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POSITIVE SOLUTIONS OF A NONLINEAR STURM–LIOUVILLE BOUNDARY-VALUE PROBLEM

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Abstract We establish the existence of positive solutions of the Sturm–Liouville problem

$$\begin{aligned} -(p(s,u)u')' &= \hat{q}(s)u^p h(s,u,u') \quad \text{in } (0,1), \\ u(0) &= 0 = u(1), \end{aligned}$$

where

$$p(s,u) = 1/(a(s) + cg(u)).$$

We assume g and $\hat{q}$ to be non-negative, continuous functions, $a(s)$ is a positive continuous function, $c \geqslant 0$, $p > 1$, and the function h is sub-quadratic with respect to u'. We combine *a priori* estimates with a fixed-point result of Krasnosel'skii to obtain the existence of a positive solution.

Keywords: Sturm–Liouville problems; positive solutions; fixed points; topological degree

2000 *Mathematics subject classification:* Primary 35J65; 34B15; 34B18

1. Introduction

We consider the second–order Sturm–Liouville problem

$$\left.\begin{aligned} -\left(\frac{u'}{a(s)+cg(u)}\right)' &= \hat{q}(s)u^p h(s,u,u') \quad \text{in } (0,1), \\ u(0) &= 0 = u(1), \end{aligned}\right\} \tag{1.1}$$

where $p > 1$, $c \geqslant 0$, $a : [0,1] \to (0,+\infty)$ is a continuous function, and $\hat{q} : [0,1] \to [0,+\infty)$ is a non-trivial, continuous function. We will assume that the function $g : [0,+\infty) \to [0,+\infty)$ is continuous and increasing, and is such that

$$\lim_{u\to+\infty} \frac{g(u)}{u^{(p-1)/(p+1)}} = 0. \tag{1.2}$$

In addition, we will assume that the nonlinearity h is continuous, as well as bounded from below, or in other words

$$c_h \leqslant h(s,u,\psi) \quad \text{for all } (s,u,\psi) \in [0,1] \times [0,\infty) \times \mathbb{R}, \tag{1.3}$$

where c_h is a positive constant. We will further assume the following sub-quadratic growth condition on the function h with respect to the derivative: given a compact set K in $[0,1]\times\mathbb{R}$, there exist positive constants A and B such that, for all $(s,u,\psi)\in K\times\mathbb{R}$, we have

$$h(s,u,\psi)\leqslant A+B\psi^2. \tag{1.4}$$

By *solutions* we will mean classical solutions, that is, $u\in C^1(0,1)\cap C^0[0,1]$ and $u'/(a(s)+cg(u))\in C^1(0,1)$, satisfying equation (1.1).

Our main result can be stated as follows.

Theorem 1.1. *The problem (1.1) has at least one positive solution.*

Certain difficulties which we may encounter while proving our main result are that the coefficient $p(s,u)=1/(a(s)+cg(u))$ is nonlinear and that it may not necessarily be bounded from below by a positive bound which is independent of u. In order to overcome these difficulties, we introduce a truncation $g_n(u)$ of the function $g(u)$ so that the new coefficient $p_n(s,u)=1/(a(s)+cg_n(u))$ becomes bounded from below by a positive constant depending on n (see (2.1)). This allows us to use a fixed-point argument for the truncation problem. Finally, we show the main result, proving that, for n sufficiently large, the solutions of the truncation problem are solutions of problem (1.1). A further difficulty in this argument is the dependence of the function h on the derivative, which leads us to the problem of establishing *a priori* bounds for the derivative. Observe that this dependence gives the problem a non-variational structure.

For further discussion on problems modelled by equations of the type $-(q(s,u)u')'=f(s,u,u')$ see, for example, [**1**, **2**, **4**–**7**, **9**, **10**, **12**–**17**]. For a study of existence of solutions when the coefficient $q(s,u)$ is constant on the variable u see, for example, [**1**, **2**, **5**, **6**, **9**, **10**, **13**, **14**, **16**]. For Sturm–Liouville problems where the coefficient $q(s,u)$ depends explicitly on the variable u, see, for example, [**4**, **7**, **15**, **17**]. Note that the problems studied in the preceding papers do not have the same structure as ours. For example, the phenomena of [**7**, **15**, **17**] are modelled by a negative nonlinearity $f(s,u,u')$, while in [**4**] the nonlinearity is bounded with respect to the variables u and u'. Observe that, in this work, $f(s,u,u')$ is both non-negative and unbounded. As a model example, consider the equation

