfrak{G}^*/\pi^m$are $\alpha^{n-m+k}/\pi^m = \alpha^k \circ \pi^m/\pi^m = \alpha^k/\pi^m$, where $\alpha^k \subsetneqq \pi^m$ and $\alpha^{n-m+k} = (\alpha^k, (\pi^m)^{\perp})$.



Therefore, in analogy to the proof of (i) we obtain

(iii) $\alpha/\pi^m|A/\pi^m$ iff $\alpha|A$.

Finally, the plane π^m is an intersection $\alpha_{-(n-m-1)}\cap\ldots\cap\alpha_0$ for some $\alpha_{-(n-m-1)}|\ldots|\alpha_0$, and hence

(iv) $\alpha\perp\pi^m$ iff $\alpha|\alpha_{-(n-m-1)}|\ldots|\alpha_0$.

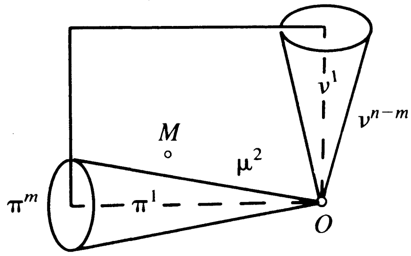
Properties (i), (ii), (iii), and (iv) imply

THEOREM 1. *$\mathfrak{G}^*/\pi^m$ is an m-dimensional isometry group with k-plane symmetries (ii). Incidence-orthogonality relations in the group space of $\mathfrak{G}^*/\pi^m$ are inherited from the group space of $\mathfrak{G}$.*

From now on we shall usually write α^k instead of α^k/π^m when $\alpha^k\subsetneq\pi^m$ and the group $\mathfrak{G}^*/\pi^m$ is considered.

LEMMA 1. *Let O be a point in a plane $\pi^m, 2\leqq m<n$. Suppose further that planes $\alpha_j^{s_j}, j=1,\ldots k$, contain O, p of them are contained in π^m and the rest contain $\nu^{n-m}=(O,(\pi^m)^\perp)$. If $f=\alpha_1^{s_1}\circ\ldots\circ\alpha_k^{s_k}$ and $f/\pi^m=1$, then $f=1$ for even p and $f=\pi^m$ for odd p.*

PROOF. We may assume that $\alpha_1^{s_1},\ldots,\alpha_p^{s_p}\subset\pi^m$ and $\alpha_j^{s_j}\supset\nu^{n-m}$ for $j>p$. If we denote $\alpha_j^{t_j}=(\alpha_j^{s_j},\nu^{n-m}), j\leqq p$, and $g=\alpha_1^{t_1}\circ\ldots\circ\alpha_p^{t_p}\circ\alpha_{p+1}^{s_{p+1}}\circ\ldots\circ\alpha_k^{s_k}$, then $g=\alpha_1^{s_1}\circ\pi^m\circ\ldots\circ\alpha_p^{s_p}\circ\pi^m\circ\alpha_{p+1}^{s_{p+1}}\circ\ldots\circ\alpha_k^{s_k}=f\circ(\pi^m)^p$ and $g/\pi^m=f/\pi^m=1$. Hence we are done if we show that $g=1$.



First, note that g fixes π^m and ν^{n-m} pointwise. Next, given a point $M\notin\pi^m\cup\nu^{n-m}$, g

