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On Birational Maps and Jacobian Matrices

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Abstract. One is concerned with Cremona-like transformations, i.e., rational maps from $\mathbb{P}^n$ to $\mathbb{P}^m$ that are birational onto the image $Y \subset \mathbb{P}^m$ and, moreover, the inverse map from Y to $\mathbb{P}^n$ lifts to $\mathbb{P}^m$. We establish a handy criterion of birationality in terms of certain syzygies and ranks of appropriate matrices and, moreover, give an effective method to explicitly obtaining the inverse map. A handful of classes of Cremona and Cremona-like transformations follow as applications.

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Introduction

Let $F: \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ be a rational map and let $Y \subset \mathbb{P}^m$ be its image. We consider the question as when F admits an inverse rational map G defined on Y by the restrictions of forms of the same degree on the ambient $\mathbb{P}^m$. Our main result is Theorem 1.4 below which gives a criterion in order that F admit an inverse G and, moreover, tells how to compute it. The criterion relies in a strong way on the notion of *Jacobian dual* introduced in [14] and gives insight even in the case of ordinary Cremona transformations. Quite a bit of the present work hinges on this criterion as such, although we also give new structure results on representation of rational varieties by projective spaces which invokes a mix of the criterion and some other arguments.

A word on the terminology. For convenience, rational maps such as G above will be called *liftable* (with reference to the fixed projective embedding $Y \subset \mathbb{P}^m$). For a birational map $F: \mathbb{P}^n \dashrightarrow Y \subset \mathbb{P}^m$, whose inverse F^{-1} is liftable, one can define its *type* to be the pair (k, k'), where k (resp. k') is the degree of the forms defining F (resp. the degree of the forms defining F^{-1}).

Fixing the embedding $Y \subset \mathbb{P}^m$, in order to have the condition that *all* rational maps with source Y be liftable, one could express it in terms of sheaves and sections, namely: first, every locally free $\mathcal{O}_Y$-module $\mathcal{L}$ ought to be the restriction to Y of $\mathcal{O}_{\mathbb{P}^m}(d)$, for some $d \geqslant 0$; second, each individual section defining the rational map ought to be lifted to an element of $H^0(\mathcal{O}_{\mathbb{P}^m}(d))$, i.e., the restriction maps

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$H^0(\mathcal{O}_{\mathbb{P}^m}(d)) \to H^0(\mathcal{O}_Y(d))$ ought to be surjective. Examples of embeddings $Y \subset \mathbb{P}^n$ enjoying these properties are projectively normal embedded varieties whose Picard group is free of rank one with generator $\mathcal{O}_{\mathbb{P}^m}(1)$ restricted to Y. It is well known that if Y has dimension at least one and is smooth locally in codimension one, the latter conditions are equivalent to saying that the homogeneous coordinate ring of the embedding $Y \subset \mathbb{P}^m$ is a unique factorization domain. Several important examples, such as projective spaces and Grassmannians in their Plücker embedding, fulfill this condition. Still, there are many embeddings which are not arithmetically factorial, but will nevertheless admit liftable birational maps: the classical theory of representation of (rational) varieties by projective spaces carries many such constructions (cf., e.g., [10]).

Following the traditional terminology in the subject, the scheme defined by the coordinate forms of F will be called the *base locus* of F. A birational map with source a projective space, whose base locus is smooth and connected is called *special*.

We now briefly describe the contents of each Section.

The first Section establishes the main criterion of birationality as applied to a map with source a projective space. This is done by a careful consideration of the syzygies of the ideal generated by the coordinate forms defining the rational map and the bilinear forms occurring in the ideal of definition of a convenient blowup, so the main criterion is first translated in a purely algebraic form. The punch line here as regards previous work is a systematic way of checking whether the map is birational (onto the image) and, moreover, a highly effective method of explicitly obtaining the inverse map. The main tool comes from the notion of jacobian dual as introduced in [14], appropriately adapted to our current needs.

In the next section one narrows down the consideration to Cremona transformations proper. After sufficiently rephrasing the main criterion in this case, one applies it to show that when the ideal generated by the coordinate forms defining the rational map is of *linear type* and its first syzygy module is generated in degree one, then the map is birational. Since ideals of linear type have been extensively considered in recent literature, this adds quite a bit to the known classes of Cremona transformations. Thus, for example, codimension two Cohen–Macaulay ideals and ideals generated by Pfaffians of skew symmetric matrices of odd size belong to these classes provided they are sufficiently generic.

Ideals generated by squarefree monomials form a natural test ground for the main criterium. We give very nearly a complete survey of the case of squarefree monomials of degree 2, that may lead to a curious characterization of the Cremona transformations among these. As it turns, there issues a series of Cremona transformations of $\mathbb{P}^n$ of type $(2, \frac{1}{2}(n+2))$ for every even value of n (this retrieves for $n = 2$ the ordinary quadratic transformation of $\mathbb{P}^2$). For n odd the map is not Cremona, rather what eventually comes out is a birational transformation of a suitable hyperplane onto a hypersurface of $\mathbb{P}^n$ of degree $(n+1)/2$ (this retrieves for $n = 5$ the classical birational representation of the Perazzo hypersurface by a linear space). There may be a similar theory in higher degrees as well, however, as of now a complete classification seems out of reach. We content ourselves in making a few general comments on these. By and large, the class of Cremona transformations defined by squarefree monomials have a combinatorial flavor and should probably be given an appropriate algorithmic treatment.

The third Section is largely experimental. Issuing from rational maps $\mathbb{P}^n \dashrightarrow \mathbb{P}^n$ defined by quadrics, whose image is a quadric of maximal rank, one derives a nearly mechanical procedure for producing involutive Cremona transformations $\mathbb{P}^n \dashrightarrow \mathbb{P}^n$ of type $(n-2, n-2)$. Again here the mechanism is effective though a bit costly.

In Section 4 one discusses rational maps whose coordinates are the minors of a certain size of a *catalecticant* matrix of generalized kind, a subject that has several connections with the classical theory of Cremona transformations. In the case of the 2×2 minors of a $2 \times (m+1)$ such matrix one is able to completely describe the birational maps among these.

The last two sections collect scattered examples which can be uniformly treated using recent theorems of Zak ([19]) and of Alzati and Russo ([1]). In analogy with a result of Ein and Shepherd-Barron ([4]), where quadro-quadric special Cremona transformations were characterized as the maps given by the linear system of quadrics through the so called Severi varieties, one succeeds in characterizing the special quadro-quadric birational transformations between $\mathbb{P}^{2m-2}$ and the Grassmannian of lines $\mathbb{G}(1, m)$ as those given by the linear system of quadrics through $(m-2)$-dimensional smooth varieties of minimal degree, i.e., smooth rational normal scrolls of dimension $m-2$ and degree $m+1$ in $\mathbb{P}^{2m-2}$ (equivalently as the maps defined by the 2×2 minors of a $2 \times (m+1)$ catalecticant matrix).

After this work was finished our attention has been called up to the recent preprint of Vermeire [17]. His condition (K_2) is somewhat related to one of our technical assumptions in case the rational map is defined by quadrics. From the point of view of the present work (K_2) is often too strong a condition as the reader will easily realize. The exact overlap between condition (K_2) and ours is not immediate. Nevertheless, both imply that the module of linear syzygies of the quadrics has maximal possible rank.

1. Birational Maps with Source a Projective Space

We will make use of the following basic criterion, which is nearly tautological.

LEMMA 1.1. *Let $X \subset \mathbb{P}^n$, let $F: X \dashrightarrow \mathbb{P}^m$ be a rational map and let Y be the image of F. Assume that F is liftable to forms $f_0, \ldots, f_m$ of the same degree. Let $\mathcal{B}_{Z\cap X}(X) \subset X \times_k Y \subset X \times_k \mathbb{P}^m$ be the blowup of X along the scheme $Z \cap X$, where $Z \subset \mathbb{P}^n$ is the scheme defined by the homogeneous ideal $I = (f_0, \ldots, f_m)$. Then the following are equivalent:*

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(i) *F is birational with liftable inverse.*
(ii) *There exist forms $g_0, \ldots, g_n \in k[\mathbf{y}]$ of the same degree in the homogeneous coordinate ring of $\mathbb{P}^m$ such that $\mathcal{B}_{Z\cap X}(X) = \mathcal{B}_{Z'\cap Y}(Y)$, where $Z' \subset \mathbb{P}^m$ is the scheme defined by the ideal $(g_0, \ldots, g_n) \subset k[\mathbf{y}]$.*

Proof. (i) $\Rightarrow$ (ii) By definition, F restricts to a biregular map between nonempty open sets of X and Y. Clearly, this biregular map has the same graph in $X \times Y$ as its inverse G. Since G is liftable, it can be given by the restriction of forms $g_0, \ldots, g_n \in k[\mathbf{y}]$. But the closure of either graph in $X \times Y$ is the blowup of X (resp. Y) along the scheme defined by $(f_0, \ldots, f_m)$ (resp $g_0, \ldots, g_n$).

(ii) $\Rightarrow$ (i) Since the structural projections $\mathcal{B}_{Z\cap X}(X) \to X$ and $\mathcal{B}_{Z'\cap Y}(Y) \to Y$ are birational, the equality $\mathcal{B}_{Z\cap X}(X) = \mathcal{B}_{Z'\cap Y}(Y)$ yields a composite birational map $Y \dashrightarrow X$. Clearly, this map is defined by the very forms $g_0, \ldots, g_n \in k[\mathbf{y}]$, hence is liftable. □

Let $F = (f_0 : \ldots : f_m) : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ be a rational map. We assume, without loss of generality, that $f_0, \ldots, f_m$ are linearly independent forms of the same degree $\geqslant 2$.

We establish the basic algebraic notation to be used throughout. Set $R := k[\mathbf{x}] = k[x_0, \ldots, x_n] = \mathrm{Sym}(V^*)$, where $\mathbb{P}^n = \mathbb{P}(V^*)$ and, similarly, $S := k[\mathbf{y}] = k[y_0, \ldots, y_m] = \mathrm{Sym}(W^*)$, where $\mathbb{P}^m = \mathbb{P}(W^*)$. Thus, the k-subalgebra $k[f_0, \ldots, f_m] \subset R$ is homogeneous and is the coordinate ring of the image $Y \subset \mathbb{P}^m$ of F. Let $\mathfrak{a} \subset S$ be the ideal generated by the defining equations of Y in $\mathbb{P}^m$. Thus, $S/\mathfrak{a} \simeq k[f_0, \ldots, f_m]$ as graded k-algebras.