$$-\left(\frac{u'}{a(s)+cu^q}\right)'=\hat{q}(s)u^p(c_0+c_1|u'|^\theta)\quad\text{in }(0,1),$$
$$u(0)=0=u(1),$$

where $1<p$, $0<q<(p-1)/(p+1)$, $0\leqslant\theta\leqslant 2$, $c\geqslant 0$, $c_0>0$ and $c_1\geqslant 0$.

Note that the partial case was studied in [**3**] by considering a linear elliptic operator and some growth assumption on the nonlinearity. This work is more in the line of [**11**], dealing with existence of solutions for some semilinear Sturm–Liouville equation without dependency on the derivative.

The paper is organized as follows. In § 2, we show the existence of positive solutions of the truncation problem. In § 3, we show that, for n sufficiently large, the solutions of the truncation problem are solutions of problem (1.1), which proves our main result, Theorem 1.1.

2. The truncation problem

Given $n \in \mathbb{N}$, we introduce a truncation $g_n(u)$ of the function $g(u)$ defined by

$$g_n(u) = \begin{cases} g(u) & \text{if } 0 \leqslant u \leqslant n, \\ g(n) & \text{if } u \geqslant n. \end{cases} \tag{2.1}$$

Consider the truncation problem

$$\left.\begin{aligned} -\left(\frac{u'}{a(s) + cg_n(u)}\right)' = \hat{q}(s)u^p h(s, u, u') \quad \text{in } (0,1), \\ u(0) = 0 = u(1). \end{aligned}\right\} \tag{2.2}$$

The following is an existence result for the truncation problem.

Theorem 2.1. *Suppose hypotheses (1.3) and (1.4) hold. Then problem (2.2) has at least one positive solution.*

The proof of Theorem 2.1 is based on the following well-known fixed-point result due to Krasnosel'skii, which we state without proof (cf. [**8**]).

Lemma 2.2. *Let C be a cone in a Banach space, and let $F : C \to C$ be a compact operator such that $F(0) = 0$. Suppose there exists an $r > 0$ verifying the condition*

(a) *$u \neq tF(u)$ for all $u \in C$ such that $\|u\| = r$ and $t \in [0,1]$.*

Suppose further that there exist a compact homotopy $H : [0,1] \times C \to C$ and an $R > r$ such that

(b) *$F(u) = H(0,u)$ for all $u \in C$,*

(c) *$H(t,u) \neq u$ for all $u \in C$ such that $\|u\| = R$ and $t \in [0,1]$,*

(d) *$H(1,u) \neq u$ for all $u \in C$ such that $\|u\| \leqslant R$.*

Then F has a fixed point $u_0 \in C$ verifying $r < \|u_0\| < R$.

In order to apply the preceding lemma, we need to establish *a priori* bounds for the solutions of a family of problems parametrized by $\lambda \geqslant 0$. In fact, consider the family

$$\left.\begin{aligned} -\left(\frac{u'}{a(s) + cg_n(u)}\right)' = \hat{q}(s)u^p h(s, u, u') + \lambda \quad \text{in } (0,1), \\ u(0) = 0 = u(1). \end{aligned}\right\} \tag{2.3}$$

We now present three lemmas which lead to the proof of Theorem 2.1. We begin with a result concerned with *a priori* bounds for the positive solutions of problem (2.3).

Lemma 2.3. *Suppose hypothesis (1.3) holds. Then there exists a positive constant B, which does not depend on λ, such that, for every positive solution u of problem (2.3), we have*

$$\|u\|_\infty \leqslant B. \tag{2.4}$$

Proof. It is not difficult to show that every positive solution u of problem (2.3) satisfies

$$u(t) = \int_0^1 K_n(t,s)(\hat{q}(s)u^p h(s,u,u') + \lambda)\,\mathrm{d}s, \tag{2.5}$$

where $K_n(t,s)$ is the associated Green function

$$K_n(t,s) = \begin{cases} \dfrac{1}{\rho}\displaystyle\int_0^t (a(\tau) + cg_n(u(\tau)))\,\mathrm{d}\tau \int_s^1 (a(\tau) + cg_n(u(\tau)))\,\mathrm{d}\tau & \text{if } 0 \leqslant t \leqslant s \leqslant 1, \\ \dfrac{1}{\rho}\displaystyle\int_0^s (a(\tau) + cg_n(u(\tau)))\,\mathrm{d}\tau \int_t^1 (a(\tau) + cg_n(u(\tau)))\,\mathrm{d}\tau & \text{if } 0 \leqslant s \leqslant t \leqslant 1. \end{cases} \tag{2.6}$$

