Canad. Math. Bull. Vol. **60** (4), 2017 pp. 879–890
http://dx.doi.org/10.4153/CMB-2017-045-2

Triangulated Equivalences Involving Gorenstein Projective Modules

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Abstract. For any ring R, we show that, in the bounded derived category $D^b(\operatorname{Mod} R)$ of left R-modules, the subcategory of complexes with finite Gorenstein projective (resp. injective) dimension modulo the subcategory of complexes with finite projective (resp. injective) dimension is equivalent to the stable category $\underline{\mathbf{GP}}(\operatorname{Mod} R)$ (resp. $\overline{\mathbf{GI}}(\operatorname{Mod} R)$) of Gorenstein projective (resp. injective) modules. As a consequence, we get that if R is a left and right noetherian ring admitting a dualizing complex, then $\underline{\mathbf{GP}}(\operatorname{Mod} R)$ and $\overline{\mathbf{GI}}(\operatorname{Mod} R)$ are equivalent.

1 Introduction

As a generalization of finitely generated projective modules, Auslander and Bridger [AuB] introduced the notion of finitely generated modules having G-dimension zero over Noetherian rings. Enochs and Jenda [EJ1] generalized it to that of Gorenstein projective modules for arbitrary modules over a general ring. Then a finitely generated module having G-dimension zero is exactly Gorenstein projective.

Let R be a finite-dimensional Gorenstein algebra over a field. Buchweitz proved in [Bu] that the stable category $\underline{\mathbf{GP}}(\operatorname{mod} R)$ of all finitely generated Gorenstein projective left R-modules is equivalent to the bounded derived category of finitely generated left R-modules modulo the bounded homotopy category of all finitely generated projective left R-modules; see also [H2].

Veliche [V] defined the Gorenstein projective dimension for complexes of left modules over any ring. It is a refinement of the projective dimension for complexes. Let R be any ring. In the bounded derived category $D^b(\operatorname{Mod} R)$ of left R-modules, we first show that the subcategory of complexes with finite Gorenstein projective dimension is a triangulated category (Proposition 3.2). Then we show that the subcategory of complexes with finite Gorenstein projective dimension modulo the subcategory of complexes with finite projective dimension is equivalent to the stable category $\underline{\mathbf{GP}}(\operatorname{Mod} R)$ of Gorenstein projective modules (Theorem 3.4). It is an extension of the Buchweitz's result mentioned above (see Proposition 3.10).

The notion of dualizing complexes is important in commutative algebra (see, for example, [AvF]). A. Yekutieli [Y], and recently, Iyengar and Krause [IK], generalized

Received by the editors July 21, 2016; revised May 10, 2017.
Published electronically September 4, 2017.
This research was partially supported by NSFC (Grant No. 11571164), a Project Funded by the Priority Academic Program Development of Jiangsu Higher Education Institutions and Chinese Universities Scientific Fund (Grant No. 2452016194).
AMS subject classification: **18G25**, 16E35.
Keywords: triangulated equivalence, Gorenstein projective module, stable category, derived category, homotopy category.

it to the non-commutative case. For a left and right noetherian ring R admitting a dualizing complex, we obtain an equivalence between the stable category of Gorenstein projective left R-modules and that of Gorenstein injective left R-modules (Proposition 3.7). If R is a finite-dimensional algebra, this equivalence was obtained by Beligiannis and Reiten [BR].

The equivalence in Theorem 3.4 links a Verdier quotient and an additive quotient. After finishing the paper, we find that there exists another result associated with the two kinds of quotients arising from tilting theory [IYa]. In the appendix, we give a sketch of how our result can be obtained from [IYa, Theorem 4.7]. We thank Dong Yang for some conversations on this topic.

Note that the stable category of Gorenstein projective (resp. injective) modules can also be realized as the homotopy category of some model category. It means that the Verdier quotients we constructed can be obtained by Quillen model structures; see [DEH] for the details.

2 Preliminaries

Throughout this paper, R is an associative ring with unit, and all modules considered are left R-modules unless stated otherwise. We use $\operatorname{Mod} R$ (resp. $\operatorname{mod} R$) to denote the category of left R-modules (resp. finitely presented left R-modules). We use $\operatorname{Proj} R$ (resp. $\operatorname{proj} R$) to denote the subcategory of $\operatorname{Mod} R$ (resp. $\operatorname{mod} R$) consisting of projective modules.