Let $I = (f_0, \ldots, f_m) \subset R$. Consider the blowup $\mathcal{B} = \mathcal{B}_Z(\mathbb{P}^n)$ of $\mathbb{P}^n$ along the scheme $Z \subset \mathbb{P}^n$ defined by I. This can be looked upon as the closure of the graph of the map F on the biprojective space $\mathbb{P}^n \times \mathbb{P}^m$. Let $\mathcal{J} \subset R \otimes_k S = k[\mathbf{x}, \mathbf{y}]$ denote the ideal of definition of $\mathcal{B}$ on $\mathbb{P}^n \times \mathbb{P}^m$. Then $\mathcal{J}$ is generated by biforms of various bidegrees (d_1, d_2), $d_1 \geqslant 0$, $d_2 \geqslant 1$. Since $f_0, \ldots, f_m$ have the same degree, the minimal generators of $\mathfrak{a}$ are exactly the minimal generators of $\mathcal{J}$ of bidegree $(0, d_2)$, $d_2 \geqslant 2$. The minimal generators of $\mathcal{J}$ of bidegree $(d_1, 1)$, with $d_1 \geqslant 1$, are those that generate the defining ideal of the closely associated $\mathbb{P}(\mathrm{Sym}(I))$. A generator of bidegree $(1, 1)$ is just a bilinear form in the $\mathbf{x}, \mathbf{y}$-variables. It corresponds to a syzygy of the ideal I whose coordinates are 1-forms (also called a *linear syzygy*).

Set $\mathbf{f} = \{f_0, \ldots, f_m\}$ for short. Let d be the common degree of these forms. Consider the free graded presentation of $(\mathbf{f})$

$$\oplus_s R(-d_s) \xrightarrow{\varphi} R^m(-d) \to R. \tag{1}$$

Let φ still denote the corresponding R-matrix with respect to suitable bases - φ will be informally called a *presentation matrix* of $\mathbf{f}$ (or of the ideal $(\mathbf{f})$). To account for the degree one part of $\mathrm{Im}(\varphi)$, we decompose φ as $\varphi = (\varphi_1 \mid \varphi_2)$, where the entries of φ_1 are linear and the entries of φ_2 are homogeneous of degree at least two. Said otherwise, $(\mathcal{J}_{(1,1)}) = I_1(\mathbf{y} \cdot \varphi_1)$, the ideal generated by the entries of $\mathbf{y} \cdot \varphi_1$. This takes care of roughly half the algebraic notation we need.

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To get at the other half, let Θ denote the Jacobian matrix of the biforms $\{(\mathbf{y})\cdot\varphi_1\}$ with respect to the $\mathbf{x}$-variables. Note that the entries of this matrix all lie in the ring S. We will in the sequel be mainly concerned with the behaviour of Θ on $\bar{S} := S/\mathfrak{a}$. For convenience $\bar{\Theta}$ will denote Θ taken modulo $\mathfrak{a}$. Also, q will stand for the number of columns of φ_1).

Since $\bar{S}$ is a standard graded ring, and $\bar{\Theta}$ is a homogeneous matrix in this grading, $\mathcal{Z}(\bar{\Theta}) := \ker(\bar{S}^{n+1} \xrightarrow{\bar{\Theta}} \bar{S}^q)$ is a graded $\bar{S}$-module, and, moreover, the coordinates of a given homogeneous vector of $\mathcal{Z}(\bar{\Theta})$ have the same degree. Given any such vector of $\mathcal{Z}(\bar{\Theta})$, with coordinates $\bar{\mathbf{g}} = \{\bar{g}_0, \ldots, \bar{g}_n\}$ in $\bar{S}$, consider the k-homomorphism $R \to \bar{S}$ that sends $x_i \mapsto \bar{g}_i$, for $i = 0, \ldots, n$ and apply it to the entries of the matrix φ_1 to get a matrix over $\bar{S}$ which we will denote by $\varphi_1(\bar{\mathbf{g}})$.

Next is our basic algebraic result from which we will derive a general criterion of birationality.

PROPOSITION 1.2. *Let there be given an ideal* $I = (\mathbf{f}) \subset R = k[\mathbf{x}]$ *as above. With the notation just introduced, the following conditions are equivalent:*

(a) *One has*
  (i) $\dim \bar{S} = n + 1$
  (ii) $\operatorname{rank} \bar{\Theta} \leqslant$ n (*hence* $\mathcal{Z}(\bar{\Theta}) \neq \bar{\mathbf{0}}$)
  (iii) *For some nonzero homogeneous vector* $\bar{\mathbf{g}}$ *of* $\mathcal{Z}(\bar{\Theta})$, $\operatorname{rank} \varphi_1(\bar{\mathbf{g}}) = \operatorname{rank} \varphi$ $(= m)$.

(b) $\operatorname{rank} \varphi_1 = \operatorname{rank} \varphi\ (=$ m) *and there exist forms* $\bar{\mathbf{g}} = \{\bar{g}_0, \ldots, \bar{g}_n\} \subset \bar{S}$ *of the same degree, not all zero, such that the identity map of* $k[\mathbf{x}, \mathbf{y}]$ *induces a bigraded* k*-algebra isomorphism* $\mathcal{R}_R(I) \simeq \mathcal{R}_{\bar{S}}((\bar{\mathbf{g}}))$.

SUPPLEMENT. *If* (a) *is satisfied then any two vectors as in* (iii) *are proportional.*

Proof. (a) $\Rightarrow$ (b) The proof is modelled after that of [14, Lemma 5.1], taking care of the needed modifications.

The scheme of proof is as follows. First we observe that certainly $\dim \mathcal{R}_R(I) = \dim R + 1 = n + 2 = \dim \bar{S} + 1 = \dim \mathcal{R}_{\bar{S}}((\bar{\mathbf{g}}))$. We will define a bigraded k-algebra epimorphism $\bar{\Phi}\colon \mathcal{R}_R(I) \twoheadrightarrow \mathcal{R}_{\bar{S}}((\bar{\mathbf{g}}))$. Since both algebras are domains, this will do. For the existence of such a bigraded surjection we use only conditions (ii) and (iii).

We will first define a bigraded k-algebra epimorphism $\Phi\colon \mathcal{S}_R(I) \twoheadrightarrow \mathcal{R}_{\bar{S}}((\bar{\mathbf{g}}))$, where $\mathcal{S}_R(I)$, as noted before, is the symmetric algebra of I. Since $\mathcal{S}_R(I) \simeq R[\mathbf{y}]/I_1(\mathbf{y}\cdot\varphi)$ we first define a map $\tilde{\Phi}$ with source the polynomial ring $R[\mathbf{y}] = k[\mathbf{x}, \mathbf{y}]$. We set $\tilde{\Phi}(x_i) = \bar{g}_i T \in \bar{S}[\bar{\mathbf{g}}T] \subset \bar{S}[T]$ and $\tilde{\Phi}(Y_j) = \bar{Y}_j \in \bar{S}$, where T is a new variable.

We next show that $\tilde{\Phi}(I_1(\mathbf{y}\cdot\varphi)) = 0 \in \bar{S}$. Note that, by construction, one has

$$\mathbf{y}\cdot\varphi_1 = \mathbf{x}\cdot\Theta^t, \tag{2}$$

where Θ^t denotes the transpose of Θ. Applying $\tilde{\Phi}$ yields $\mathbf{y}\cdot\varphi_1(\bar{\mathbf{g}}T) = \bar{\mathbf{g}}T\cdot\bar{\Theta}^t = 0$, where the first equality follows from (2), observing that Θ involves only $\mathbf{y}$, while the vanishing issues from having taken $\bar{\mathbf{g}}$ such that $\bar{\Theta} \cdot \bar{\mathbf{g}}^t = 0$ in $\bar{S}$.

We have shown that $\mathbf{y} \cdot \varphi_1(\bar{\mathbf{g}}) = 0$. On the other hand, by our second assumption to the effect that $\operatorname{rank} \varphi_1(\bar{\mathbf{g}}) = \operatorname{rank} \varphi$, one has

$$\operatorname{rank} \varphi = \operatorname{rank} \varphi_1(\bar{\mathbf{g}}) \leqslant \operatorname{rank} \varphi(\bar{\mathbf{g}}) \leqslant \operatorname{rank} \varphi,$$

hence $\operatorname{rank} \varphi(\bar{\mathbf{g}}) = \operatorname{rank} \varphi_1(\bar{\mathbf{g}})$. This implies that $\mathbf{y} \cdot \varphi(\bar{\mathbf{g}}) = 0$ as well (e.g., by passing to the fraction field of $\bar{S}[T]$), hence $\tilde{\Phi}(I_1(\mathbf{y} \cdot \varphi)) = 0$ as required.

To achieve the last step, let $\Gamma = \ker(\mathcal{S}_R(I) \longrightarrow \mathcal{R}_R(I))$ - the R-torsion submodule of $\mathcal{S}_R(I)$. Since a power of the ideal I annihilates Γ and $\operatorname{rad}(I) = \operatorname{rad}(I_m(\varphi))$ (because $m = \operatorname{rank} \varphi$), then (say) $I_m(\varphi)^s \Gamma = 0$. Applying $\tilde{\Phi}$ yields $(\tilde{\Phi}(I_m(\varphi)))^s \tilde{\Phi}(\Gamma) = I_m(\varphi(\bar{\mathbf{g}}T))^s \tilde{\Phi}(\Gamma) = 0$. Since $\operatorname{rank} \varphi(\bar{\mathbf{g}}) = \mathrm{m}$, we have $I_m(\varphi(\bar{\mathbf{g}}T)) \neq 0$. Therefore, $\tilde{\Phi}(\Gamma) = 0$.

This shows the existence of a surjective (bigraded) k-algebra homomorphism $\bar{\Phi}$: $\mathcal{R}_R(I) \twoheadrightarrow \mathcal{R}_{\bar{S}}((\bar{\mathbf{g}}))$, thus concluding the proof of the first implication.

(b) $\Rightarrow$ (a) By a trivial dimension argument, the isomorphism between the Rees algebras implies condition (i) of item (a).