Here ρ is given by

$$\rho = \int_0^1 (a(\tau) + cg_n(u(\tau)))\,\mathrm{d}\tau.$$

Simple computations show that every solution u satisfies (cf. [**2**])

$$u(s) \geqslant q(s)\|u\|_\infty \quad \text{for all } s \in [0,1], \tag{2.7}$$

where

$$q(s) = \frac{1}{\rho}\min\biggl\{\int_0^s (a(\tau) + cg_n(u(\tau)))\,\mathrm{d}\tau,\ \int_s^1 (a(\tau) + cg_n(u(\tau)))\,\mathrm{d}\tau\biggr\}.$$

Moreover, recalling that g is increasing,

$$q(s) \geqslant \frac{\min a}{\|a\|_\infty + cg(n)} s(1-s) \quad \text{for all } s \in [0,1] \tag{2.8}$$

and

$$K_n(t,s) \geqslant \begin{cases} \dfrac{(\min a)^2}{\|a\|_\infty + cg(n)} t(1-s) & \text{if } 0 \leqslant t \leqslant s \leqslant 1, \\ \dfrac{(\min a)^2}{\|a\|_\infty + cg(n)} s(1-t) & \text{if } 0 \leqslant s \leqslant t \leqslant 1. \end{cases} \tag{2.9}$$

Hypothesis (1.3) implies that every positive solution u of problem (2.3) satisfies

$$u(t) \geqslant \frac{c_h(\min a)^{p+2}}{(\|a\|_\infty + cg(n))^{p+1}}\|u\|_\infty^p \int_0^1 G(t,s)\hat{q}(s)s^p(1-s)^p\,\mathrm{d}s \tag{2.10}$$

for all $t \in [0,1]$, where

$$G(t,s) = \begin{cases} t(1-s) & \text{if } 0 \leqslant t \leqslant s \leqslant 1, \\ s(1-t) & \text{if } 0 \leqslant s \leqslant t \leqslant 1. \end{cases} \tag{2.11}$$

The existence of *a priori* bounds B for the positive solutions u now follows. □

The following shows the existence of *a priori* bounds for the derivatives of the solutions.

Lemma 2.4. *Suppose hypotheses (1.3) and (1.4) hold. Then, for all $\bar\lambda$ positive, there exists a constant B' such that, for $\lambda \in [0,\bar\lambda]$, every positive solution of problem (2.3) satisfies*

$$\|u'\|_\infty \leqslant B'. \tag{2.12}$$

Proof. By Lemma 2.3 and hypothesis (1.4) we know that there exist positive constants A_n and B_n such that, for all $s \in [0,1]$, every positive solution u of problem (2.3) satisfies

$$h(s,u(s),u'(s)) \leqslant A_n + B_n u'(s)^2. \tag{2.13}$$

Therefore,

$$h(s,u(s),u'(s)) \leqslant A_n + C_n\biggl(\frac{u'(s)}{a(s)+cg_n(u)}\biggr)^2, \tag{2.14}$$

where $C_n = B_n(\|a\|_\infty + cg(n))^2$.