Let R be a ring. We write a complex as

$$X^\bullet := \cdots \longrightarrow X^{i-1} \xrightarrow{d^{i-1}_{X^\bullet}} X^i \xrightarrow{d^{i}_{X^\bullet}} X^{i+1} \xrightarrow{d^{i+1}_{X^\bullet}} X^{i+2} \longrightarrow \cdots,$$

where $d^i_{X^\bullet}$ is the i-th differential of $X^\bullet$. Any left R-module M is regarded as the *stalk complex*, that is, a complex concentrated in degree zero. We denote the homology complex of $X^\bullet$ by $H(X^\bullet)$. A *chain map* (or *a morphism*) from $X^\bullet$ to $Y^\bullet$ is a collection $\{f^i\}_{i\in\mathbb{Z}}$, where $f^i\colon X^i \to Y^i$ is a morphism in $\operatorname{Mod} R$ commuting with the differentials for any $i \in \mathbb{Z}$. Let $f\colon X^\bullet \to Y^\bullet$ be a chain map. We denote by $X^\bullet[1]$ the 1-*shift* of $X^\bullet$ to the left with $X^i[1] = X^{i+1}$ and $d^i_{X^\bullet[1]} = -d^{i+1}_{X^\bullet}$ for any $i \in \mathbb{Z}$ (the ring of integers). The chain map f is called a *quasi-isomorphism* if $H(f)\colon H(X^\bullet) \to H(Y^\bullet)$ is an isomorphism.

A complex $T^\bullet$ of left R-modules is called *totally acyclic* if it consists of projective modules such that $\operatorname{Hom}_R(T^\bullet, P)$ remains exact for any projective module P. A left R-module M is called *Gorenstein projective* if there exists a totally acyclic complex $T^\bullet$ such that M is isomorphic to some cokernel of $T^\bullet$. Dually, the notions of *cototally acyclic complexes* and *Gorenstein injective modules* are defined.

Let $X^\bullet$ be a complex of left R-modules. A *projective resolution* of $X^\bullet$ is a quasi-isomorphism $f\colon P^\bullet \to X^\bullet$ such that $P^\bullet$ consists of projective modules and $\operatorname{Hom}_R(P^\bullet, \cdot)$ preserves exact complexes. The projective dimension of $X^\bullet$ is *finite* if there exists some projective resolution $P^\bullet$ such that $P^i = 0$ for $i \ll 0$. Note that in the bounded derived category $D^b(\operatorname{Mod} R)$, the subcategory of complexes with finite projective dimension is isomorphic to the bounded homotopy category $K^b(\operatorname{Proj} R)$ of all projective left R-modules.

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With every complex $X^\bullet$ we associate two numbers:

$$\sup X^\bullet := \sup\{i \mid X^i \neq 0\} \quad \text{and} \quad \inf X^\bullet := \inf\{i \mid X^i \neq 0\}.$$

$X^\bullet$ is called *bounded above* if $\sup X^\bullet < \infty$; $X^\bullet$ is called *bounded below* if $\inf X^\bullet > -\infty$; and $X^\bullet$ is called *bounded* if it is bounded above and below. For every complex $X^\bullet$ and $n \in \mathbb{Z}$, there is a kind of operation called the *hard truncation*:

$$X^\bullet_{\sqsubset n} := \cdots \longrightarrow X^{n-1} \longrightarrow X^n \longrightarrow 0 \longrightarrow \cdots,$$
$$X^\bullet_{\sqsupset n} := \cdots \longrightarrow 0 \longrightarrow X^n \longrightarrow X^{n+1} \longrightarrow \cdots.$$

There certainly exists a triangle

$$X^\bullet_{\sqsupset n} \longrightarrow X^\bullet \longrightarrow X^\bullet_{\sqsubset n-1} \longrightarrow X^\bullet_{\sqsupset n}[1]$$

in $K(\operatorname{Mod} R)$.

For an additive category $\mathcal{A}$, we denote by $C(\mathcal{A})$ and $K(\mathcal{A})$ the category of complexes of $\mathcal{A}$ and the homotopy category of complexes of $\mathcal{A}$, respectively. If $\mathcal{A}$ is abelian, we denote by $D(\mathcal{A})$ the derived category of $\mathcal{A}$. The bounded versions of $C(\mathcal{A})$, $K(\mathcal{A})$ and $D(\mathcal{A})$ are denoted by $C^b(\mathcal{A})$, $K^b(\mathcal{A})$ and $D^b(\mathcal{A})$, respectively.