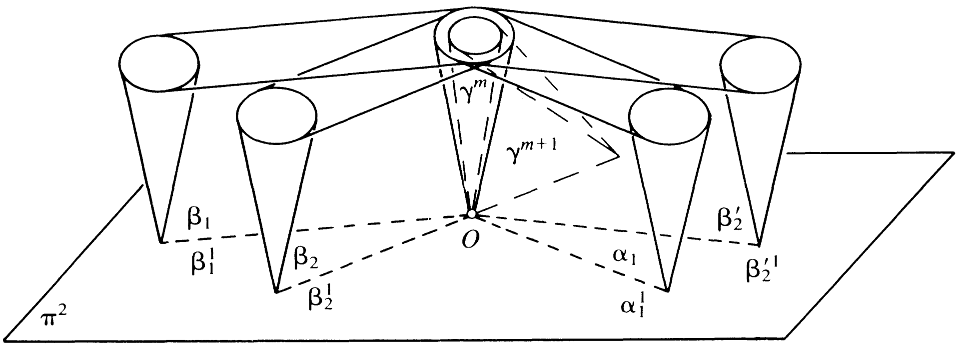
fixes planes (M, π^m) and (M, ν^{n-m}) and so their intersection μ^2. In the plane μ^2 lines $\nu^1 = (M, \pi^m) \cap \nu^{n-m}$ and $\pi^1 = (M, \nu^{n-m}) \cap \pi^m$ are fixed pointwise, therefore lines orthogonal to them are fixed. This implies that all points of μ^2 having different perpendiculars onto $\nu^\perp$ and $\pi^\perp$ are fixed. Hence, every line of μ^2 is fixed (because it contains two such points), which fixes points of μ^2 as their intersections. □

The implications of Lemma 1 are going to be very extensive but easily recognizable; therefore we shall use it without any reference.

LEMMA 2. *If $O^f = O$ for some $f \in \mathfrak{G}$ and every point O of an l-plane $\gamma^l, 0 \leqq l \leqq n-1$, then $f = \alpha_1 \circ \ldots \circ \alpha_k$ where $\alpha_i \supset \gamma^l$.*

PROOF. Let $l = 0$. Theorem 8.4 of [7] implies that given O, α, β we can find γ, δ such that $\alpha \circ \beta = \gamma \circ \delta$ and $O \in \gamma$. Hence, given $f = \beta_1 \circ \ldots \circ \beta_m$ we can replace inductively $\beta_1, \ldots, \beta_{m-1}, \beta_m$ with some $\alpha_1, \ldots, \alpha_{m-1}, \beta'_m$ such that $f = \alpha_1 \circ \ldots \circ \alpha_{m-1} \circ \beta'_m$ and $\alpha_i \ni O$. Finally, $f \circ O = O \circ f$ and $\alpha_i \ni O$ imply $\alpha_{m-1} \circ \ldots \circ \alpha_1 \circ f = \beta'_m \ni O$ or $\beta'_m = O$. If $\beta'_m \ni O$ we are done. If $\beta'_m = O$ we replace β'_m with a product $\alpha_m \circ \ldots \circ \alpha_{m+n-1} = O$ for some $\alpha_m | \ldots \alpha_{m+n-1} | O$.

Inductively, we assume that Lemma 2 holds for all l with $0 \leqq l \leqq m < n-1$. Let $l = m+1$ and $\gamma^m \subset \gamma^{m+1}$. Our assumption implies $f = \beta_1 \circ \ldots \circ \beta_k$ for some $\beta_1, \ldots, \beta_k \supset \gamma^m$. Let π^2 be the plane orthogonal to β_1 and β_2 at $O \in \gamma^m$. Since $\gamma^m \subset (O, (\pi^2)^\perp)$, there is a hyperplane α_1 containing γ^{m+1} and orthogonal to π^2. Denote $\beta_1^1 = \beta_1 \cap \pi^2, \beta_2^1 = \beta_2 \cap \pi^2$ and $\alpha_1^1 = \alpha_1 \cap \pi^2$.



In the plane π^2 there exists a line $\beta_2'^1 \ni O$ such that $\beta_1^1 \circ \beta_2^1 = \alpha_1^1 \circ \beta_2'^1$ (Axiom 3_2). Therefore (by Lemma 1) $\beta_1 \circ \beta_2 = \alpha_1 \circ \beta'_2$, where $\beta'_2 \supset \beta_2'^1$ and $\beta'_2 \perp \pi^2$ (so $\beta'_2 \supset \gamma^m$). Hence, given $f = \beta_1 \circ \ldots \circ \beta_k$ with $\beta_i \supset \gamma^m$, we can replace inductively $\beta_1, \ldots, \beta_k$ with $\alpha_1, \ldots, \alpha_k$ such that $f = \alpha_1 \circ \ldots \circ \alpha_k$ and $\alpha_1, \ldots, \alpha_{k-1} \supset \gamma^{m+1}$. Finally, $\alpha_k (= \alpha_{k-1} \circ \ldots \circ \alpha_1 \circ f)$ fixes γ^{m+1} pointwise, which implies $\alpha_k \supset \gamma^{m+1}$. □