Let u be a positive solution of problem (2.3). Note that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\ln\biggl(\lambda + B^p\|\hat q\|_\infty A_n + B^p\|\hat q\|_\infty C_n\biggl(\frac{u'(s)}{a(s)+cg_n(u)}\biggr)^2\biggr)&\\ = \frac{-2B^p\|\hat q\|_\infty C_n u'(s)/(a(s)+cg_n(u))(\hat q(s)u^p h(s,u,u')+\lambda)}{\lambda + B^p\|\hat q\|_\infty A_n + B^p\|\hat q\|_\infty C_n(u'(s)/(a(s)+cg_n(u)))^2}&.\end{aligned}$$

According to inequality (2.14) and Lemma 2.3, if $u'(s) < 0$, then

$$\frac{\mathrm{d}}{\mathrm{d}s}\ln\biggl(\lambda + B^p\|\hat q\|_\infty A_n + B^p\|\hat q\|_\infty C_n\biggl(\frac{u'(s)}{a(s)+cg_n(u)}\biggr)^2\biggr) \leqslant -2\frac{C_n B^p\|\hat q\|_\infty}{\min a}u'(s), \tag{2.15}$$

and if $u'(s) > 0$, then

$$\frac{\mathrm{d}}{\mathrm{d}s}\ln\biggl(\lambda + B^p\|\hat q\|_\infty A_n + B^p\|\hat q\|_\infty C_n\biggl(\frac{u'(s)}{a(s)+cg_n(u)}\biggr)^2\biggr) \geqslant -2\frac{C_n B^p\|\hat q\|_\infty}{\min a}u'(s). \tag{2.16}$$

On the other hand, let $\hat f(s) = \hat q(s)u^p(s)h(s,u(s),u'(s)) + \lambda$, $\hat p(s) = a(s) + cg_n(u(s))$ and let t_* be such that $u'(t_*) = 0$. Then identity (2.5) implies that

$$\int_0^{t_*}\int_0^s \hat p(\tau)\hat f(s)\,\mathrm{d}\tau\,\mathrm{d}s = \int_{t_*}^1\int_s^1 \hat p(\tau)\hat f(s)\,\mathrm{d}\tau\,\mathrm{d}s.$$

Hence, t_* is the unique number in $(0,1)$ such that $u(t_*) = \|u\|_\infty$, and such that u is increasing on $[0,t_*)$, while decreasing on $(t_*,1]$. Integration of (2.15) and (2.16) on the intervals $[t_*,s]$ and $[s,t_*]$, respectively, yields

$$\begin{aligned}\ln\biggl(\lambda + B^p\|\hat q\|_\infty A_n + B^p\|\hat q\|_\infty C_n\biggl(\frac{u'(s)}{a(s)+cg_n(u)}\biggr)^2\biggr)&\\ \leqslant 2\frac{C_n B^p\|\hat q\|_\infty}{\min a}(\|u\|_\infty - u(s)) + \ln(\bar\lambda + B^p\|\hat q\|_\infty A_n)&. \end{aligned}\tag{2.17}$$

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Therefore,

$$\ln(\lambda + B^p\|\hat{q}\|_\infty A_n + B^p\|\hat{q}\|_\infty B_n u'(s)^2) \leqslant 2\frac{C_n B^{p+1}\|\hat{q}\|_\infty}{\min a} + \ln(\bar{\lambda} + B^p\|\hat{q}\|_\infty A_n) \quad (2.18)$$

for every $s \in [0,1]$ and for every $\lambda \in [0,\bar{\lambda}]$.

The lemma clearly results from this inequality. □

We need the following result. Consider the Banach space

$$X = \mathcal{C}^1([0,1],\mathbb{R})$$

endowed with the norm $\|u\|_1 = \|u\|_\infty + \|u'\|_\infty$.

Define the cone C by

$$C = \{u \in X : u \geqslant 0 \text{ and } u(0) = u(1) = 0\}$$

and the operator $\mathcal{F}_\lambda : X \to X$ by

$$\mathcal{F}_\lambda(u)(s) = \int_0^1 K_n(s,\tau)(\hat{q}(\tau)u(\tau)^p h(\tau,u(\tau),u'(\tau)) + \lambda)\,\mathrm{d}\tau.$$

Lemma 2.5. *The operator $\mathcal{F}_\lambda : X \to X$ is compact, and the cone C is invariant under $\mathcal{F}_\lambda$.*

Outline of the proof. The compactness of $\mathcal{F}_\lambda$ follows from the well-known Arzelà–Ascoli theorem. The invariance of the cone C is a consequence of the fact that the nonlinearities are non-negative.