3 Main Results

Let $X^\bullet$ be a complex of left R-modules. Following [V], $X^\bullet$ is said to *have finite Gorenstein projective dimension* if there exists a diagram of morphisms of complexes

$$T^\bullet \xrightarrow{\tau} P^\bullet \xrightarrow{\pi} X^\bullet, \tag{3.1}$$

where $P^\bullet \xrightarrow{\pi} X^\bullet$ is a projective resolution of $X^\bullet$ and $T^\bullet$ is a totally acyclic complex with τ^i bijective for $i \ll 0$. In this case, (3.1) is called a *complete resolution* of $X^\bullet$. Clearly, the finiteness of the projective dimension of a complex implies the finiteness of its Gorenstein projective dimension.

Note that, if $X^\bullet$ has finite Gorenstein projective dimension, then $H(X^\bullet)$ is bounded below. If, in addition, $H(X^\bullet)$ is bounded above, then there exists a bounded complex $G^\bullet$ consisting of Gorenstein projective modules such that $G^\bullet \cong X^\bullet$ in $D(\operatorname{Mod} R)$ (see [V, Construction 5.5]).

A *Gorenstein projective resolution* of $X^\bullet$ is a complex $G^\bullet$ consisting of Gorenstein projective modules such that $G^\bullet \cong X^\bullet$ in $D(\operatorname{Mod} R)$. Such a resolution is *finite* if $G^i = 0$ for all $|i| \gg 0$; and it is *special* if it is finite, $\sup G^\bullet = \sup H(X^\bullet)$ and G^i is projective for any $i < \sup H(X^\bullet)$. Also by [V, Construction 5.7], if $H(X^\bullet)$ is bounded and $X^\bullet$ admits a special Gorenstein projective resolution, then $X^\bullet$ has finite Gorenstein projective dimension.

Remark 3.1 Let $X^\bullet$ be a complex of left R-modules. If $H(X^\bullet)$ is bounded, then $X^\bullet$ has finite Gorenstein projective dimension if and only if it admits a special Gorenstein projective resolution. Thus, the definition of finite Gorenstein projective dimension for complexes is well defined in $D^b(\operatorname{Mod} R)$.

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Proposition 3.2 *All homology bounded complexes with finite Gorenstein projective dimension form a triangulated full subcategory of $D^b(\operatorname{Mod} R)$, denoted by $D^b(\operatorname{Mod} R)_{fGP}$.*

Proof As noted in Remark 3.1, $D^b(\operatorname{Mod} R)_{fGP}$ is closed under isomorphisms in $D^b(\operatorname{Mod} R)$; moreover, it is closed under shifts. We only need to show that $D^b(\operatorname{Mod} R)_{fGP}$ is closed under mapping cones. Let

$$X^\bullet \longrightarrow Y^\bullet \longrightarrow Z^\bullet \longrightarrow X^\bullet[1]$$

be a triangle in $D^b(\operatorname{Mod} R)$. We can assume that it is induced by an exact sequence of complexes

$$0 \longrightarrow X^\bullet \longrightarrow Y^\bullet \longrightarrow Z^\bullet \longrightarrow 0$$

in $C(\operatorname{Mod} R)$. Now the assertion follows from [V, Theorem 3.9]. ∎

Let $\mathcal{A}$ be an exact category in the sense of [Q]. If $\mathcal{A}$ has enough projectives and injectives and the projectives coincide with the injectives, then $\mathcal{A}$ is called a *Frobenius category*. We use $\mathcal{I}$ to denote the class of projective-injective objects of $\mathcal{A}$. Then the stable category $\mathcal{A}/\mathcal{I}$ is a triangulated category by [H1]. The triangles are induced by the pushout as follows:

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d, "u"'] & I(X) \arrow[r] \arrow[d] & T(X) \arrow[r] \arrow[d, "="'] & 0 \\
0 \arrow[r] & Y \arrow[r] & C(u) \arrow[r] & T(X) \arrow[r] & 0,
\end{tikzcd}$$

where $I(X)$ is an injective object, $C(u)$ is the pushout, and $T(X)$ is the first cosyzygy of X. Then

$$X \xrightarrow{u} Y \longrightarrow C(u) \longrightarrow T(X)$$

is a triangle in $\mathcal{A}/\mathcal{I}$, where T is the shift functor in $\mathcal{A}/\mathcal{I}$.