We say that M is a *midpoint* for points $A \neq B$ if $A^M = B$ and $M \in (A, B)$. If

$A = B$ we define A to be the midpoint for A and B. (In the elliptic case $A^M = A$ does not imply $M = A$.) Not every two points have a midpoint! In the elliptic case two different points have 0 or 2 midpoints, i.e., if M is a midpoint for A and B, then its conjugate point also is. More generally, if $A = B \in \sigma^k$, or $A \neq B$ and $A^{\sigma^k} = B$, we say that σ^k is a *plane of symmetry* for A and B. It is clear that once there is a midpoint M for $A \neq B$ then every plane $\sigma^k \ni M, \sigma^k \perp (A,B)$ is a plane of symmetry for them. Conversely, every plane of symmetry for $A \neq B$ is at their midpoint orthogonal to (A,B). We now prove two easy generalizations of a theorem of Hjelmslev (Theorem 3.28 of [2]).

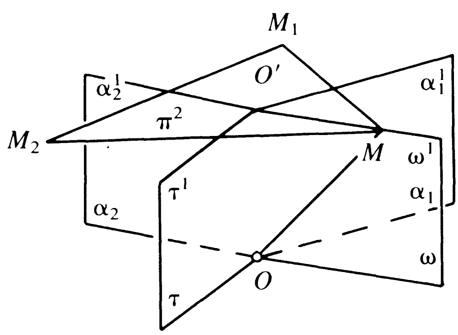
THEOREM 2. *If $A = B^f$ for some points A, B and an isometry $f \in \mathfrak{G}$, then A and B have a midpoint.*

PROOF. It is sufficient to show that A and B, where $A \neq B$, have a hyperplane of symmetry. The isometry f is a product $\alpha_k \circ \ldots \circ \alpha_1$ of hyperplane symmetries. We obtain a sequence of points $B, B_1 = B^{\alpha_1}, \ldots, B_k = B_{k-1}^{\alpha_k} = A$ and we are done (by induction) if we prove that there is a hyperplane τ for which $B^\tau = B^{\alpha_2 \circ \alpha_1} = B_2$ (i.e. $B^{\tau \circ \alpha_2 \circ \alpha_1} = B$).

In a plane π^2 containing B, B_1 and B_2 we have $B^{\alpha_1^1} = B_1$ and $B_1^{\alpha_2^1} = B_2$. where $\alpha^1 - i = \alpha_i \cap \pi^2$. By Theorem 4.1 of [2], $B^{\tau^1} = B_2$ for some $\tau^1 \subset \pi^2$. Hence $B^\tau = B_2$, where $\tau = (\tau^1, (\pi^2)^\perp)$. □

THEOREM 3. *If $O^f = O$ and $(\mu^1)^f = \lambda^1$ for some isometry f, lines λ^1, μ^1 and a point $O \in \lambda^1$, then there is a hyperplane $\sigma \ni O$ such that $(\lambda^1)^\sigma = \mu^1$. Moreover, σ can be chosen so that $f \circ \sigma$ fixes λ^1 pointwise.*

PROOF. By Lemma 2, $f = \alpha_k \circ \ldots \circ \alpha_1$ for some $\alpha_i \ni O$. As in Theorem 2, we are done if we prove that there is a hyperplane $\tau \ni O$ such that $(\alpha_2 \circ \alpha_1)/\mu^1 = \tau/\mu^1$ (i.e. such that $\tau \circ \alpha_2 \circ \alpha_1$ fixes μ^1 pointwise). Let $M \neq O$ be a point of μ^1 not conjugate to $O, M_1 = M^{\alpha_1}, M_2 = M_1^{\alpha_2}$ and π^2 some plane containing M, M_1 and M_2. Denote $\alpha_i^1 = \alpha_i \cap \pi^2, i = 1, 2$. If O' is the orthogonal projection of O onto π^2, then $O' \in \alpha_1^1 \cap \alpha_2^1$ since O is not conjugate to $M \in \pi^2$) and there are lines $\omega^1 \ni O'$, M and τ^1 such that $\tau^1 \circ \alpha_2^1 \circ \alpha_1^1 = \omega^1$ (again by Theorem 4.1 of [2]). Therefore, if $\tau = (\tau^1, (\pi^2)^\perp)$ and $\omega = (\omega^1, (\pi^2)^\perp)$, then $\tau \circ \alpha_2 \circ \alpha_1 = \omega$ fixes μ^1 pointwise. □