The isomorphism between the Rees algebras being induced by the identity of $k[\mathbf{x}, \mathbf{y}]$, it follows that both algebras share the same bihomogeneous presentation ideal over $k[\mathbf{x}, \mathbf{y}]$. Among a minimal set of generators of this ideal, let $F_0(\mathbf{x}, \mathbf{y}), \ldots, F_q(\mathbf{x}, \mathbf{y})$ be a minimal set of biforms of bidgree $(1, 1)$. Using the same notation as in the proof of the first implication and rewriting $F_k(\mathbf{x}, \mathbf{y})$ as $\sum_{j=0}^{m+1} \ell_{kj}(\mathbf{x})\, y_j$, the matrix whose entries are the $\mathbf{x}$-linear forms $\ell_{kj}(\mathbf{x})$ $(0 \leqslant k \leqslant q,\ 0 \leqslant j \leqslant m)$ is the linear part φ_1 of a syzygy matrix of the ideal I. On the other hand, rewriting $F_k(\mathbf{x}, \mathbf{y})$ as $\sum_{i=0}^{n+1} \tilde{\ell}_{k\,i}(\mathbf{y})\, x_i$, one readily sees that the matrix whose entries are the $\mathbf{y}$-linear forms $\ell_{k\,i}(\mathbf{y})$ $(0 \leqslant k \leqslant q,\ 0 \leqslant i \leqslant n)$ is the Jacobian matrix Θ of the biforms $F_0(\mathbf{x}, \mathbf{y}), \ldots, F_q(\mathbf{x}, \mathbf{y})$ with respect to the $\mathbf{x}$- variables. Moreover, since these biforms are relations of the Rees algebra $\mathcal{R}_{\bar{S}}(\bar{\mathbf{g}})$, it follows that $\sum_{i=0}^{n+1} \tilde{\ell}_{k\,i}(\mathbf{y})\, \bar{g}_i T = F_k(\bar{\mathbf{g}}T, \mathbf{y}) = 0$ in $\bar{S}[T]$. This means that $\bar{\Theta} \cdot \bar{\mathbf{g}} = 0$ in $\bar{S}$, hence $\bar{\mathbf{g}}$ is a nonzero vector of $\mathcal{Z}(\bar{\Theta})$. It follows that $\operatorname{rank} \bar{\Theta} \leqslant \mathrm{n}$. This shows condition (ii) of (a).

To see (iii), let $\mathfrak{b} = \ker(R = k[\mathbf{x}] \to k[\bar{g}_0, \ldots, \bar{g}_n])$, where $x_i \longmapsto \bar{g}_i$ as before. Now, since the elements of $\mathbf{g}$ are homogeneous of the same degree, an element of $\mathfrak{b}$ belongs necessarily to the presentation ideal of $\mathcal{R}_{\bar{S}}(\bar{\mathbf{g}})$. The latter is the same as the presentation ideal of $\mathcal{R}_R(I)$ which contains R as a subring. Therefore, $\mathfrak{b} = 0$ (i.e., $\bar{\mathbf{g}}$ is algebraically independent over k). Thus, evaluating as above is an isomorphism and, in particular, ranks remain unchanged. That is to say, $\operatorname{rank} \varphi_1(\bar{\mathbf{g}}) = \operatorname{rank} \varphi_1 = \operatorname{rank} \varphi$, as was to be shown.

Finally, to prove the contention of the supplement, we note that another choice of a vector $\bar{\mathbf{h}}$ such as $\bar{\mathbf{g}}$ implies that the corresponding Rees algebras are isomorphic by a bigraded k-isomorphism. This isomorphism induces an isomorphism of the graded k-algebras $k[\bar{\mathbf{g}}]$ and $k[\bar{\mathbf{h}}]$. Therefore, as vectors over $\bar{S}$, $\bar{\mathbf{g}}$ and $\bar{\mathbf{h}}$ are proportional. $\square$

A vector $\bar{\mathbf{g}} \in \bar{S}^{n+1}$ satisfying condition (a),(iii) of Proposition 1.2 will be named here a *Jacobian dual vector* of $I = (\mathbf{f})$. The terminology is borrowed from [14], where the theory of Jacobian dual modules has been fully developed.

DEFINITION 1.3. We will say that an ideal $I \subset R$ as above has the *strong rank property* if the matrix $\bar{\Theta}$ has rank at most $n = \dim R - 1$ and, for some minimal homogeneous generator $\mathbf{v} \in \mathcal{Z}(\bar{\Theta})$, if we set $\bar{\mathbf{g}} = \mathbf{v}^t$, the evaluated matrix $\varphi_1(\bar{\mathbf{g}})$ has rank $m = \operatorname{rank} \varphi = \mu(\mathrm{I}) - 1$.

We now state our main geometric result giving a criterion for birationality.

THEOREM 1.4. *Let $F: \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ be a rational map represented by linearly independent forms $\mathbf{f} = \{f_0, \ldots, f_m\} \subset k[\mathbf{x}]$ of the same degree $\geqslant 2$ and let Y be the image of F. The following conditions are equivalent:*

(a) *One has*
  (i) $\dim Y = n$
  (ii) *The ideal $I = (\mathbf{f})$ has the strong rank property.*
(b) *F is birational onto Y, with liftable inverse map, and* $\operatorname{rank} \varphi_1 = \operatorname{rank} \varphi$.

Moreover, in these conditions, if $(\bar{\mathbf{g}})$ and $(\bar{\mathbf{h}})$ are two Jacobian dual vectors of I then their respective generators are representatives of the same rational map $G: Y \dashrightarrow \mathbb{P}^n$ and G thus defined is the inverse map to F.

Proof. The equivalence of (a) and (b) is a consequence of Proposition 1.2 and Lemma 1.1, since the equality of the blowups (thought of as graphs in biprojective space) corresponds to a bigraded k-algebra isomorphism $\mathcal{R}_R(I) \simeq \mathcal{R}_{\bar{S}}((\bar{\mathbf{g}}))$ induced by the identity map of $k[\mathbf{x}, \mathbf{y}]$.

On the other hand, by the supplement to Proposition 1.2 it follows that $(\bar{\mathbf{g}})$ and $(\bar{\mathbf{h}})$ are proportional as vectors with coordinates in $\bar{S}$, hence they define the same rational map $Y \dashrightarrow \mathbb{P}^n$. □

A question arises as to whether one has concrete conditions under which I has the strong rank property. Given forms of the same degree $\bar{\mathbf{g}} = \{\bar{g}_0, \ldots, \bar{g}_n\} \subset \bar{S}$, let $\mathfrak{b}(\bar{\mathbf{g}}) := \ker(k[\mathbf{x}] \to k[\bar{\mathbf{g}}] \subset \bar{S})$, with $x_i \mapsto \bar{g}_i$.

PROPOSITION 1.5. *Let $I = (\mathbf{f})$, with $\mathbf{f} = \{f_0, \ldots, f_m\} \subset k[\mathbf{x}] = k[x_0, \ldots, x_n]$ forms of the same degree $\geqslant 2$, φ be a graded presentation matrix of I and let φ_1 denote its submatrix in degree one. Let $\bar{\mathbf{g}} = \{\bar{g}_0, \ldots, \bar{g}_n\} \subset \bar{S}$ be forms of the same degree. Then* $\operatorname{rank} \varphi_1(\bar{\mathbf{g}}) \geqslant \mathrm{m}$ *if and only if $I_m(\varphi_1) \not\subset \mathfrak{b}(\bar{\mathbf{g}})$. Moreover, if I has linear presentation (i.e., if $\varphi = \varphi_1$) then this condition is equivalent to $I \not\subset \mathfrak{b}(\bar{\mathbf{g}})$.*

Proof. The first equivalence is obvious since $k[\mathbf{x}]/\mathfrak{b}(\bar{\mathbf{g}}) \simeq k[\bar{\mathbf{g}}]$. The second follows because $I_m(\varphi)$ and I have the same radical. □

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2. Cremona Transformations

In this section we consider a dominant rational map $F\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^n$. If F is birational, it is called a *Cremona transformation*.

For convenience, we restate the criterion of the previous section in this more restricted context.

PROPOSITION 2.1. *Let* $F\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ *be a dominant rational map represented by forms* $\mathbf{f} = \{f_0, \ldots, f_n\} \subset k[\mathbf{x}]$ *of the same degree* $\geqslant 2$. *Let* φ *be a graded presentation matrix of* $(\mathbf{f})$ *and let* φ_1 *be its degree one submatrix. Then* F *is birational and* $\operatorname{rank}\varphi_1 = \operatorname{rank}\varphi$ *if and only if the ideal* $I = (\mathbf{f})$ *has the strong rank property.*

2.1. CREMONA TRANSFORMATIONS OF LINEAR TYPE

We give a major case of the previous proposition. An ideal $I \subset A$ in a ring A is said to be *of linear type* if the natural A-algebra homomorphism $\mathcal{R}_A(I) \twoheadrightarrow \mathcal{S}_A(I)$ is injective or, equivalently, if the symmetric algebra of I has trivial A-torsion. It follows immediately that the minimal number of generators of such an ideal locally at a prime p is at most $\dim A_p$, a condition frequently called G_∞. This forces (in fact, is equivalent) to a condition in terms of the Fitting ideals of φ of various sizes:

$$\operatorname{ht} I_t(\varphi) \geqslant n - t + 2, \quad \text{for} \quad 1 \leqslant t \leqslant n \tag{3}$$

This condition has occasionally been dubbed $\mathcal{F}_1$ (cf. [14, Definition 2.5]).

PROPOSITION 2.2. *Let* $I \subset R = k[\mathbf{x}] = k[x_0, \ldots, x_n]$ *be a homogeneous ideal of linear type, minimally generated by forms* $\mathbf{f} = \{f_0, \ldots, f_n\}$, *whose graded presentation matrix has only linear entries. Then* $\mathbf{f}$ *is algebraically independent over* k *and* I *has the strong rank property. In particular, the map* $F\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ *is a Cremona map.*

Proof. The result is essentially the core of the first part of [14]. For convenience, we reproduce it for the present needs.