Let $\mathcal{T}$ be a triangulated category and $\mathcal{K}$ a triangulated subcategory of $\mathcal{T}$ closed under summands, that is, a *thick subcategory*. Then by [GM] we can form the Verdier quotient $\mathcal{T}/\mathcal{K}$. It is also a triangulated category. Note that the category of Gorenstein projective R-modules, denoted by $\mathbf{GP}(\operatorname{Mod} R)$, is a Frobenius category whose projective-injective objects are all projective modules. Hence the stable category $\underline{\mathbf{GP}}(\operatorname{Mod} R)$ is a triangulated category. Also by Proposition 3.2, we have that $D^b(\operatorname{Mod} R)_{fGP}$ is a triangulated category. Obviously, $K^b(\operatorname{Proj} R)$ is a triangulated subcategory of $D^b(\operatorname{Mod} R)_{fGP}$; moreover, it is closed under direct summands. Thus, $D^b(\operatorname{Mod} R)_{fGP}/K^b(\operatorname{Proj} R)$ is also a triangulated category.

Lemma 3.3 *Let M be Gorenstein projective in* $\operatorname{Mod} R$. *If $P^\bullet \in K^b(\operatorname{Proj} R)$ with $P^i = 0$ for any $i \geq 0$, then $\operatorname{Hom}_{D(\operatorname{Mod} R)}(M, P^\bullet) = 0$.*

Proof Note that $\operatorname{Hom}_{D(\operatorname{Mod} R)}(M, P[i]) \cong \operatorname{Ext}^i_R(M, P) = 0$ for any projective left R-module P and $i \geq 1$. Now we can get the assertion by using induction on the width of $P^\bullet$ in $K^b(\operatorname{Proj} R)$. ∎

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Notice that any Gorenstein projective module as a complex has finite Gorenstein projective dimension, so there exists an embedding: $\mathbf{GP}(\mathrm{Mod}\,R) \hookrightarrow D^b(\mathrm{Mod}\,R)_{fGP}$. Let F be the composition

$$\mathbf{GP}(\mathrm{Mod}\,R) \hookrightarrow D^b(\mathrm{Mod}\,R)_{fGP} \longrightarrow D^b(\mathrm{Mod}\,R)_{fGP}/K^b(\mathrm{Proj}\,R),$$

where the latter is the natural quotient functor. It is clear that F sends projective modules to 0 in $D^b(\mathrm{Mod}\,R)_{fGP}/K^b(\mathrm{Proj}\,R)$, so it factors through $\underline{\mathbf{GP}}(\mathrm{Mod}\,R)$. As a consequence, there exists a functor $\overline{F}\colon \underline{\mathbf{GP}}(\mathrm{Mod}\,R) \to D^b(\mathrm{Mod}\,R)_{fGP}/K^b(\mathrm{Proj}\,R)$ such that $F = \overline{F}\pi$, where $\pi\colon \mathbf{GP}(\mathrm{Mod}\,R) \to \underline{\mathbf{GP}}(\mathrm{Mod}\,R)$ is the natural quotient functor. Our main result is the following theorem.

Theorem 3.4 *The functor*

$$\overline{F}\colon \underline{\mathbf{GP}}(\mathrm{Mod}\,R) \longrightarrow D^b(\mathrm{Mod}\,R)_{fGP}/K^b(\mathrm{Proj}\,R)$$

is a triangulated equivalence.

Proof We will show that $\overline{F}$ is a triangulated functor and it is essentially surjective (or dense), full and faithful.

(1) $\overline{F}$ is a triangulated functor.

Let

$$X \xrightarrow{u} Y \to C(u) \to T(X)$$

be a triangle in $\underline{\mathbf{GP}}(\mathrm{Mod}\,R)$. Then it comes from a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d, "u"'] & I(X) \arrow[r] \arrow[d] & T(X) \arrow[r] \arrow[d, "="'] & 0 \\
0 \arrow[r] & Y \arrow[r] & C(u) \arrow[r] & T(X) \arrow[r] & 0
\end{tikzcd}$$

in $\mathbf{GP}(\mathrm{Mod}\,R)$. This yields a commutative diagram of triangles

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "u"'] & I(X) \arrow[r] \arrow[d] & T(X) \arrow[r] \arrow[d, "="'] & X[1] \arrow[d, "u[1]"'] \\
Y \arrow[r] & C(u) \arrow[r] & T(X) \arrow[r] & Y[1]
\end{tikzcd}$$

in $D^b(\mathrm{Mod}\,R)_{fGP}$ with $[1]$ the shift functor. It is sent to a commutative diagram of triangles in $D^b(\mathrm{Mod}\,R)_{fGP}/K^b(\mathrm{Proj}\,R)$. We have $T(X) \cong X[1]$, since $I(X)$ is zero in $D^b(\mathrm{Mod}\,R)_{fGP}/K^b(\mathrm{Proj}\,R)$. Thus

$$X \longrightarrow Y \longrightarrow C(u) \longrightarrow X[1]$$

is a triangle in $D^b(\mathrm{Mod}\,R)_{fGP}/K^b(\mathrm{Proj}\,R)$ and $\overline{F}$ is a triangulated functor.

