J. Austral. Math. Soc. **23** (Series A) (1977), 348–353.

CONSTRUCTION OF BALANCED INCOMPLETE BLOCK DESIGNS

ELIZABETH J. MORGAN

(Received 24 June 1976)

Communicated by W. D. Wallis

Abstract

Given a symmetric balanced incomplete block design or a finite plane, we recursively construct balanced incomplete block designs by taking unions of certain blocks and points of the original design to be the blocks of the new design.

For definitions of *balanced incomplete block design* (BIBD), *symmetric balanced incomplete block design* (SBIBD), finite *projective*, *affine* and *inversive* planes, see Dembowski (1968) and Hall (1967). The usual notation for the parameters of these designs will be used:

BIBD — (v, b, r, k, λ); SBIBD — (v, k, λ);

projective plane of order q — $(q^2+q+1, q+1, 1)$;

affine plane of order q — $(q^2, q(q+1), q+1, q, 1)$;

inverse plane of order q — $(q^2+1, q(q^2+1), q(q+1), q+1, q+1)$.

In what follows we use certain unions of blocks and elements of SBIBDs and geometrical designs to construct new blocks which themselves form BIBDs. Since the method of proof is essentially the same in many cases, not all proofs are given.

We remark that although in some of the BIBDs constructed the greatest common divisor of b, r and λ is greater than 1, in fact the BIBDs do not have repeated blocks.

THEOREM 1. *If there exists a projective plane of order q then there exists a balanced incomplete block design with parameters*

$$\left(q^2+q+1, \binom{q+1}{n}(q^2+q+1), \binom{q+1}{n}(nq+1), nq+1, \binom{q}{n-1}(nq+1)\right),$$

for all n such that $2 \leqq n \leqq q$.

PROOF. The elements of the new design are the q^2+q+1 points of the plane. The blocks for the new design are constructed as follows. Let P be any point, and take the union of n lines through P as one of the new blocks. Do this for all possible distinct sets of n lines through P, and for all points P. Then the new design has block size $n(q+1)-(n-1)=nq+1$, and since there are $q+1$ lines through a point and q^2+q+1 points altogether, there are $\binom{q+1}{n}(q^2+q+1)$ blocks formed in this way.

Any one point P belongs to $\binom{q+1}{n}+(q^2+q)\binom{q}{n-1}$ new blocks, so the new design has replication number $\binom{q+1}{n}(nq+1)$.

Any pair P, Q of points occurs in $\binom{q}{n-1}$ new blocks formed from unions of lines through P, and in $\binom{q}{n-1}$ new blocks formed from unions of lines through Q. And if R is a point collinear with P and Q, then $\binom{q}{n-1}$ of the new blocks formed from the union of n lines through R contain P and Q; there are $q-1$ such points R, not equal to P or Q. Finally there are q^2 points S which are distinct from P and Q and not collinear with them, and $\binom{q-1}{n-2}$ of the new blocks formed from the union of n lines through S contain both P and Q. So altogether the elements P and Q occur together in λ new blocks, where

$$\lambda=\binom{q}{n-1}+\binom{q}{n-1}+\binom{q}{n-1}(q-1)+\binom{q-1}{n-2}q^2$$

$$=\binom{q}{n-1}(nq+1). \qquad \square$$

The same construction will work for an affine plane of order q; each new block is formed by taking the union of n lines through some point P, where $2\leqq n\leqq q$. So we have the following result.

THEOREM 2. *If there exists an affine plane of order q, then there exists a balanced incomplete block design with parameters*

$$\left(q^2,\binom{q+1}{n}q^2,\binom{q+1}{n}(nq-n+1),nq-n+1,\binom{q}{n-1}(nq-n+1)\right),$$

for all n such that $2\leqq n\leqq q$. $\square$

https://doi.org/10.1017/S1446788700018978 Published online by Cambridge University Press

In an inversive plane the blocks are called circles, and any three points determine a unique circle. (So a finite inversive plane is a 3-design.) A *bundle* of circles is the set of all circles through two distinct points P and Q, and will be denoted by $[P, Q]$; the points P and Q are the *carriers* of the bundle. A *pencil* is any maximal set of mutually tangent circles at a point P, and P is the carrier of the pencil. A *flock* is a set of mutually disjoint circles such that, with the exception of precisely two points P and Q, every point of the plane belongs to a unique circle of the set. The points P and Q are the carriers, and the flock will be denoted by $\langle P, Q\rangle$. It is known that, in a plane of order q, each bundle consists of $q+1$ circles, each pencil consists of q circles, and each flock, if it exists, consists of $q-1$ circles. (Dembowski (1964, 1968).)