It is clear, first of all, that $\mathbf{f}$ is algebraically independent over k as a nonzero polynomial relation over k cannot belong to the presentation ideal of the symmetric algebra of I. We let Θ be defined as before, i.e., $\mathbf{x} \cdot \varphi = \mathbf{y} \cdot \Theta^t$. Setting $S = k[\mathbf{y}]$, surely $\mathcal{S}_R(I) = \mathcal{S}_S(\operatorname{coker}\Theta^t)$. Since I is of linear type, the right hand side is a domain, hence $\operatorname{coker}\Theta^t$ is torsionfree and a dimension count implies that its rank is 1. Therefore, $\operatorname{coker}\Theta^t \simeq (\mathbf{g})$, for some ideal $\mathbf{g} \subset S$ with same presentation matrix Θ^t. Clearly, $\mathcal{R}_R(I) = \mathcal{R}_S((\mathbf{g}))$, showing that the map defined by $\mathbf{f}$ is a Cremona map with inverse defined by $\mathbf{g}$. We also note that, as a bonus, $\mathbf{g}$ is of linear type as well and, in particular $\operatorname{ht} I_n(\Theta) \geqslant 2$. □

Remark 2.3. A complete intersection of forms of degree $\geqslant 2$ is of linear type but does not define a Cremona map. Thus, being Cremona often may require quite a bit of linear syzygies.

On the other hand, the condition of being of linear type is not necessary for birationality. Here is an instance:

$$\varphi = \begin{pmatrix} 0 & 0 & -x_1 \\ -x_0 & x_0 - x_1 & x_1 \\ x_0 & 0 & 0 \\ x_2 & -x_3 & x_3 \end{pmatrix}$$

The determinantal locus has the correct codimension ($=2$) and so does $I_1(\varphi)$. However, $\operatorname{ht} I_2(\varphi) = 2$ which is one less than the required bound for condition (3) to hold. Yet, the determinants $\mathbf{f}$ are still algebraically independent though not of linear type. The map is a Cremona map with inverse given by the quadrics

$$\mathbf{g} = y_0y_3,\ y_1y_3,\ y_0(y_1 - y_2),\ y_1(y_0 - y_1),$$

hence of type $(3, 2)$. The base locus scheme of the inverse is reduced (the union of one simple line and three distinct points not lying on the line). Note that the presentation matrix of the quadrics is not entirely linear. This is as it should be, since the minimal relations of higher $\mathbf{y}$-degree of $\mathbf{f}$ come exactly from the minimal homogeneous syzygies of degree $\geqslant 2$ of $\mathbf{g}$. It is conceivable that type (n, n) imply that $(\mathbf{f})$ be of linear type.

It should be noticed that this quadro-cubic Cremona transformation of $\mathbb{P}^3$ was known to M. Nöther back in the early 1870's (cf. [8]). Nöther actually gave the construction of a series of such transformations of type $(2r - s + 1, r + 1)$, for any given positive integers r, s such that $s \leqslant \frac{3}{2}r$.

EXAMPLES

We give two large classes of instances of the linear type / linear presentation situation. These classes have been largely treated in the literature (cf. [16] for a general overview). Let

$$R(-d-1)^q \xrightarrow{\varphi} R^{n+1}(-d) \xrightarrow{\mathbf{f}} R$$

be the graded linear presentation of the ideal $I = (\mathbf{f})$. The simplest case has $q = n$ or $q = n + 1$.

EXAMPLES 2.4 (*Maximal minors of an* $(n+1) \times n$ *linear matrix*). Let $q = n$ above and suppose that φ satisfy condition (3) above. Then the map $(f_0 : \cdots : f_n)$: $\mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is a Cremona map of type (n, n) and the inverse map define a base locus of the same kind (i.e., a codimension two arithmetically Cohen–Macaulay variety).

To see that the map is Cremona it suffices, by Proposition 2.2, to check that I is of linear type. The latter follows, e.g., from [7, Corollary 7.4 and Theorem 9.1]. Next, the proof of Proposition 2.2 showed that the transpose of Θ is the presentation matrix of the inverse map $\mathbf{g}$ and the ideal $(\mathbf{g})$ is necessarily of linear type. Therefore,

$\mathbf{g}$ satisfies condition (3) and, in particular $\operatorname{ht} I_n(\Theta) \geqslant 2$. Since $\mathbf{g}$ is obtained as the n-minors of Θ divided by its common gcd we must have $\mathbf{g} = I_n(\Theta)$. This shows that the inverse map is of the same degree.

EXAMPLE 2.5 (*Pfaffians of an odd size skew symmetric matrix*). Let $n \geqslant 2$ be even and suppose that φ is an $(n+1) \times (n+1)$ skew symmetric matrix φ over $k[\mathbf{x}]$ satisfying condition (3) above. Let $\mathbf{f} = \{f_0, \dots, f_n\}$ be the Pfaffians of φ, where we assume that $\operatorname{ht}(\mathbf{f}) = 3$ (the maximal possible value). Then the map $\mathbf{f}\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is a Cremona map of type $(n/2, n-1)$ and the base locus of the inverse map is a codimension 2 arithmetically Buchsbaum variety of degree $(n+1)(n-2)/2$.

Indeed, in the present conditions, it follows that φ is the presentation matrix of the ideal I generated by the Pfaffians and I is of linear type (cf. [7, Corollary 7.4 and Theorem 9.1]). By Proposition 2.2, the map defined by $\mathbf{f}$ is a Cremona map.

To guess the type is a little more convoluted in this case. Let $\mathbf{g}$ define the inverse map. We claim that the free homogeneous resolution of $S/(\mathbf{g})$ over S is

$$0 \longrightarrow S(-r-2) \stackrel{\psi}{\longrightarrow} S(-r-1)^{n+1} \stackrel{\Theta^t}{\longrightarrow} S(-r)^{n+1} \stackrel{\mathbf{g}}{\longrightarrow} S \longrightarrow S/(\mathbf{g}) \longrightarrow 0,$$

where $\psi = \mathbf{y}^t$ and $r = \deg g_i$.

We first check that the above sequence of maps is a complex and then apply the acyclicity criterion. By construction, we have $\Theta \cdot \mathbf{g}^t = 0$, hence $\mathbf{g}\Theta^t = 0$ as well. This shows that the sequence is a complex at the relevant beginning.

Since φ is skew symmetric, we have $\mathbf{v}\varphi\mathbf{v}^t = 0$ for any vector $\mathbf{v}$, so, in particular, $\mathbf{y}\varphi\mathbf{y}^t = 0$. Then it follows that $\mathbf{x}\Theta^t\mathbf{y}^t = 0$, hence the entries of the column matrix $\Theta^t\mathbf{y}^t$ are the coordinates of a syzygy of $\mathbf{x}$, which is impossible since they involve only $\mathbf{y}$-variables. We must conclude that $\Theta^t\mathbf{y}^t = 0$, which shows that the sequence is a complex at the second step too.

We apply the acyclicity criterion: first, $\operatorname{rank} \Theta = \mathrm{n}$ by the supplement to Proposition 1.2, or directly in the present case. Since $\operatorname{ht}(\mathbf{y}) \geqslant 3$, it only remains to check that $\operatorname{ht} I_n(\Theta) \geqslant 2$. But $\mathbf{g}$ is of linear type as well, hence this must be the case over and over.

Having shown that the ideal $(\mathbf{g})$ has such a free resolution, we can apply [14, Lemma 4.7] to get the stated numerical and algebraic information on the base locus.

Actually, an ideal such as $\mathbf{g}$, defining a Cremona map, must have pfaffians as the inverse map. This follows from [14, Proposition 2.4]. Therefore, in an indirect way, this takes care of other situations in which the presentation matrix is $(n+1) \times (n+1)$, with n even. We don't know if these exhaust the possibilities for Cremona maps with linear square presentation matrix.

Remark 2.6. Since condition (3) is equivalent to saying that $\mu(I_\wp) \leqslant \operatorname{ht} \wp$ for every prime ideal $\wp \supset I = (\mathbf{f})$, it is clear that smoothness of $\operatorname{Proj}(R/I)$ implies that condition when I is the ideal of Pfaffians. However, the singular locus is defined by the pfaffians of degree $(n-2)/2$ of φ. Therefore, only the case $n = 4$ qualifies as

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a *special* (i.e., with connected smooth base locus) Cremona transformation. It is a quadro-cubic (i.e., type (2, 3)) Cremona map whose base locus is an elliptic curve (cf. [10]).

2.2. MONOMIAL CREMONA TRANSFORMATIONS

We agree to call a rational map $F\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ *monomial* provided it is defined by monomials in the coordinates of the source. This means of course that the base locus is a union of 'fat' linear subspaces, but we will in this part mainly exploit the algebraic data of such maps. If, moreover, the defining monomials are squarefree then the base locus is reduced. We will in this case obtain a whole series of Cremona transformations whose base locus is reduced yet the syzygies are not generated by linear ones.

Of course, in principle, one could develop an algorithm that takes all mutual quotients of the defining monomials (and further quotients of these) for eventual retrieval of the affine field generators x_i/x_j (j fixed, i variable). Still, this sort of algorithm would hide the subtle symmetries in the data that explain the result, if it happens at all to be the case. The case of the products of the variables taken n at the time is quite deceptive in this sense as dividing out by a convenient monomial (the product of all variables) will readily show that it is the Cremona map defined by the inverses of the variables – the notorious *reciprocal* Cremona transformation.

Nevertheless, it may be of interest to understand how the previous criteria work in the monomial case. For this we recall the following combinatorial notion. To a monomial $\mathbb{X}^{\mathbf{a}}$ one associates its exponent vector $\mathbf{a} \in \mathbb{N}^{n}$. Given a finite set $\mathbf{f} = \mathbb{X}^{\mathbf{a}_0}, \ldots, \mathbb{X}^{\mathbf{a}_n}$ of monomials, the integer matrix $(\mathbf{a}_0, \ldots, \mathbf{a}_n)$ is often called the *log-matrix* of $\mathbf{f}$.

Most of the simplification in this setup is embodied in the following result.

LEMMA 2.7 (char $k = 0$). *Let $F\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be a rational monomial map, defined by monomials* $\mathbf{f}$ *of the same degree. With previous notation, assume that the linear part φ_1 of the syzygy matrix of* $\mathbf{f}$ *has rank n. Then F is a Cremona transformation if and only if the following conditions are satisfied:*

(i) *The log-matrix of* $\mathbf{f}$ *has rank $n+1$*
(ii) *The jacobian matrix Θ has rank $\leqslant n$.*

In this case, the inverse of F is also monomial.