(2) $\overline{F}$ is essentially surjective (or dense).

Let $X^\bullet$ be any complex in $D^b(\mathrm{Mod}\,R)_{fGP}/K^b(\mathrm{Proj}\,R)$. We can assume that $X^\bullet$ is bounded on both sides. By definition, there exists a complete resolution

$$T^\bullet \xrightarrow{\tau} P^\bullet \xrightarrow{\pi} X^\bullet$$

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of $X^\bullet$. Suppose that τ^i is bijective in degree $\le t$. Note that $P^\bullet$ can be selected to be bounded on the right. Write

$$P^\bullet := \cdots \longrightarrow P^{t-1} \longrightarrow P^t \longrightarrow \cdots \longrightarrow P^{s-2} \longrightarrow P^{s-1} \longrightarrow P^s \longrightarrow 0.$$

Then we have a triangle

$$P^\bullet_{\sqsupset_{t+1}} \longrightarrow P^\bullet \longrightarrow P^\bullet_{\sqsubset_t} \longrightarrow P^\bullet_{\sqsupset_{t+1}}[1]$$

in $K(\operatorname{Mod} R)$, which is sent to a triangle in $D^b(\operatorname{Mod} R)_{fGP}/K^b(\operatorname{Proj} R)$. Hence, $P^\bullet \cong P^\bullet_{\sqsubset_t}$ in $D^b(\operatorname{Mod} R)_{fGP}/K^b(\operatorname{Proj} R)$. We see that $P^\bullet_{\sqsubset_t}$ is a projective resolution of $C^{t-1}_{P^\bullet}$ and $C^{t-1}_{P^\bullet}$ is Gorenstein projective, where $C^{t-1}_{P^\bullet} = \operatorname{Coker} d^{t-1}_{P^\bullet}$. Hence, $X^\bullet \cong \mathbf{C}^{t-1}_{P^\bullet}$ in $D^b(\operatorname{Mod} R)_{fGP}/K^b(\operatorname{Proj} R)$. This implies that $\overline{F}$ is dense.

(3) F and $\overline{F}$ are full.

Let

$$X \overset{f}{\longleftarrow} Z^\bullet \overset{g}{\longrightarrow} Y$$

be a diagram in $D^b(\operatorname{Mod} R)_{fGP}/K^b(\operatorname{Proj} R)$ with X, Y Gorenstein projective, where f lies in the compatible saturated multiplicative system corresponding to $K^b(\operatorname{Proj} R)$. Complete f to a triangle

$$X[-1] \overset{\omega}{\longrightarrow} Q^\bullet \longrightarrow Z^\bullet \overset{f}{\longrightarrow} X$$

with $Q^\bullet \in K^b(\operatorname{Proj} R)$. Consider the triangle

$$Q^\bullet_{\sqsupset 1} \overset{\iota}{\longrightarrow} Q^\bullet \overset{\varphi}{\longrightarrow} Q^\bullet_{\sqsubset 0} \longrightarrow Q^\bullet_{\sqsupset 1}[1]$$

in $K^b(\operatorname{Proj} R)$. Since $\operatorname{Hom}_{D(\operatorname{Mod} R)}(X[-1], Q^\bullet_{\sqsubset 0}) = \operatorname{Hom}_{D(\operatorname{Mod} R)}(X, Q^\bullet_{\sqsubset 0}[1]) = 0$ by Lemma 3.3, it follows that $\varphi\omega = 0$ and ω factors through ι. Consider the diagram

$$\begin{tikzcd}
X[-1] \arrow[r] \arrow[d, "="'] & Q^\bullet_{\sqsupset 1} \arrow[r] \arrow[d, "\iota"'] & Z'^\bullet \arrow[r, "s"] \arrow[d, "l"'] & X \arrow[d, "="'] \\
X[-1] \arrow[r, "\omega"] & Q^\bullet \arrow[r] & Z^\bullet \arrow[r, "f"] & X,
\end{tikzcd}$$

where s, l, f are all in the compatible saturated multiplicative system corresponding to $K^b(\operatorname{Proj} R)$. Since $\operatorname{Hom}_{D(R)}(Q^\bullet_{\sqsupset 1}, Y) = \operatorname{Hom}_{K(R)}(Q^\bullet_{\sqsupset 1}, Y) = 0$, there exists some $h: X \to Y$ such that $gl = hs = hfl$. So we have that $h = gf^{-1}$ and F is full. Since $F = \overline{F}\pi$, it follows that $\overline{F}$ is also full.