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In the proof of Lemma 4 we shall need the following result.

LEMMA 3. *If planes α^k and $\beta^l, k, l \neq 0, k+l \geqq n$, contain some point O and isometry $f = \alpha^k \circ \beta^l$ fixes some line $\lambda^1 \ni O$ pointwise, then α^k and β^l have a common line.*

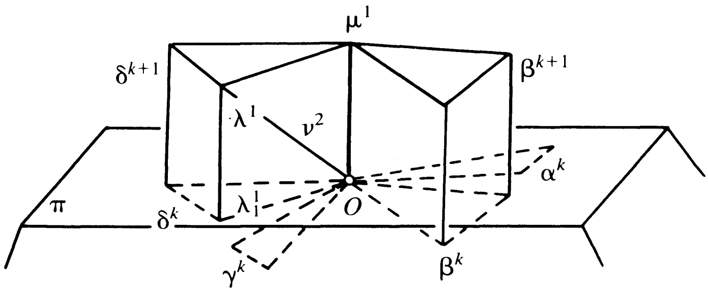
PROOF. Let A be some point of λ^1 not conjugate to O. Planes α^k and β^l belong to some hyperplanes of symmetry for A and $B = A^{\alpha^k} = A^{\beta^l}$. If $A = B$, then $\lambda^1 \subset \alpha^k, \beta^l$. If $A \neq B$, then α^k and β^l are in the same hyperplane σ of symmetry (for A and B) because otherwise O would be conjugate to A. In this case α^k and β^l have a common line in σ. □

We prove Lemma 4 and Theorem 4 simultaneously by induction. Lemma 4 holds for $n = m+1$ provided Lemma 4 and Theorem 4 hold for $n \leqq m$. Hence, the inductive hypothesis of Theorem 4 ($n \leqq m$) enables us to apply Lemma 4 for $n = m+1$.

LEMMA 4. *Suppose $f = \alpha^k \circ \beta^l$ for some isometry $f \in \mathfrak{G}$ and planes α^k, β^l such that $\alpha^k, \beta^l \ni O, l-k \in \{0,1\}$ and $l+k = n$. Given a line $\lambda^1 \ni O$ there exist planes γ^k, δ^l such that $f = \gamma^k \circ \delta^l, \gamma^k \ni O$ and $\delta^l \supset \lambda^1$.*

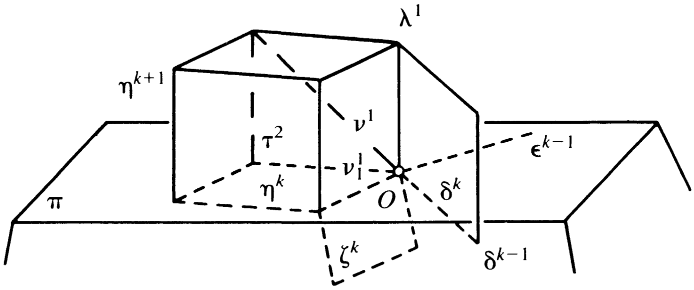
PROOF. For $n = 2$ the statement is proved in [2]. At the inductive step we assume that Lemma 4 and Theorem 4 are true for $2 \leqq n \leqq m$.