Proof. We apply Proposition 2.1. By [11, Proposition 1.2], the log-matrix of $\mathbf{f}$ and the jacobian matrix of $\mathbf{f}$ have the same rank. Therefore, by (i), $\mathbf{f}$ is algebraically independent over k. Thus, we only have to check that the ideal $(\mathbf{f})$ has the strong rank property. Let $\mathbf{g} \in \ker(\Theta)$ be a non-zero homogeneous vector. To see that $\operatorname{rank}\varphi_1(\mathbf{g}) \geqslant n$, we use Proposition 1.5. First we claim that $I_n(\varphi)$ is generated by monomials. This follows from the fact that φ is graded in the fine $\mathbb{N}^{n+1}$-grading of the polynomial ring $k[\mathbf{x}]$ (cf. [11, Lemma 1.1], where a similar argument was used).

Of course, then also $I_n(\varphi_1)$ is generated by monomials. Since $\operatorname{rank}\varphi_1 = \mathrm{n}$ by assumption, $I_n(\varphi_1) \neq 0$. Since no nonzero monomial can be a polynomial relation, it must be the case that $I_n(\varphi_1)$ is not contained in the ideal of polynomial relations of $\mathbf{g}$. This concludes the proof that F is birational under the stated conditions.

To conclude, we recall that $\ker(\Theta)$ has rank one, hence is cyclic, and that the inverse map is given by the coordinates of the uniquely (up to a field constant) defined homogeneous generator of $\ker(\Theta)$. However, by a similar argument as before, one sees that Θ is graded in the fine $\mathbb{N}^{n+1}$-grading of the polynomial ring $k[\mathbf{y}]$. Therefore, $\mathbf{g}$ has monomial coordinates, as was to be shown. □

In the special case of rational maps defined by squarefree monomials of degree 2 we come very near to a complete characterization of the Cremona ones.

Let $F: \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be the rational map defined by a set of monomials of the form $x_i x_j$, with $i \neq j$. For the sake of simplicity, we assume the following connectedness condition: for any pair of distinct indices $i, j \in \{0, \ldots, n\}$, there are indices $i_1 = i, \ldots, i_r = j \in \{0, \ldots, n\}$ such that $x_{i_1}x_{i_2}, \ldots, x_{i_{r-1}}x_{i_r}$ are monomials in the set. Thus, the monomials correspond to the edges of a simple connected graph on $n+1$ vertices, which we denote by $\mathcal{G}(F)$.

It looks reasonable to pose the following

CONJECTURE 2.8. *Let $F: \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be a rational monomial map, defined by distinct squarefree monomials* $\mathbf{f}$ *of degree* 2 *defining a* (*connected*) *graph* $\mathcal{G}(F)$. *Set* $I = (\mathbf{f})$. *The following conditions are equivalent:*

(i) *I is of linear type,*
(ii) *G has exactly one cycle and this cycle is odd,*
(iii) *F is a Cremona transformation.*

The result may not be difficult to prove. On one side, (i) and (ii) are known to be equivalent (cf. [18]). That (iii) implies (ii) is easy: since there are $n+1$ edges as well, the graph must contain a subgraph which is a *cycle*. Now, in such a setup, it is easy or well known that the log-matrix of these monomials is the incidence matrix of the corresponding graph. Moreover, the rank of the incidence matrix is maximal (i.e., $n+1$) if and only if the graph contains at least one subgraph which is a cycle with an odd number of vertices (i.e., if and only if the graph is not bipartite). Thus only non-bipartite graphs with $n+1$ vertices and $n+1$ edges stand a chance of yielding Cremona transformations $\mathbb{P}^n \dashrightarrow \mathbb{P}^n$. Furthermore, such graphs can contain at most one subgraph which is a cycle. Summing up, only graphs with $n+1$ vertices containing exactly one odd cycle (necessarily having $n+1$ edges) are candidates for yielding Cremona transformations $\mathbb{P}^n \dashrightarrow \mathbb{P}^n$.

The core is the implication is (ii) $\Rightarrow$ (iii) (or, equivalently, (i) $\Rightarrow$ (iii)). Checking whether the linear part of the syzygy matrix has the right rank may well turn out to be an easy combinatorial task, but estimating the rank of the jacobian matrix Θ does not seem to immediately yield.

In the special case where the whole graph is itself a cycle, we can give complete results. Moreover, the even case still leads to a Cremona map by restriction to a suitable hyperplane. Assume, without loss of generality, that the monomials are $\mathbf{f} = \{x_0x_1, x_1x_2, \ldots, x_nx_0\}$.

We distinguish the even and odd cases.

For n even, the map is Cremona of type $(2, \frac{1}{2}(n+2))$, as we now proceed to verify. One can do much better, since the monomials actually generate an ideal of linear type in this case ([18]) – however, it will be of little help trying to apply Proposition 2.2 directly as the presentation matrix of the monomials has as many linear as quadratic syzygies.

It is not difficult to see that the following syzygies form the linear part of a presentation matrix of these monomials

$$\varphi_1 = \begin{pmatrix} -x_n & 0 & 0 & 0 & 0 & 0 & -x_2 \\ 0 & 0 & 0 & 0 & 0 & -x_3 & x_0 \\ 0 & 0 & 0 & 0 & -x_4 & x_1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & x_0 & x_{n-2} & 0 & 0 & 0 & 0 \\ x_1 & -x_{n-1} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

It is more laborious but also straightforward to write down the Jacobian matrix Θ of the biforms $\mathbf{y} \cdot \varphi_1$ with respect to the $\mathbf{x}$-variables. Direct calculation then shows that the following monomials of degree $(n+2)/2$ are the coordinates of a vector in $\ker(\Theta)$:

$$y_0y_1y_3\cdots y_{n-1},\quad y_1y_2y_4\cdots y_n,\quad y_2y_3y_5\cdots y_{n-1}y_0,$$
$$\ldots, y_{n-1}y_ny_1\cdots y_{n-3},\quad y_ny_0y_2\cdots y_{n-2}.$$

In particular, rank $\Theta \leqslant$ n. Finally, the ideal $\mathfrak{b} \subset k[\mathbf{x}]$ of polynomial relations of these monomials is prime, hence contains no monomials. Direct inspection shows that some (actually, all) n-minor of φ_1 is a monomial. Therefore, $I_n(\varphi_1) \not\subset \mathfrak{b}$. By Proposition 1.5, the ideal $(\mathbf{f})$ has the strong rank property.

Thus, the map is a Cremona map and the $(n+2)/2$-forms are the coordinates of the inverse map. Graph theoretically, these monomials correspond to minimal coverings of the odd cycle, the variables appearing in one such monomial corresponding to the vertices of the corresponding covering.

For n odd, the image variety is a hypersurface (of degree $(n+1)/2$ with equation $y_0y_2\cdots y_{n-1} - y_1y_3\cdots y_n = 0$), so the map is not even dominant. In this case, the restriction of the map to a hyperplane, say $x_{n-1} = x_n$, is birational onto the hypersurface (notice that, after renaming variables, one of the coordinates will be the square of a variable). To see this, one proceeds very closely to the even case since the data are very similar. To guess a jacobian dual vector – notice that now there will be various choices as the homogeneous coordinate ring of the hypersurface is definitely not factorial – one takes again the squarefree monomials corresponding to the minimal coverings of a cycle, only now of a cycle of order n. It follows similarly that the ideal generated by the coordinates of the map has the strong rank property and the restriction of the above monomials to the hypersurface will give the inverse map. In particular, the map is of type $(2, (n+1)/2)$.

We note *en passant* that the case $n = 5$ is the representation of the so called *Perazzo hypersurface* by quadrics of $\mathbb{P}^4$ (cf. [10, Chapter 8, Example 7]).

Remark 2.9. A similar discussion can be carried for the natural generalization of the preceding maps, namely, those defined by monomials corresponding to the *paths* of a fixed length of a connected graph (the above case yielding paths of length one). The case where the length of the path is one less than the cardinal of a minimal cover (the latter coinciding with the codimension of the base locus) retrieves, up to variables renaming, the inverse of the case of length one considered above. Further, assume that n is even. The case where the length of the path is two less than the codimension of the base locus yields an arithmetically Gorenstein reduced base locus whose defining equations are the Pfaffians of an $(n+1) \times (n+1)$ skew symmetric matrix. The corresponding ideal of Pfaffians is of linear type, hence the type of the Cremona transformation is $(n/2, n-1)$ by Example 2.5. Since in principle one can compute the degree of the coordinates of the inverse map of a Cremona transformation out of the given map (e.g., by the obvious extension of the method of the residual curves given in [10, Chapter VIII, §, Theorem VII]), it would be quite curious to understand why the expected number is the same number one gets by sheer combinatorics in these monomial transformations.

3. Involutive Cremona Transformations Out of Quadrics

Let $q_0, \ldots, q_n \in k[\mathbf{x}]$ be linearly independent quadrics. If one asks when the corresponding rational map $\mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is a Cremona transformation one will soon realize that the quadrics have to be sufficiently special. For one thing, one needs sufficiently many linear syzygies which is a phenomenon that imposes rather strong restrictions – e.g., the quadrics ought to at least not generate an $(\mathbf{x})$-primary ideal (i.e., not generate a complete intersection). It is not difficult to write examples of zero-dimensional schemes ideal-theoretically cut by $n+1$ quadrics whose syzygies are all generated in degree $\geqslant 2$. Moreover, it is quite conceivable that such examples abound in any codimension. For that matter even less of these turn out to be involutive $(2, 2)$ Cremona transformations (see, e.g., [4] where the special $(2, 2)$ Cremona transformations are completely classified).

However, quadrics can still lead to involutive Cremona transformations in a curious way.

PROPOSITION 3.1 (char $k = 0$). *Let* $\mathbf{q} := \{q_0, \ldots, q_n\} \subset R = k[\mathbf{x}] = k[x_0, \ldots, x_n]$ $(n \geqslant 3)$ *be quadrics, let* $\theta = \theta(\mathbf{q})$ *stand for the Jacobian matrix of* $\mathbf{q}$ *and let*

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$\{f_0, \ldots, f_n\} \subset R$ be the n-minors of any $n \times (n+1)$ submatrix of θ divided by their gcd. Assume that:

(i) *$f_0, \ldots, f_n$ are algebraically independent,*
(ii) *The rational map $(q_0 : \cdots : q_n) : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ has no fixed component and its image is a quadric hypersurface of maximal rank,*
(iii) *$\operatorname{ht} I_n(\theta) \geqslant 2$ and $\operatorname{coker}(R^{n+1} \xrightarrow{\theta^t} R^{n+1})$ is a torsionfree R-module.*

Then $(f_0 : \cdots : f_n) : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is an involutive Cremona transformation of type $(n-2, n-2)$.