(4) $\overline{F}$ is faithful.

Suppose that there exists a morphism $f: X \to Y$ in $\underline{\mathbf{GP}}(\operatorname{Mod} R)$ such that $\overline{F}(f) = 0$. Complete it to a triangle

$$X \overset{f}{\longrightarrow} Y \overset{g}{\longrightarrow} Z \to X[1]$$

in $\underline{\mathbf{GP}}(\operatorname{Mod} R)$. Since $\overline{F}(f) = 0$, we have that $\overline{F}(g)$ is a section. By (3), $\overline{F}$ is full. So there exists some map $\alpha: Z \to Y$ such that $1_{\overline{F}(Y)} = \overline{F}(\alpha g)$, and also there exists some map $\beta: Y \to Y$ satisfying $\overline{F}(\beta) = 1_{\overline{F}(Y)}$ such that $\beta = \alpha g$. Consider the triangle

$$Y \overset{\beta}{\longrightarrow} Y \longrightarrow C(\beta) \longrightarrow Y[1]$$

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in $\underline{\mathbf{GP}}(\operatorname{Mod} R)$. We have that $\overline{F}[C(\beta)] \in K^b(\operatorname{Proj} R)$. By [EJ2, Proposition 10.2.3], any Gorenstein projective module is either of infinite projective dimension or projective. It follows that $C(\beta)$ is projective and β is an isomorphism. So g is a section and hence $f = 0$. This completes the proof. ∎

Remark 3.5 By using the results in [V, AS], we see that all of the above results have (Gorenstein) injective counterparts. The corresponding categories are denoted by $\mathbf{GI}(\operatorname{Mod} R)$, $\overline{\mathbf{GI}}(\operatorname{Mod} R)$, and $D^b(\operatorname{Mod} R)_{fGI}$, respectively.

Let R be a left and right noetherian ring and let $I^\bullet$ be a complex of R-bimodules. Following [IK], $I^\bullet$ is called a *dualizing complex* if the following three conditions are satisfied.

(a) $I^\bullet$ is bounded and each I^n is injective both as an R-module and as an R^{op}-module.
(b) $H^n(I^\bullet)$ is finitely generated both as an R-module and as an R^{op}-module for each n.
(c) The canonical maps
$$R \longrightarrow \operatorname{Hom}_R(I^\bullet, I^\bullet) \quad \text{and} \quad R \longrightarrow \operatorname{Hom}_{R^{op}}(I^\bullet, I^\bullet)$$
are quasi-isomorphisms.

Lemma 3.6 *Let R be a left and right noetherian ring admitting a dualizing complex $I^\bullet$. Then*
$$I^\bullet \otimes_R^L - : D^b(\operatorname{Mod} R)_{fGP} \longrightarrow D^b(\operatorname{Mod} R)_{fGI}$$
is a triangulated equivalence, where $I^\bullet \otimes_R^L -$ is the left derived functor of $I^\bullet \otimes_R -$.

Proof Combine [IK, Proposition 7.2] (see also [AvF, Theorem 3.2]) with [IK, Theorem 8.1]. ∎

Because the subcategory of $K(\operatorname{Proj} R)$ (resp. $K(\operatorname{Inj} R)$) consisting of totally acyclic complexes is clearly triangulated equivalent to $\underline{\mathbf{GP}}(\operatorname{Mod} R)$ (resp. $\overline{\mathbf{GI}}(\operatorname{Mod} R)$), the equivalence in the following result can be deduced from [IK, 5.12]. We give here an alternative proof.

Proposition 3.7 *Let R be a left and right noetherian ring admitting a dualizing complex $I^\bullet$. Then we have a triangulated equivalence*
$$\underline{\mathbf{GP}}(\operatorname{Mod} R) \simeq \overline{\mathbf{GI}}(\operatorname{Mod} R).$$

Proof We have the commutative diagram
$$\begin{tikzcd} K^b(\operatorname{Proj}R) \arrow[r, "I^\bullet \otimes_R -"] \arrow[d] & K^b(\operatorname{Inj}R) \arrow[d] \\ D^b(\operatorname{Mod} R)_{fGP} \arrow[r, "I^\bullet \otimes_R^L -"] & D^b(\operatorname{Mod} R)_{fGI} \end{tikzcd}$$
in $D^b(\operatorname{Mod} R)$, where both columns are embeddings. By [IK, Theorem 4.2], we have that $I^\bullet \otimes_R - : K(\operatorname{Proj} R) \to K(\operatorname{Inj} R)$ is an equivalence, which induces an equivalence

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$I^\bullet \otimes_R -\colon K^b(\operatorname{Proj} R) \to K^b(\operatorname{Inj} R)$ in $D^b(\operatorname{Mod} R)$. Combine Theorem 3.4 and its dual version with Lemma 3.6, and we have $\underline{\mathbf{GP}}(\operatorname{Mod} R) \simeq \overline{\mathbf{GI}}(\operatorname{Mod} R)$. ■

As a consequence of this proposition, we have the following corollary.