Proof of Theorem 2.1. To prove Theorem 2.1, it suffices to show that $\mathcal{F}_0$ has a fixed point. For this, we will check that the four conditions of Lemma 2.2 are satisfied. Fix a suitable positive constant λ and consider the homotopy $H : [0,1] \times C \to C$ given by

$$H(t,u)(s) = \mathcal{F}_{\lambda t}(u)(s).$$

Note that $H(t,u)$ is a compact homotopy, and since $H(0,u) = \mathcal{F}_0(u)$ we have that condition (b) is satisfied.

Concerning condition (a), by continuity, there exists a $M > 0$ such that, if $\|u\|_1 \leqslant 1$, then

$$|K_n(s,\tau)h(\tau,u(\tau),u'(\tau))| \leqslant M. \quad (2.19)$$

Note that there exists a constant $\tilde{c}$ such that

$$\|\mathcal{F}_0(u)\|_\infty, \|\mathcal{F}_0(u)'\|_\infty \leqslant \tilde{c}\|u\|_\infty^{p-1}.$$

Thus, if $\|u\|_1 = \delta$, with $0 < \delta \leqslant 1$, then

$$\|\mathcal{F}_0(u)\|_1 = \|\mathcal{F}_0(u)\|_\infty + \|\mathcal{F}_0(u)'\|_\infty \leqslant C_0\delta^{p-1}\|u\|_1,$$

where C_0 is a positive constant. Taking $\delta \in (0, C_0^{-1/(p-1)})$, we have

$$\|\mathcal{F}_0(u)\|_1 < \|u\|_1. \quad (2.20)$$

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From Lemmas 2.3 and 2.4 we conclude that there exists an R sufficiently large that condition (c) is satisfied.

In order to verify condition (d) we use the following subsidiary lemma.

Lemma 2.6. *There exists a $\lambda_0 > 0$ such that problem (2.3) has no positive solutions for $\lambda > \lambda_0$.*

Proof. Let u be a positive solution of problem (2.3), or in other words

$$u(t) = \int_0^1 K_n(s,t)(\hat{q}(s)u(s)^p h(s,u(s),u'(s)) + \lambda)\,\mathrm{d}s.$$

Then

$$\|u\|_\infty \geqslant \lambda \int_0^1 K_n(s,\tfrac{1}{2})\,\mathrm{d}s.$$

By Lemma 2.3 we know that $\|u\|_\infty \leqslant B$, and hence

$$\lambda \leqslant \frac{B}{\int_0^1 K_n(s,\frac{1}{2})\,\mathrm{d}s}.$$

Therefore, for

$$\lambda > \frac{B}{\int_0^1 K_n(s,\frac{1}{2})\,\mathrm{d}s}$$

there are no positive solutions of problem (2.3). □

So, choosing $\lambda > \lambda_0$ in the homotopy $H(t,u)$, we see that condition (d) is satisfied by Lemma 2.6.

Thus, all of Krasnosel'skii's conditions are satisfied. □

3. Proof of Theorem 1.1

The proof of Theorem 1.1 is a direct consequence of the following.

Lemma 3.1. *There exists an $n_0 \in \mathbb{N}$ such that every positive solution u of problem $(2.2)_{n_0}$ satisfies*

$$\|u\|_\infty < n_0. \tag{3.1}$$

Proof. By contradiction, there would exist a sequence of solutions $\{u_n\}_n$ of problem (2.2) such that $\|u_n\|_\infty \geqslant n$ for all $n \in \mathbb{N}$. Using the same argument as in Lemma 2.3 (see (2.10)), we would obtain the estimate

$$\begin{aligned} 1 &\geqslant \frac{(\min a)^2}{\|a\|_\infty + cg(n)}\left(\frac{\min a}{\|a\|_\infty + cg(n)}\right)^p c_h \|u_n\|_\infty^{p-1} \max_{t\in[0,1]} \int_0^1 G(t,s)\hat{q}(s)s^p(1-s)^p\,\mathrm{d}s \\ &\geqslant (\min a)^{p+2} c_h \left(\frac{n^{(p-1)/(p+1)}}{\|a\|_\infty + cg(n)}\right)^{p+1} \max_{t\in[0,1]} \int_0^1 G(t,s)\hat{q}(s)s^p(1-s)^p\,\mathrm{d}s. \end{aligned}$$

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But this is impossible, since

$$\lim_{n\to+\infty} \frac{n^{(p-1)/(p+1)}}{\|a\|_\infty + cg(n)} = +\infty$$

by hypothesis (1.2). □

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