Proof. By (ii), the subalgebra $k[\mathbf{q}] \subset R$ has dimension n. Since char $k = 0$, rank$\theta = $ n. By the basic property of the cofactors, θ annihilates its matrix of cofactors. If $\mathbf{f} := \{f_0, \ldots, f_n\} \subset R$ are the n-minors of any $n \times (n+1)$ submatrix of θ divided by their gcd, it follows that $\mathbf{f}^t$ is a syzygy of θ and that $0 \longrightarrow R \xrightarrow{\mathbf{f}^t} R^{n+1} \xrightarrow{\theta} R^{n+1}$ is an exact sequence. Dualizing yields a complex

$$R^{n+1} \xrightarrow{\theta^t} R^{n+1} \longrightarrow I = (\mathbf{f}) \longrightarrow 0, \tag{4}$$

which is clearly exact at I. But $\operatorname{Im}(\theta^t)$ is a reflexive R-module by assumption (iii). Therefore, $\operatorname{Im}(\theta^t) \subset \ker(R^{n+1} \longrightarrow I)$ are reflexive R-modules which coincide locally in codimension one because $\operatorname{ht} I_n(\theta) \geqslant 2$ by the other half of (iii). Thus, (4) is exact.

On the other hand, taking the Jacobian matrix Θ of the forms $\mathbf{y} \cdot \theta^t$ with respect to the $\mathbf{x}$-variables yields back the original matrix θ (cf. [12, Lemma 7.28]). Therefore, the map will be birational, with same inverse, provided we show that $\mathbf{f}$ has the characteristic property of a Jacobian dual vector. However, by Proposition 1.5, for that it suffices to show that the generators $\mathbf{f}$ are algebraically independent, which is our assumption (i). This shows that $(f_0 : \cdots : f_n) : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is an involutive Cremona transformation.

To obtain its type, we use (5) again to compute the Hilbert series of R/I. We first complete (4) to a free resolution. We contend that there is an invertible linear $\mathbf{q} \to \mathbf{q}'$ transformation of the k-vector space $(\mathbf{q})_2$ such that

$$0 \longrightarrow R \xrightarrow{\mathbf{q}'^t} R^{n+1} \xrightarrow{\theta^t} R^{n+1} \longrightarrow I = (\mathbf{f}) \longrightarrow 0, \tag{5}$$

is the free resolution of I. Indeed, let $Q(q_0, \ldots, q_n) = \sum_{0 \leqslant i \leqslant j \leqslant n} \lambda_{ij} q_i q_j = 0$ be the quadratic relation satisfied by the points in the image of the rational map defined by $\mathbf{q}$ (i.e., the generator of the presentation ideal of $k[\mathbf{q}]$). For any $0 \leqslant \ell \leqslant n$, taking derivatives $\partial/\partial x_\ell$, one immediately sees that $\mathbf{q}'^t$ is a syzygy of θ^t, with $\mathbf{q}'^t = \mathcal{Q} \cdot \mathbf{q}^t$, where $\mathcal{Q}$ is the matrix of the quadratic form Q, which is invertible by the rank assumption. Thus, (5) is a complex and it is exact in the left as well since rank $\theta = $ n and since $I_1(\mathbf{q}'^t) = (\mathbf{q})$ has codimension at least 2 because $\mathbf{q}$ defines a rational map without fixed component by assumption.

From the resolution, we see that the Hilbert series of R/I is

$$\frac{1-(n+1)t^s+(n+1)t^{s+1}-t^{s+3}}{(1-t)^{n+1}},$$

where s is the common degree of the generators of I. Since I has codimension at least 2 by (iii) then it has codimension exactly 2 ([2, Theorem 2.1]), it follows that I has codimension 2. Therefore, the Hilbert series has a pole of order $n+1-2=n-1$ at $t=1$, hence $s=n-2$, as required (cf. [14, Proof of Lemma 4.7] for a similar computation). □

Remark 3.2. A word about the assumptions of Proposition 3.1. The requirement (ii) that the quadric obtained as the image of $\mathbf{q}$ be smooth is essential as easily constructed examples show. The torsionfreeness requirement in (iii) can be circumvented by killing the torsion of $\operatorname{coker}(R^{n+1}\xrightarrow{\theta^t}R^{n+1})$, i.e., by directly looking at the ideal generated by the maximal minors of an $n\times(n+1)$ submatrix of θ^t divided by their gcd. However, the free resolution of this ideal will not have the same data as (5) and the resulting Cremona map will conceivably no longer be of type $(n-2,n-2)$. Finally, condition (i) is obviously necessary for the map to be birational onto $\mathbb{P}^n$. It is however conceivable that this falls off a stronger condition on the codimension of the remaining Fitting ideals of θ.

Next we give some examples to illustrate the method. We note that if the assumptions of the proposition hold then the ideal generated by the quadrics $\mathbf{q}$ necessarily has codimension at least 3.

EXAMPLE 3.3. Let $\mathbf{q}$ be the 2×2 minors of the catalecticant

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3\\ x_2 & x_3 & x_4 & x_5\end{pmatrix}$$

In this range, the image of the rational map $\mathbb{P}^5 \dashrightarrow \mathbb{P}^5$ is the Grassmann variety of lines in $\mathbb{P}^3$, defined by the known Plücker relation, hence of maximal rank. Condition (ii) of the Proposition can be verified directly or by using *Macaulay*. The coordinates of the resulting Cremona map cut a reduced base locus which is not smooth in codimension one.

EXAMPLE 3.4 ([13, Example 1.7]). Consider the following quadrics in $k[x_0,\dots,x_5]$:

$$x_2x_4+x_0x_5,\ x_1x_3+x_2x_5,\ x_0x_1-x_1x_4+x_2x_4-x_4x_5,$$
$$x_0x_2+x_0x_3+x_3x_4+x_0x_5,\ x_1x_2+x_1x_5-x_2x_5+x_5^2,\ x_2^2+x_2x_3+x_2x_5-x_3x_5,$$

A computation with the aid of *Macaulay* tells us that these quadrics fill in all the conditions of Proposition 3.1. We note that the given quadrics cut ideal-theoretically

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a smooth arithmetically Cohen–Macaulay surface of degree 4. However, the Cremona transformation obtained is far from being special in the terminology of [4].

4. Birational Maps Coming from Catalecticant Matrices

In this section we will consider rational maps defined by minors of catalecticant matrices. We recall their definition for the reader's convenience.

DEFINITION 4.1. Fix integers $r, m \geqslant 1$. Let $m+1 = \sum_{i=1}^{r} a_i$ be an r-partition of $m+1$, with $a_1, \dots, a_r$ integers such that $1 \leqslant a_1 \leqslant \dots \leqslant a_r$. The *generic catalecticant matrix of type* $(a_1, \dots, a_r)$ is the $2 \times (m+1)$ matrix consisting of Hankel blocks as follows:

$$\mathbb{X}(a_1, \dots, a_r) = \left(\begin{array}{ccc|ccc|c|ccc} x_{1,0} & \dots & x_{1,a_1-1} & x_{2,0} & \dots & x_{2,a_2-1} & \dots & x_{r,0} & \dots & x_{a_r-1} \\ x_{1,1} & \dots & x_{1,a_1} & x_{2,1} & \dots & x_{2,a_2} & \dots & x_{r,1} & \dots & x_{a_r} \end{array} \right).$$

The $2\times$ 2 minors of $\mathbb{X}(a_1, \dots, a_r)$ define ideal theoretically a smooth algebraic variety $S(a_1, \dots, a_r) \subset \mathbb{P}^{m+r}$ of dimension r and degree $d = \sum_{i=1}^{r} a_i = m+1$, that happens to be a rational normal scroll (generated by the r rational normal curves $C_i \subset \mathbb{P}^{a_i}$ defined each by a Hankel block). A classical result of Bertini and Del Pezzo shows that, besides the Veronese surface in $\mathbb{P}^5$, the rational normal scrolls are the only smooth varieties of minimal degree (see [5, Theorem 1] for a modern account on these varieties).

The following two statements reformulate classical results of Semple (see [9], [15] and also [1]). The reason for their insertion here is that they fit the general pattern of the results. In particular, the inverse map is given by the coordinates of a jacobian dual.

PROPOSITION 4.2 (char $k = 0$). *Let* $F\colon \mathbb{P}^{m+r} \dashrightarrow \mathbb{P}^{\binom{m+1}{2}-1}$ *denote the rational map defined by the minors of the generic catalecticant matrix* $\mathbb{X}(a_1, \dots, a_r)$ *as introduced above. Assume that* $m \geqslant 3$ *and that* $r \geqslant m-2$. *Then:*

- (i) *The image of* F *is the Plücker embedding into* $\mathbb{P}^{\binom{m+1}{2}-1}$ *of the Grassmannian* $\mathbb{G}(1, m)$ *of lines of* $\mathbb{P}^m$.
- (ii) *The general fiber of* F *is a linear space of dimension* $r - m + 2 \leqslant 3$; *in particular,* F *is birational if and only if* $r = m-2$. *Moreover, this is the case if and only if* $(a_1, \dots, a_{m-2})$ *is one amongst* $(1, \dots, 1, 4)$, $(1, \dots, 1, 2, 3)$ *and* $(1, \dots, 1, 2, 2, 2)$.
- (iii) *If* F *is birational, the type of* F *is* $(2, 2)$ *and the inverse is given by the coordinates of a (uniquely determined) Jacobian dual vector.*

Proof. Since $\mathbb{X}(a_1, \dots, a_r)$ is a specialization of the generic $2 \times (m+1)$ matrix, the image of F is contained in $\mathbb{G}(1, m) \subset \mathbb{P}^{\binom{m+1}{2}-1}$. Equivalently, for a point $P \in \mathbb{P}^{m+r}$ outside $S(a_1, \dots, a_r)$ the matrix $\mathbb{X}(a_1, \dots, a_r)$ evaluated in P has rank 2 and F

contracts its orbit under $SL(2, k)$ to a point representing the corresponding 2-dimensional subspace of k^{m+1}.