Corollary 3.8 (*cf.* [BR, Chapter X, Proposition 1.4]) *Let R be a finite-dimensional algebra over a field K. Then we have a triangulated equivalence*

$$\underline{\mathbf{GP}}(\operatorname{Mod} R) \simeq \overline{\mathbf{GI}}(\operatorname{Mod} R),$$

which restricts to a triangulated equivalence

$$\underline{\mathbf{GP}}(\operatorname{mod} R) \simeq \overline{\mathbf{GI}}(\operatorname{mod} R).$$

Proof It is easy to see that $\mathbb{D}R$ as a complex of R-bimodules is a dualizing complex for R, where $\mathbb{D} = \operatorname{Hom}_K(\cdot, K)$. So by Proposition 3.7, we get the first triangulated equivalence. Note that the functor $\mathbb{D}R \otimes_R -$ is isomorphic to $\mathbb{D}\operatorname{Hom}_R(\cdot, R)$ naturally in $\operatorname{mod} R$. It is clear that $\operatorname{Hom}_R(\cdot, R)$ gives a duality from totally acyclic complexes in $\operatorname{mod} R$ to totally acyclic complexes in $\operatorname{mod} R^{op}$. It follows that the first triangulated equivalence restricts to the second one. ■

Recall that a finite-dimensional algebra R is called *Gorenstein* if the left and right self-injective dimensions of R are finite. In this case, the left and right self-injective dimensions of R are identical.

Lemma 3.9 *Let R be a finite-dimensional Gorenstein algebra. Then we have a triangulated equivalence*

$$D^b(\operatorname{Mod} R)_{fGP} = D^b(\operatorname{Mod} R).$$

Proof Let the left and right self-injective dimensions of R are equal to $n (< \infty)$. Let $X^\bullet$ be a homologically bounded complex and $P^\bullet$ its projective resolution. Then we have the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & P^{s-n} \arrow[r, "d_{P^\bullet}^{s-n}"] \arrow[d, "\pi^{s-n}"'] & P^{s-n+1} \arrow[r] \arrow[d, "\pi^{s-n+1}"'] & \cdots \arrow[r] \arrow[d] & P^{s-1} \arrow[r, "d_{P^\bullet}^{s-1}"] \arrow[d, "\pi^{s-1}"'] & P^s \arrow[r] \arrow[d, "\pi^s"'] & \cdots \\
\cdots \arrow[r] & X^{s-n} \arrow[r, "d_{X^\bullet}^{s-n}"] & X^{s-n+1} \arrow[r] & \cdots \arrow[r] & X^{s-1} \arrow[r, "d_{X^\bullet}^{s-1}"] & X^s \arrow[r] & \cdots.
\end{tikzcd}$$

Let $\inf H(X^\bullet) = s$. Then $P^\bullet$ and $X^\bullet$ are exact in degree $\leq s-1$. So we have an exact sequence

$$\cdots \longrightarrow P^{s-n} \longrightarrow P^{s-n+1} \longrightarrow \cdots \longrightarrow P^{s-2} \longrightarrow P^{s-1} \longrightarrow \operatorname{Im} d_{P^\bullet}^{s-1} \longrightarrow 0,$$

which is a projective resolution of $\operatorname{Im} d_{P^\bullet}^{s-1}$. It follows from [EJ2, Theorem 10.2.14] that $\operatorname{Ker} d_{P^\bullet}^{s-n}$ is Gorenstein projective. Thus, $X^\bullet$ has finite Gorenstein projective dimension by [V, Theorem 3.4]. ■

Buchweitz proved in [Bu, Theorem 4.4.1(2)] that for a finite-dimensional Gorenstein algebra R, there exists a triangulated equivalence:

$$\underline{\mathbf{GP}}(\operatorname{mod} R) \simeq D^b(\operatorname{mod} R)/K^b(\operatorname{proj} R).$$

The following proposition extends this result.