THEOREM 3. *Suppose there exists an inversive plane of order q. Then there exist balanced incomplete block designs with parameters*

(i) $\left(q^2+1, \frac{q^2(q^2+1)}{2}\binom{q+1}{n}, \frac{q^2(nq-n+2)}{2}\binom{q+1}{n}, nq-n+2, \binom{nq-n+2}{2}\binom{q+1}{n}\right)$ *for all n such that* $2 \leqq n \leqq q$;

(ii) $\left(q^2+1, (q^2+1)(q+1)\binom{q}{n}, (q+1)(nq+1)\binom{q}{n}, nq+1, (q+1)(nq+1)\binom{q-1}{n-1}\right)$ *for all n such that* $2 \leqq n \leqq q$;

(iii) $\left(q^2+1, \frac{q^2(q^2+1)(q-1)}{2}, \frac{q^2(q-1)(q+3)}{2}, q+3, \frac{(q-1)(q+2)(q+3)}{2}\right)$ *if q is even.*

PROOF. In each case the elements of the BIBD are the q^2+1 points of the inversive plane.

For case (i), a block is formed by taking the union of n circles chosen from the $q+1$ circles in a bundle $[P, Q]$. Doing this for all sets of n circles chosen from the bundle, and then for all bundles as P and Q vary, gives all the blocks.

For case (ii), a block is formed by taking the union of n circles chosen from the q in a pencil with carrier P. Doing this for all sets of n circles chosen from the pencil and for all pencils as P varies produces all the blocks.

For case (iii), we need q to be even to ensure that a flock exists for every pair of points P and Q (see Dembowski (1964)). To form the blocks of the BIBD, for each flock $\langle P, Q\rangle$ adjoin both P and Q to each one of the $q-1$ circles in the flock, thus producing $q-1$ blocks of size $q+3$ from each flock.

It is straightforward to verify that the resulting designs are BIBDs; we shall merely check that the designs (i) and (iii) are balanced. (Case (ii) is very similar to case (i).)

CASE (i): Consider an arbitrary pair of points P and Q. They occur together in $\binom{q+1}{n}$ new blocks formed from the bundle $[P, Q]$, in $\binom{q}{n-1}$ new blocks formed from bundle $[P, R]$ and in $\binom{q}{n-1}$ new blocks formed from bundle $[Q, R]$, where R is any point not equal to P or Q; there are $q^2 - 1$ such points R.

Now consider bundles $[R, S]$ where P, Q, R and S are distinct points. We have two cases to consider. Firstly, suppose that all four of these points lie on one circle. Since there are $q + 1$ circles through both P and Q and since for each of these circles there are $\binom{q-1}{2}$ pairs of points not equal to P or Q, such points R and S may be chosen in $(q+1)\binom{q-1}{2}$ ways. The remaining $n - 1$ circles through R and S may be chosen in $\binom{q}{n-1}$ ways. So there are $(q+1)\binom{q-1}{2}\binom{q}{n-1}$ new blocks from bundles $[R, S]$ which contain P and Q, where P, Q, R and S lie on one circle. Secondly, consider the case with R, S and P on one circle and R, S and Q on a different circle. There are $q^2 - 1$ circles through P that do not pass through Q; choose one, and then choose points R and S on this circle, with R and S not equal to P. As a second circle through R and S, take the unique circle through R, S and Q, and then choose $n - 2$ more circles through R and S from a possible $q - 1$ circles. This gives $(q^2 - 1)\binom{q}{2}\binom{q-1}{n-2}$ new blocks containing P and Q from bundles $[R, S]$ with P, Q, R and S not on one circle.

Therefore

$$\begin{aligned}\lambda &= \binom{q+1}{n} + 2\binom{q}{n-1}(q^2-1) + (q+1)\binom{q-1}{2}\binom{q}{n-1} \\ &\quad + (q^2-1)\binom{q}{2}\binom{q-1}{n-2} \\ &= \binom{q+1}{n}\binom{nq-n+2}{2}\end{aligned}$$

after simplification.