We deal with the case $r = m - 2$. The cases $r > m - 2$ are similarly treated. In this situation $\dim \mathbb{G}(1, m) = \dim \mathbb{P}^{m+r}$, so it suffices to show that F is generically $1 - 1$. For that, we have to show that a general point of $\mathbb{G}(1, m)$ (i.e., a general line of $\mathbb{P}^m$) is the image by F of a unique point of $\mathbb{P}^{2m-2}$. To translate this condition in terms of $2 \times (m + 1)$ matrices with entries in k, we let $SL(2, k)$ act onto the set of these matrices by left multiplication. Then the question becomes whether, given a general 2-rowed matrix A with entries in k, say,

$$\begin{pmatrix} \alpha_0 & \dots & \alpha_m \\ \alpha'_0 & \dots & \alpha'_m \end{pmatrix},$$

there exists a unique matrix $\Lambda \in SL(2, k)$ such that ΛA has the form of a (special) catalecticant matrix of type $(a_1, \dots, a_{m-2})$ with entries in k. Now, since $\sum_{i=1}^{m-2} a_i = m + 1$, then $\sum_{i=1}^{m-2}(a_i - 1) = m + 1 - m + 2 = 3$, hence the only possible catalecticant matrices in this case are of type $(1, \dots, 1, 4)$, $(1, \dots, 1, 2, 3)$ $(m \geqslant 4)$ and $(1, \dots, 1, 2, 2, 2)$ $(m \geqslant 5)$.

Therefore, the above claim is equivalent to, given general A as above, finding unique $\lambda, \lambda', \mu, \mu' \in k$ such that $\lambda\mu' - \lambda'\mu = 1$, satisfying, respectively, the following sets of equalities

$$\begin{aligned} \mu\alpha_{m-3} + \mu'\alpha'_{m-3} &= \lambda\alpha_{m-2} + \lambda'\alpha'_{m-2}, \\ \mu\alpha_{m-2} + \mu'\alpha'_{m-2} &= \lambda\alpha_{m-1} + \lambda'\alpha'_{m-1}, \\ \mu\alpha_{m-1} + \mu'\alpha'_{m-1} &= \lambda\alpha_m + \lambda'\alpha'_m, \end{aligned}$$

$$\begin{aligned} \mu\alpha_{m-4} + \mu'\alpha'_{m-4} &= \lambda\alpha_{m-3} + \lambda'\alpha'_{m-3}, \\ \mu\alpha_{m-2} + \mu'\alpha'_{m-2} &= \lambda\alpha_{m-1} + \lambda'\alpha'_{m-1}, \\ \mu\alpha_{m-1} + \mu'\alpha'_{m-1} &= \lambda\alpha_m + \lambda'\alpha'_m, \end{aligned}$$

or

$$\begin{aligned} \mu\alpha_{m-5} + \mu'\alpha'_{m-5} &= \lambda\alpha_{m-4} + \lambda'\alpha'_{m-4}, \\ \mu\alpha_{m-3} + \mu'\alpha'_{m-3} &= \lambda\alpha_{m-2} + \lambda'\alpha'_{m-2}, \\ \mu\alpha_{m-1} + \mu'\alpha'_{m-1} &= \lambda\alpha_m + \lambda'\alpha'_m. \end{aligned}$$

This is an easy exercise in linear algebra, by noting that for general A any solution matrix

$$\Lambda = \begin{pmatrix} \lambda & \lambda' \\ \mu & \mu' \end{pmatrix}$$

lies in $GL(2, k)$. Thus, we have shown that F is birational onto $\mathbb{G}(1, m)$.

Finally, we prove (iii). First note that the ideal I has linear presentation because it is a specialization of the ideal generated by the 2-minors of a generic $2 \times (m + 1)$

matrix (alternatively, cutting the variety defined by I by a general m-dimensional linear subspace $L \simeq \mathbb{P}^m$ yields $m+1$ points in linear general position in $\mathbb{P}^m$). On the other hand, since Grassmannians are arithmetically factorial in their Plücker embedding, the inverse map is liftable. It follows from Theorem 1.4 that I has the strong rank property and the inverse map is given by a Jacobian dual vector of I.

It remains to show that the coordinates of the jacobian dual vector are quadrics. For this sake, the argument of [4, Proposition 2.3] can be applied (see also Lemma 5.1 below), to wit, if F is a birational transformation of $\mathbb{P}^{m-2}$ onto $\mathbb{G}(1,m)$ of type $(2,d)$ with smooth base locus X, then the secant variety to X in $\mathbb{P}^{2m-2}$ is a hypersurface of degree $2d-1$. Since the secant variety to a rational normal scroll $S(a_1,\dots,a_{m-2}) \subset \mathbb{P}^{2m-2}$ is a cubic hypersurface, it follows that F^{-1} is given by the restriction to $\mathbb{G}(1,m)$ of quadratic forms (a geometric proof of this fact actually appears in [9, p. 205], where it is shown that the total transform of a general line $L \subset \mathbb{G}(1,m)$ by F^{-1} is a conic, which is clearly equivalent to saying that F^{-1} is defined by quadratic forms). □

There is a version in which the base locus is a degenerate Segre manifold.

PROPOSITION 4.3 (char $k=0$). *Let* $F\colon \mathbb{P}^{2m-2} \dashrightarrow \mathbb{P}^{\binom{m+1}{2}-1}$ *denote the rational map defined by the maximal minors of the* $2\times(m+1)$ *matrix*

$$\left(\begin{array}{ccc|cc} x_0 & \dots & x_{m-2} & 0 & x_{2m-2} \\ x_{m-1} & \dots & x_{2m-3} & x_{2m-2} & 0 \end{array}\right).$$

Assume that $m \geqslant 3$. *Then:*

(i) *The image of F is the the Plücker embedding of* $\mathbb{G}(1,m)$ *into* $\mathbb{P}^{\binom{m+1}{2}-1}$

(ii) *The base locus of F is the Segre embedding of* $\mathbb{P}^1\times\mathbb{P}^{m-2}$ *as a degenerate subvariety contained in the hyperplane* $x_{2m-2}=0$.

(iii) *The map F is birational of type* $(2,1)$*; in particular, its inverse is the restriction to* $\mathbb{G}(1,m)$ *of a linear projection. Moreover, the center of this projection is defined by the coordinates of a (uniquely determined) Jacobian dual vector.*

Proof. The proof of (i) is similar to the previous one, hence will be omitted.

For (ii), one notes that the saturation of the ideal I generated by the 2×2 minors of the given matrix is generated by the maximal minors of its initial $2\times(m-1)$ submatrix and the variable x_{2m-2}. This clearly shows the contention of this item.

The argument to the effect that I has the strong rank property and the inverse map is given by a Jacobian dual vector of I is the same as the one given in the proof of Proposition 4.2. To find the degree of its coordinates we still argue as in the proof of Proposition 4.2: since the secant variety to the Segre variety is the hyperplane $x_{2m-2}=0$, one has $2d-1=1$, i.e., $d=1$. (A geometric argument to find the center of the projection can be found in [9, p. 213].) □

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The previous result can be generalized to matrices of the following form.

$$\mathbb{X} = \begin{pmatrix} x_0 & x_1 & \dots & x_m \\ x_r & x_{1+r} & \dots & x_{m+r} \\ x_{2r} & x_{1+2r} & \dots & x_{m+2r} \\ \vdots & \vdots & & \vdots \\ x_{nr} & x_{1+nr} & \dots & x_{m+nr} \end{pmatrix}$$

Such an $(n+1) \times (m+1)$ matrix is called a *generic r-catalecticant matrix*.

PROPOSITION 4.4. *For $s \geqslant 1$, the $(s+1)$-minors of the generic $(m-s-1)$-catalecticant matrix $\mathbb{X}$ of size $(s+1) \times (m+1)$ define a birational map of type $(s+1, s+1)$*

$$F: \mathbb{P}^{(m-s)(s+1)} \dashrightarrow \mathbb{G}(s, m) \subset \mathbb{P}^{\binom{m+1}{s+1}-1},$$

whose image is the Grassmannian of s-spaces in $\mathbb{P}^m$.

Proof. The proof is exactly the same as for $s = 1$. For a general choice of $s+1$ points $P_0, \dots, P_s$, generating a s-plane given by a general $(s+1) \times (m+1)$ matrix A, there exists a unique matrix $\Lambda \in \mathrm{SL}(s+1, k)$ such that ΛA is an r-catalecticant matrix with entries in k.

The birational transformation is seen to be of type $(s+1, s+1)$ (see [9]) and, as in the proof of Proposition 4.2, the inverse map is given by the Jacobian dual of the ideal generated by the minors of order $s+1$ of the matrix. We leave the details to the reader. □

5. A Classification Result

In [4] quadro-quadric special Cremona transformations (those whose base locus is smooth and connected) were characterized as the maps given by the systems of quadrics through Severi varieties. By a well known result of Zak, there are just four of these.

In the previous section we have seen that, for every $m \geqslant 3$, there exist special quadro-quadric birational maps $\mathbb{P}^{2m-2} \dashrightarrow \mathbb{P}^{\binom{m+1}{2}-1}$ with image $\mathbb{G}(1, m)$, given by the 2×2 minors of the $2 \times (m+1)$ generic catalecticant matrix $\mathbb{X}(a_1, \dots, a_{m-2})$, where $(a_1, \dots, a_{m-2})$ is one of the following: $(1, \dots, 1, 4)$, $(1, \dots, 1, 2, 3)$ or $(1, \dots, 1, 2, 2, 2)$.

Here, by appealing to a recent result of Zak ([19]), we characterize special quadro-quadric birational transformations $\mathbb{P}^{2m-2} \dashrightarrow \mathbb{P}^{\binom{m+1}{2}-1}$ with image $\mathbb{G}(1, m)$ as the maps given by the systems of quadrics through $(m-2)$-dimensional smooth varieties of minimal degree in $\mathbb{P}^{m-2}$, that is to say, smooth rational normal scrolls of dimension $m-2$ and degree $m+1$. Equivalently, these will be the maps defined by the 2×2 minors of a $2 \times (m+1)$ generic catalecticant matrix of type $(a_1, \dots, a_{m-2})$ with $\sum_{i=1}^{m-2} a_i = m+1$.