(i) Suppose $n = m+1$ is odd, hence $l = k+1$ and $n = 2k+1$. Let π be a hyperplane containing α^k and orthogonal to β^{k+1}, μ^1 the line orthogonal to π at O, and ν^2 a plane containing λ^1 and μ^1. Denote $\beta^k = \beta^{k+1} \cap \pi, \lambda_1^1 = \nu^2 \cap \pi$. The isometry f induces the isometry $f/\pi = \alpha^k \circ \beta^k$ of π, and by induction $f/\pi = \gamma^k \circ \delta^k$ for some $\gamma^k \ni O$ and $\delta^k \supset \lambda_1^1$. So $f = \gamma^k \circ \delta^{k+1}$, where $\delta^{k+1} = (\delta^k, \mu^1)$ and (therefore) $\delta^{k+1} \supset \lambda^1$.



(ii) Assume now $n = m+1$ is even, hence $l = k$ and $n = 2k$. Applying Theorem 3, we obtain a hyperplane $\sigma \ni O$ such that $f \circ \sigma$ fixes λ^1 pointwise. By Lemma 2, $f \circ \sigma$ can be represented as $\alpha_1 \circ \ldots \circ \alpha_j, \alpha_i \supset \lambda^1$. Therefore $f \circ \sigma$ induces the isometry $(f \circ \sigma)/\pi = \alpha_1^{n-2} \circ \ldots \circ \alpha_j^{n-2}$ of $\pi = (O, (\lambda^1)^\perp)$, where $\alpha_i^{n-2} = \alpha_i \cap \pi$. By induction $(f \circ \sigma)/\pi = \epsilon^s \circ \zeta^t = f \circ \sigma$ for some $\epsilon^s, \zeta^t \subset \pi$ such that $\epsilon^s, \zeta^t \ni O$ and $t-s, 2k-1-(t+s) \in \{0,1\}$. The remaining part of the proof depends on $t+s$.

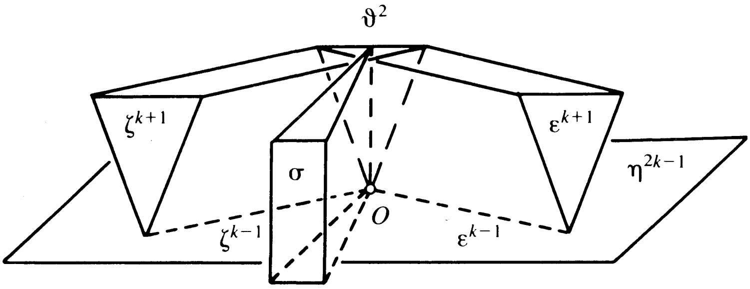
(a) First, suppose $s = k-1$ and $t = k$. Denote by ν^1 the line orthogonal to σ at O, by τ^2 a plane containing λ^1 and ν^1, by ν_1^1 the line $\tau^2 \cap \pi$.



By induction, $\epsilon^{k-1} \circ \zeta^k$ can be replaced by $\delta^{k-1} \circ \eta^k$, where $\delta^{k-1} \ni O$ and $\eta^k \supset \nu_1^1$. Hence, for $\eta^{k+1} = (\eta^k, \lambda^1) \perp \sigma, \delta^k = (\delta^{k-1}, \lambda^1)$ and $\vartheta^k = \eta^{k+1} \cap \sigma$, we have $f = \delta^{k-1} \circ \eta^k \circ \sigma = \delta^k \circ \eta^{k+1} \circ \sigma = \delta^k \circ \vartheta^K$. Finally, $f = \delta^k \circ \vartheta^k = (\vartheta^k)^{\delta^k} \circ \delta^k = \gamma^k \circ \delta^k$, and we are done.

(b) Suppose now that $s = t = k - 1$. We first show that f fixes pointwise some line through O.