CASE (iii): Let P and Q be an arbitrary pair of points. There are $q-1$ new blocks containing P and Q formed from the flock $\langle P, Q\rangle$. From flock $\langle P, R\rangle$ there is one new block containing P and Q, and also one from flock $\langle Q, R\rangle$ for any point R not equal to P or Q; there are q^2-1 such points R. Now suppose neither P nor Q is a carrier. We shall use the fact that in an inversive plane of even order, given any two distinct circles there is a unique carrier pair containing these two circles in the flock (see Dembowski (1964)). We can choose one circle containing both P and Q in $q+1$ ways, and then choose any circle disjoint from this in $q(q-1)(q-2)/2$ ways (see Dembowski (1964)). Then the unique flock containing these two circles consists of $q-1$ circles altogether, and $q-2$ of them do not contain P or Q. So there are $\{(q+1)q(q-1)(q-2)/2\}/(q-2)$ new blocks arising in this way which contain both P and Q. So P and Q appear together in λ blocks where

$$\lambda = (q-1) + 2(q^2-1) + \frac{q(q^2-1)}{2}$$

$$= \frac{(q-1)(q+2)(q+3)}{2}. \qquad \square$$

NOTE. If we take the strong union (retaining repetitions) of $n-1$ circles chosen from each bundle $[P, Q]$, where $3 \leqq n \leqq q+1$, then the resulting design is a balanced n-ary design (see Morgan (1977)) with parameters

$$V = q^2+1,\ B = \frac{q^2(q^2+1)}{2}\binom{q+1}{n-1};\ r_1 = \frac{q^2(q^2-1)}{2}\binom{q}{n-2},$$

$$r_2 = \cdots = r_{n-2} = 0,\ r_{n-1} = q^2\binom{q+1}{n-1},\ R = \frac{q^2(q+1)(n-1)}{2}\binom{q+1}{n-1};$$

$$K = (q+1)(n-1),\ \Lambda = \binom{q+1}{n-1}\frac{(n-1)}{2}\{(n-1)(q+1)^2 - 2n - q + 3\}.$$

Also, if instead of one copy of the two carrier points P and Q of a flock $\langle P, Q\rangle$ being adjoined to each circle of the flock, $n-1$ copies of P and Q are adjoined (for each flock), then, for $n \geqq 3$, the resulting design is a balanced n-ary design with parameters

$$V = q^2+1,\ B = \frac{q^2(q^2+1)(q-1)}{2};\ r_1 = \frac{q^2(q^2-1)}{2},\ r_2 = \cdots = r_{n-2} = 0,$$

$$r_{n-1} = q^2(q-1),\ R = \frac{q^2(q-1)}{2}(q+2n-1);\ K = q+2n-1,$$

$$\Lambda = \tfrac{1}{2}(q-1)(q^2+4nq-3q+2n^2-2).$$

Theorem 1 with $n = 2$ will generalise from a projective plane to an arbitrary SBIBD.

THEOREM 4. *If there exists a symmetric balanced incomplete block design with parameters* (v, k, λ) *then there exists a balanced incomplete block design with parameters*

$$\left(v, \binom{v}{2}, \frac{(2k-\lambda)(v-1)}{2}, 2k-\lambda, \binom{2k-\lambda}{2}\right).$$

PROOF. Elements of the new design are the same as those of the SBIBD. Each block of the new design is the union of a pair of blocks of the original design, with the λ elements common to the pair counted once only, and all $\binom{v}{2}$ possible pairs are used. Since any element appears in k blocks of the SBIBD and not in $v-k$ blocks, each element appears in $k(v-k)+k(k-1)/2$ new blocks; since $\lambda(v-1) = k(k-1)$ this simplifies to the replication number $(2k-\lambda)(v-1)/2$.

Finally, consider any two elements x and y:

x and y occur together in λ blocks of the SBIBD,

x but not y occurs in $k-\lambda$ blocks of the SBIBD,

y but not x occurs in $k-\lambda$ blocks of the SBIBD,

and there are $v-2k+\lambda$ blocks not containing x or y.

Consequently, the number of new blocks containing both x and y is

$$\frac{\lambda(\lambda-1)}{2} + 2\lambda(k-\lambda) + \lambda(v-2k+\lambda) + (k-\lambda)^2,$$

which simplifies to $(2k-\lambda)(2k-\lambda-1)/2$. □

References

P. Dembowski (1964), 'Möbiusebenen gerader Ordnung', *Math. Annalen* **157**, 179–205.

P. Dembowski (1968), *Finite Geometries* (Springer-Verlag, New York).

M. Hall (1967), *Combinatorial Theory* (Blaisdell, Waltham, Mass.).

Elizabeth J. Morgan (1977), 'Construction of balanced n-ary designs'. *Utilitas Mathematica*, **11**, 3–31.

Department of Mathematics,
University of Queensland,
St. Lucia, Q 4067,
Australia.