INVARIANTS OF FINITE REFLECTION GROUPS

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Let us define a reflection to be a unitary transformation, other than the identity, which leaves fixed, pointwise, a (reflecting) hyperplane, that is, a subspace of deficiency 1, and a reflection group to be a group generated by reflections. Chevalley **(1)** (and also Coxeter **(2)** together with Shephard and Todd **(4)**) has shown that a reflection group G, acting on a space of n dimensions, possesses a set of n algebraically independent (polynomial) invariants which form a polynomial basis for the set of all invariants of G. Our aim here is to prove:

THEOREM. *Let G be a finite reflection group, acting on a space V of finite dimension. Let J be the Jacobian* (*matrix*) *of a basic set of invariants of G, computed relative to any basis of V. Let p be any point of V. Then the following numbers are equal:*

(a) *the maximum number of linearly independent reflecting hyperplanes containing p;*

(b) *the maximum rank of $1 - x$ for all x in G for which $xp = p$;*

(c) *the nullity of J at p.*

The equality of the numbers defined in (b) and (c) is the essence of a conjecture of Shephard **(3)**.

Throughout the paper, G is a reflection group, of finite order g, acting on a space V of n dimensions. The symbols $L_1, \ldots, L_v$ denote the hyperplanes in which reflections of G take place, as well as non-zero linear forms which vanish on the corresponding hyperplanes, and for each i, a_i is a corresponding non-zero normal vector, r_i is the order of the (cyclic) subgroup of G which leaves L_i fixed pointwise, and R_i is a generator of this subgroup. Finally, $I_1, \ldots, I_n$ are basic invariants of G; $d_1, \ldots, d_n$ are their degrees; and J generically denotes their Jacobian, relative to whatever basis is at hand.

LEMMA. *For some non-zero scalar c,*

$$\det J = c \prod_{i=1}^{v} L_i^{r_i - 1}.$$

A proof of this well-known result will be included because it and the corollary below play a key role in the proof of the theorem. Choose an orthonormal basis of V so that the first co-ordinate x_1 is a multiple of L_1. If I is any invariant of G, the equation $R_1 I = I$ implies that I is a polynomial in ${x_1}^{r_1}$, whence

$$x_1^{r_1 - 1} \text{ divides } \partial I / \partial x_1.$$

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Thus the first row of J, and hence also $\det J$, is divisible by

$$x_1^{r_1-1}, \text{ and hence also by } L_1^{r_1-1}.$$

Similarly, $\det J$ is divisible by each $L_i^{r_i-1}$. Using the formula

$$\sum_{j=1}^{n} (d_j - 1) = \sum_{i=1}^{v} (r_i - 1),$$

proved in (**4**, p. 290, l. 12), a comparison of degrees shows that the factor c in the statement of the lemma is a scalar, non-zero because the I_j are algebraically independent.

From the first part of the proof we have:

COROLLARY. *The determinant of the Jacobian of any n invariants of G is divisible by* $\prod L_i^{r_i-1}$.

Proof of the theorem. If k, l, and m denote the respective numbers defined by (a), (b), and (c), we prove in turn that $m \leqslant k$, $k \leqslant l$, and $l \leqslant m$.

First label the L's so that $L_1, \ldots, L_u$ are those which contain p, and then choose an orthonormal basis $p_1, \ldots, p_n$ of V so that $p_1, \ldots, p_k$ span the same subspace as $a_1, \ldots, a_u$, the normals to the L's. Let G' be the (reflection) group generated by $R_1, \ldots, R_u$. The co-ordinates $x_{k+1} = I_{k+1}', \ldots, x_n = I_n'$ are invariants of G'. If $I_1', \ldots, I_k'$ are any invariants of G, they are also invariants of G', and the corollary above shows that

$$\prod_{1}^{u} L_i^{r_i-1}$$

divides

$$\partial(I_1', \ldots, I_n')/\partial(x_1, \ldots, x_n),$$

that is, divides

$$\partial(I_1', \ldots, I_k')/\partial(x_1, \ldots, x_k).$$

Consider now the expansion of $\det J$ across the first k rows:

$$\det J = \sum \pm J'(i_1, \ldots, i_k) J''(i_{k+1}, \ldots, i_n),$$

with $J'(i_1, \ldots, i_k)$ denoting the minor corresponding to the rows $1, \ldots, k$ and columns $i_1, \ldots, i_k$ of J, $J''(i_{k+1}, \ldots, i_n)$ denoting the minor corresponding to the rows $k+1, \ldots, n$ and columns $i_{k+1}, \ldots, i_n$, and the sum being over all permutations $i_1, \ldots, i_n$ of $1, \ldots, n$ for which $i_1 < \ldots < i_k$ and $i_{k+1} < \ldots < i_n$. By what has just been shown, each J' is divisible by

$$\prod_{1}^{u} L_i^{r_i-1},$$

so that, by the lemma, there are polynomials $M(i_1, \ldots, i_k)$ such that

$$\prod_{u+1}^{v} L_i^{r_i-1} = \sum M(i_1, \ldots, i_k) J''(i_{k+1}, \ldots, i_n).$$

Since the left side of this equation is not 0 at p, we conclude that some J'' is not 0 at p, whence J has rank $n - k$ at least and nullity k at most at p. Thus $m \leqslant k$.

Next, assume that the labelling is such that $L_1, \ldots, L_k$ contain p and are linearly independent. Set $x = R_1 R_2 \ldots R_k$. Suppose $xq = q$, with $q \in V$. Then $R_1^{-1} q = R_2 \ldots R_k q$ implies that

$$q + c_1 a_1 = q + c_2 a_2 + \ldots + c_k a_k$$

for suitable scalars c_j, whence, because of the linear independence of the a_j, we conclude that $c_1 = 0$ and $R_1 q = q$. Similarly $R_2 q = q, \ldots, R_k q = q$, hence q lies in each of $L_1, \ldots, L_k$, and the solution space of the equation $xq = q$ has dimension $n - k$. Thus $1 - x$ has rank k, and the inequality $k \leqslant l$ has been established.

Finally choose $x \in G$ so that $1 - x$ has rank l and $xp = p$, and then an orthonormal basis $p_1, \ldots, p_n$ of V so that $xp_j = c_j p_j$ with $c_j \neq 1$ for $1 \leqslant j \leqslant l$ and $c_j = 1$ for $l + 1 \leqslant j \leqslant n$. If I is an invariant of G, the equation $xI = I$ implies that each term of I has a total exponent in the co-ordinates $x_1, \ldots, x_l$ which is either 0 or at least 2. Thus for each j such that $1 \leqslant j \leqslant l$, $\partial I/\partial x_j$ is 0 at any point at which $x_1, \ldots, x_l$ are all 0, in particular, at p. This implies that the first l rows of J vanish at p, whence $l \leqslant m$.

Thus the theorem is completely proved.

References

1. C. Chevalley, *Invariants of finite groups generated by reflections*, Amer. J. Math., *77* (1955), 778.

2. H. S. M. Coxeter, *The product of the generators of a finite group generated by reflections*, Duke Math. J., *18* (1951), 765.

3. G. C. Shephard, *Some problems of finite reflection groups*, Enseignement Math., *II* (1956), 42.

4. G. C. Shephard and J. A. Todd, *Finite unitary reflection groups*, Can. J. Math., *6* (1954), 274.

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