Thus, let $F\colon \mathbb{P}^{2m-2} \dashrightarrow \mathbb{P}^{\binom{m+1}{2}-1}$ be a birational map with image $\mathbb{G}(1,m)$, of type (d_1,d_2), and let X be the base locus of F. Let $g\colon Y = Bl_X(\mathbb{P}^{2m-2}) \to \mathbb{P}^{2m-2}$ stand for the structural map. There is a canonical diagram

$$\begin{array}{ccccc} & & Y & & \\ & g \swarrow & & \searrow f & \\ \mathbb{P}^{2m-2} & \overset{F}{\dashrightarrow} & & \mathbb{G}(1,m) & \subset \ \mathbb{P}^{\binom{m+1}{2}-1} \end{array},$$

where f is the projection onto the second factor.

Let Z denote the base locus of F^{-1}, $E_1 = g^{-1}(X)$ and $E_2 = f^{-1}(Z)$. Take $H_1 \in |g^*(\mathcal{O}_{\mathbb{P}^{2m-2}}(1))|$, $H_2 \in |f^*(\mathcal{O}_{\mathbb{G}(1,m)}(1))|$ and let T_1 and T_2 stand, respectively, for the strict transform on Y of a general line of $\mathbb{P}^{2m-2}$ and of a general line of $\mathbb{G}(1,m)$. Let $N_{T_2/Y}$ be the normal bundle of T_2 in Y. Then $Pic(Y) \simeq \mathbb{Z}[H_1] \oplus \mathbb{Z}[H_2]$ and $\Lambda^{2m-3} N_{T_2/Y} = \mathcal{O}_{\mathbb{P}^1}(a_m)$, for some $a_m \in \mathbb{Z}$.

Let us recall that, given a subvariety $W \subset \mathbb{P}^n$ and an integer $s \geqslant 2$, an *s-secant line* to W is a line in $\mathbb{P}^n$ not contained in W and intersecting W in at least s points (counted multiplicities). The closure $\mathrm{Sec}_s(W) \subset \mathbb{P}^n$ of the union of all s-secant lines to W is called the *sth secant variety* of W. Of course, $\mathrm{Sec}_2(W)$ is the ordinary secant variety of W.

The following lemma is based on a clever result of [4].

LEMMA 5.1. *Let notations be as above. Then*

(i) $H_2 \sim d_1H_1 - E_1$, $H_1 \sim d_2H_2 - E_2$, *and* $T_1 \cdot H_2 = d_1$, $T_2 \cdot H_1 = d_2$, $E_1 \cdot T_2 = E_2 \cdot T_1 = d_1d_2 - 1$.

(ii) $(d_1d_2 - 1)(2m - 3 - dim(X)) + d_2(1 - 2m) + a_m + 2 = 0$.

(iii) $g(E_2) = \mathrm{Sec}_{d_1}(X)$ *is a hypersurface of degree* $d_1d_2 - 1$.

Proof. The proof of (i) and (iii) are exactly the ones given in [4, Lemma 2.4,2.3], hence will be omitted. To obtain (ii), consider the adjunction formula $\mathcal{O}_{\mathbb{P}^1}(-2) = \omega_{T_2} = \Lambda^{2m-3} N_{T_2/Y} \otimes \omega_Y$ and the equality $\omega_Y = (-2m+1)H_1 + (2m - 3 - \dim X)E_1$. From these, using (i), we see that

$$a_m + d_2(1 - 2m) + (d_1d_2 - 1)(2m - 3 - \dim X) = [a_mH_2 + (-2m+1)H_1 + (2m - 3 - \dim X)E_1] \cdot T_2 = 2$$

This proves the contention of the lemma. □

THEOREM 5.2. *Let* $F\colon \mathbb{P}^{2m-2} \dashrightarrow \mathbb{G}(1,m) \subset \mathbb{P}^{\binom{m+1}{2}-1}$ *be a birational map of type* $(2,2)$ *having smooth irreducible base locus. Then F is given by the* 2×2 *minors of a generic catalecticant matrix* $\mathbb{X}(a_1, \ldots, a_{m-2})$, *where* $(a_1, \ldots, a_{m-2})$ *is one of the following*

$$(1, \ldots, 1, 4), \quad (1, \ldots, 1, 2, 3), \quad (1, \ldots, 1, 2, 2, 2),$$

Proof. By Lemma 5.1 (ii), $\dim X$ satisfies a linear equation whose coefficients depend on the fixed integers d_1, d_2, m. This equation has therefore a unique solution in rational numbers. This solution, if it happens to be an integer, must return $\dim X$ as the base locus X depends only on F. But, we have seen in

Proposition 4.2 that, for every $m \geqslant 3$ and $d_1 = d_2 = 2$, the integer $m-2$ is a solution of the afore-mentioned equation. It therefore follows that the base locus X of F has dimension $m-2$, hence is of codimension m in $\mathbb{P}^{2m-2}$; the variety X is non-degenerated because $\mathrm{Sec}_2(X)$ is a hypersurface of degree 3 by Lemma 5.1(iii). Since X is cut by $\binom{m+1}{2}$ quadrics, it is a variety of minimal degree (cf. [19, Corollary 5.8]). However, a smooth variety of dimension $m-2$ and minimal degree $m+1$ in $\mathbb{P}^{2m-2}$ is a rational normal scroll and the latter is defined by the ideal generated by the 2×2 minors of a $2 \times (m+1)$ generic catalecticant matrix of type $(a_1, \ldots, a_{m-2})$ with $\sum_{i=1}^{m-2} a_i = m+1$. This proves the theorem. □

One can in the same fashion classify the special birational maps $\mathbb{P}^{2m-2} \dashrightarrow \mathbb{G}(1, m) \subset \mathbb{P}^{\binom{m+1}{2}-1}$ of type $(2, 1)$.

THEOREM 5.3. *Let* $F: \mathbb{P}^{2m-2} \dashrightarrow \mathbb{P}^{\binom{m+1}{2}-1}$ *be a birational map of type* $(2, 1)$ *with image* $\mathbb{G}(1, m)$ *and smooth irreducible base locus. Then* F *is given by the* 2×2 *minors of a* $2 \times (m+1)$ *matrix of the form*

$$\left(\begin{array}{ccc|cc} x_0 & \ldots & x_{m-2} & 0 & x_{2m-2} \\ x_{m-1} & \ldots & x_{2m-3} & x_{2m-2} & 0 \end{array}\right).$$

Proof. By Lemma 5.1 (ii), $\dim X$ satisfies a linear equation whose coefficients depend on the fixed integers d_1, d_2, m. This equation has therefore a unique solution in rational numbers. This solution, if it happens to be an integer, must return $\dim X$ as the base locus X depends only on F. But, we have seen in Proposition 4.3 that, for every $m \geqslant 3$ and $d_1 = 2$, $d_2 = 1$ the integer $m-1$ is a solution of the aforementioned equation. It therefore follows that the base locus X of F has dimension $m-1$, hence is of codimension m in $\mathbb{P}^{2m-2}$; the variety X is degenerated because $\mathrm{Sec}_2(X)$ is a hyperplane H in $\mathbb{P}^{2m-2}$ by Lemma 5.1(iii) and X spans this $\mathbb{P}^{2m-3}$. Since X is cut by $\binom{m+1}{2}$ quadrics in $\mathbb{P}^{2m-2}$, then X is cut by $\binom{m+1}{2} - 2m + 1 = \binom{m-1}{2}$ quadrics in H. Since X has codimension $m-2$ in H, it is a variety of minimal degree (cf. [19, Corollary 5.8]). However, a smooth variety of dimension $m-1$ and minimal degree $m-1$ in $\mathbb{P}^{2m-3}$ is a rational normal scroll of type $(a_1, \ldots, a_{m-1}) = (1, \ldots, 1)$ and the latter is defined by the ideal generated by the 2×2 minors of a $2 \times (m-1)$ generic catalecticant matrix of type $(1, \ldots, 1)$. This proves the theorem. □

6. Other Examples

Birational maps defined by quadrics that generate ideals having the strong rank property abound in the classical literature.

The earliest examples of this sort cases were the maps defined by quadrics through a rational normal curve of degree 4, whose image is $\mathbb{G}(1, 3) \subset \mathbb{P}^5$ (see Proposition 4.2) and by quadrics trough the normal elliptic curves of degree 5 – the quadro-cubic

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Cremona transformation of $\mathbb{P}^4$ (cf. also Remark 2.6). Other classical examples are given by quadrics through the Veronese surface in $\mathbb{P}^5$, by quadrics through the Segre variety $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$, which define an involutive Cremona transformation of $\mathbb{P}^5$, respectively $\mathbb{P}^8$, or by the maps described in Proposition 4.2. One easily sees that the varieties which are the base locus of these examples are smooth varieties in $\mathbb{P}^r$ of codimension s, degree less than or equal to $2s$-1 and with $\mathrm{Sec}(X) \subset \mathbb{P}^r$; moreover they are scheme theoretically defined by the quadrics containing them.

This has been generalized in the following way.

THEOREM 6.1 ([1]). *Let $X \subset \mathbb{P}^r$ be a smooth linearly normal variety. Assume that*

(a) $\mathrm{Sec}(X) \neq \mathbb{P}^r$,
(b) $h^1(\mathcal{O}_X) = 0$ *if* $\dim(X) \geqslant 2$.

If $\deg X \leqslant 2\mathrm{cod}X - 1$, *then the quadrics through X define a rational map F from* $\mathbb{P}^r$ *to* $\mathbb{P}(H^0(\mathcal{I}_X(2)))$, *which is birational onto the image and whose base locus is X.*

In order to obtain examples of such birational maps F defined by ideals I with the strong rank property, one ought to verify that F is liftable. This is the case for quite many arithmetically Cohen Macaulay varieties. Moreover, also quite often I will have linear presentation as it will have the same Betti numbers as the ideal defining $d = \deg X \leqslant 2s - 1$ ($s = \mathrm{cod}X$) points in $\mathbb{P}^s$ in linear general position. As is well known, the latter proliferate.

For yet a different class of examples, consider the quadrics through a linearly normal smooth curve C of genus g, embedded in $\mathbb{P}^{g+s}$ with degree $d = 2g + s$, $s \geqslant 3$, $g + s \geqslant 4$. These define birational maps onto the image by the above theorem. If $C \subset \mathbb{P}^{g+s}$ is a linearly normal curve embedded by a line bundle of degree $d = 2g + s$ and if $s \geqslant 2$, then the ideal I defining C is generated by quadrics. Further, if $s \geqslant 3$ then I has linear presentation (see [6]).

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