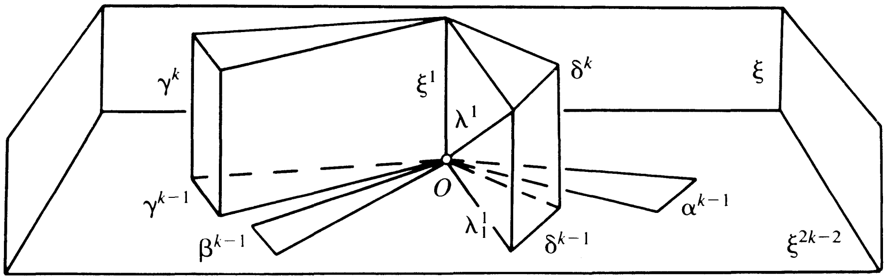
If η^{2k-2} is a plane containing ϵ^{k-1} and $\zeta^{k-1}, \vartheta^2 = (O, (\eta^{2k-2})^{\perp}), \epsilon^{k+1} = (\epsilon^{k-1}, \vartheta^2)$ and $\zeta^{k+1} = (\zeta^{k-1}, \vartheta^2)$, then $f \circ \sigma = \epsilon^{k-1} \circ \zeta^{k-1} = \epsilon^{k-1} \circ \eta^{k-2} \circ \eta^{k-2} \circ \zeta^{k-1} = \epsilon^{k+1} \circ \eta^{k-1}$. Hence, $f \circ \sigma$ fixes ϑ^2 pointwise which implies that f fixes pointwise a line through O in $\vartheta^2 \cap \sigma$.



Therefore, using Lemma 3, we can conclude that α^k and β^k have a common line $\nu^1 \ni O$.

Let ν be the hyperplane orthogonal to ν^1 at O. Then $f/\nu = \alpha^{k-1} \circ \beta^{k-1} = f$, where $\alpha^{k-1} = \alpha^k \cap \nu$ and $\beta^{k-1} = \beta^k \cap \nu$. Let ξ^{2k-2} be a plane containing α^{k-1} and β^{k-1}, ξ a hyperplane containing ξ^{2k-2} and λ^1, ξ^1 the line in ξ orthogonal to ξ^{2k-2} at O. As usual, we construct a line λ_1^1 which belongs to ξ^{2k-2} and to a 2-plane through λ^1 and ξ^1.

Now $f/\xi^{2k-2} = \alpha^{k-1} \circ \beta^{k-1} = f/\nu = f$ and by induction $f/\xi^{2k-2} = \gamma^{k-1} \circ \delta^{k-1} = f$ for some $\gamma^{k-1}, \delta^{k-1} \subset \xi^{2k-2}, \gamma^{k-1}, \ni O$ and $\delta^{k-1} \supset \lambda_1^1$. Finally, $\gamma^k = (\gamma^{k-1}, \xi^1)$ and $\delta^k = (\delta^{k-1}, \xi^1)$ are the planes we need. □



COROLLARY 1. *Suppose* $f = \alpha^k \circ \beta^l$ *for some isometry* $f \in \mathfrak{G}$ *and planes* α^k, β^l *such that* $\alpha^k, \beta^l \ni O, l-k \in \{0,1\}$ *and* $l+k = n-1$. *Given a line* $\lambda^1 \ni O$ *there are planes* $\gamma^{k+1}, \delta^{l+1}$ *such that* $f = \gamma^{k+1} \circ \delta^{l+1}, \gamma^{k+1} \ni O$ *and* $\delta^{l+1} \supset \lambda^1$.

PROOF. . Let λ^1_1 be the intersection of a hyperplane π containing α^k and β^l with a 2-plane which is othogonal to π and contains λ^1. Applying Lemma 4 for $f/\pi = \alpha^k \circ \beta^l = f$ and the line λ^1_1 we obtain $f/\pi = \gamma^k \circ \delta^l = f$, where $\gamma^k \ni O$ and $\delta^l \supset \lambda^1_1$. So $\gamma^{k+1} = (\gamma^k, \pi^\perp)$ and $\delta^{l+1} = (\delta^l, \pi^\perp)$ are the planes we need. $\square$

THEOREM 4. *Every isometry* f *of an n-dimensional isometry group* $\mathfrak{G}$ *is a product of two plane symmetries* α^k *and* β^l, *where* $l-k, n-(l+k) \in \{0,1\}$. *If* $O^f = O$ *for some* $O \in \mathfrak{G}$, *then* α^k *and* β^l *can be chosen so that* $\alpha^k, \beta^l \ni O$.

PROOF. For $n = 2$ the statement is proved in [2]. At the inductive step we assume that Theorem 4 holds for $2 \leqq n \leqq m$. Then Lemma 4 holds for $2 \leqq n \leqq m+1$. Let $n = m+1$.

(i) Suppose f fixes some point O and a line $\lambda^1 \ni O$ pointwise. Then f induces the isometry f/π of the hyperplane π orthogonal to λ^1 at O and, as in Lemma 4, $f/\pi = \alpha^k \circ \beta^l = f$ for some $\alpha^k, \beta^l \subset \pi, \alpha^k, \beta^l \ni O$ and $l-k, m-(l+k) \in \{0,1\}$. If $k+l = m$ we are finished. If $k+l = m-1$, then $f = \alpha^k \circ \beta^l = \alpha^{k+1} \circ \beta^{l+1}$, where planes $\alpha^{k+1} = (\alpha^k, \lambda^1)$ and $\beta^{l+1} = (\beta^l, \lambda^1)$ satisfy given conditions.

(ii) Suppose now f has an invariant point O. By Theorem 3 there is a hyperplane $\sigma \ni O$ such that $f \circ \sigma$ fixes some line $\lambda^1 \ni O$ pointwise. Using (i) we get $f \circ \sigma = \gamma^k \circ \delta^l$ for some $\gamma^k, \delta^l \ni O$ and $l-k, m+1-(l+k) \in \{0,1\}$. Denote by ν^1 the line through O orthogonal to σ.

If $l+k = m+1$, then by Lemma 4 $f \circ \sigma = \epsilon^k \circ \alpha^l$ for some $\epsilon^k, \alpha^l \ni O$ and $\alpha^l \supset \nu^1$ (i.e. $\alpha^l \perp \sigma$). Hence, for $\alpha^{l-1} = \alpha^l \cap \sigma$ and $\beta^k = (\epsilon^k)^{\alpha^{l-1}}$ we obtain $f = \epsilon^k \circ \alpha^l \circ \sigma = \epsilon^k \circ \alpha^{l-1} = \alpha^{l-1} \circ \beta^k$ which is just what we wanted.

If $l+k = m$, then by Corollary 1 $f \circ \sigma = \epsilon^{k+1} \circ \alpha^{l+1}$ for some $\epsilon^{k+1}, \alpha^{l+1} \ni O$ and $\alpha^{l+1} \perp \sigma$. Again, if $\alpha^l = \alpha^{l+1} \cap \sigma$ and $\beta^{k+1} = (\epsilon^{k+1})^{\alpha^l}$, we are finished by $f = \epsilon^{k+1} \circ \alpha^{l+1} \circ \sigma = \epsilon^{k+1} \circ \alpha^l = \alpha^l \circ \beta^{k+1}$.

(iii) Suppose now that O is an arbitrary point. Theorem 1 provides us with a hyperplane σ of symmetry for O and $f^{-1}(O)$. i.e. such that $f \circ \sigma$ fixes O. Therefore, as we proved in (ii), $f \circ \sigma = \gamma^k \circ \delta^l$ for some $\gamma^k, \delta^l \ni O$ and $l+k, m+1-(l+k) \in \{0,1\}$. Denote by ν^1 a line through O orthogonal to σ. Using the same procedure as in (ii) we are done by applying Lemma 4 or Corollary 1. □

4. Final remarks. Starting with the results of this paper we derive in [11] so-called normal forms for elements of $\mathfrak{G}$ (see [9]) provided $\mathfrak{G}$ satisfies an additional

Axiom N_n. *Every two $\lfloor \frac{n-1}{2} \rfloor$-dimensional planes of an n-dimensional isometry group, $n > 2$, which span a $(2\lfloor \frac{n-1}{2} \rfloor + 1)$-plane have a common perpendicular line. Every two $\lfloor \frac{n}{2} \rfloor$-dimensional planes which span a $2\lfloor \frac{n}{2} \rfloor$-plane have a common perpendicular 2-plane intersecting them along lines.*

A natural question arises

PROBLEM. What are the fields corresponding to $\mathfrak{G}$ for which Axiom N_n holds?

In [12] we extend the results of this paper and the paper [11] to infinite-dimensional absolute geometry (see [8]).

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University of Washington
Seattle, Washington