Proposition 3.10 *Let R be a finite-dimensional Gorenstein algebra. Then there exists a triangulated equivalence*

$$\underline{\mathbf{GP}}(\operatorname{Mod} R) \simeq D^b(\operatorname{Mod} R)/K^b(\operatorname{Proj} R),$$

which restricts to a triangulated equivalence

$$\underline{\mathbf{GP}}(\operatorname{mod} R) \simeq D^b(\operatorname{mod} R)/K^b(\operatorname{proj} R).$$

Proof The first assertion follows from Theorem 3.4 and Lemma 3.9. Note that the embedding

$$\mathbf{GP}(\operatorname{mod} R) \hookrightarrow \mathbf{GP}(\operatorname{Mod} R)$$

induces a triangulated embedding

$$\underline{\mathbf{GP}}(\operatorname{mod} R) \hookrightarrow \underline{\mathbf{GP}}(\operatorname{Mod} R).$$

It is easy to see that the essential image of the composition

$$\underline{\mathbf{GP}}(\operatorname{mod} R) \hookrightarrow \underline{\mathbf{GP}}(\operatorname{Mod} R) \xrightarrow{\overline{F}} D^b(\operatorname{Mod} R)_{fGP}/K^b(\operatorname{Proj} R) (= D^b(\operatorname{Mod} R)/K^b(\operatorname{Proj} R))$$

is $D^b(\operatorname{mod} R)/K^b(\operatorname{proj} R)$, where $\overline{F}$ is as in Theorem 3.4. Thus, we get the second triangulated equivalence. ∎

A Iyama–Yang Equivalence

In this appendix, we will introduce an equivalence proved by Iyama and Yang when they studied the interplay between silting reduction and Calabi–Yau reduction in triangulated categories [IYa], and then apply the Iyama–Yang equivalence to obtain the previous main result.

Before stating the Iyama–Yang equivalence, we recall some notions. Let $\mathcal{T}$ be a triangulated category. A full subcategory $\mathcal{P}$ of $\mathcal{T}$ is called *presilting* if $\operatorname{Hom}_{\mathcal{T}}(\mathcal{P}, \mathcal{P}[i]) = 0$ for any $i \geq 1$. In the following we always assume that $\mathcal{P} = \operatorname{add}\mathcal{P}$, that is, $\mathcal{P}$ is closed under $\mathcal{T}$-summands. We write $\mathcal{S} := \operatorname{thick}_{\mathcal{T}} \mathcal{P}$ (the smallest triangulated subcategory of $\mathcal{T}$ containing $\mathcal{P}$ and closed under $\mathcal{T}$-summands). The Verdier quotient $\mathcal{U} := \mathcal{T}/\mathcal{S}$ is called the *silting reduction* of $\mathcal{T}$ with respect to $\mathcal{P}$ ([AI]).

Let $l \in \mathbb{Z}$. We define two full subcategories of $\mathcal{T}$ as follows:

$$\mathcal{S}_{\geq l} = \mathcal{S}_{>l-1} := \bigcup_{i\geq 0} \mathcal{P}[-l-i] * \cdots * \mathcal{P}[-l-1] * \mathcal{P}[-l],$$
$$\mathcal{S}_{\leq l} = \mathcal{S}_{<l+1} := \bigcup_{i\geq 0} \mathcal{P}[-l] * \mathcal{P}[-l+1] * \cdots * \mathcal{P}[-l+i],$$

where for two subcategories $\mathcal{A}, \mathcal{B}$ of $\mathcal{T}$,

$$\mathcal{A} * \mathcal{B} := \left\{ X \mid \text{ there exists a triangle } A \to X \to B \to A[1] \text{ with } A \in \mathcal{A} \text{ and } B \in \mathcal{B} \right\}.$$

By the octahedral axiom, we have $(\mathcal{A} * \mathcal{B}) * \mathcal{C} = \mathcal{A} * (\mathcal{B} * \mathcal{C})$ for subcategories $\mathcal{A}, \mathcal{B}, \mathcal{C}$ of $\mathcal{T}$. We also define a full subcategory of $\mathcal{T}$:

$$\mathcal{Z} := {}^{\perp}\mathcal{S}_{<0} \cap \mathcal{S}_{>0}^{\perp} = {}^{\perp}\mathcal{P}[>0] \cap \mathcal{P}[<0]^{\perp},$$

https://doi.org/10.4153/CMB-2017-045-2 Published online by Cambridge University Press

where for a subcategory $\mathcal{M}$ of $\mathcal{T}$,

$$
\begin{aligned}
{}^{\perp}\mathcal{M} &:= \left\{ X \mid \operatorname{Hom}_{\mathcal{T}}(X, \mathcal{M}) = 0 \right\}, \\
{}^{\perp}\mathcal{M}[> n] &:= \left\{ X \mid \operatorname{Hom}_{\mathcal{T}}(X, \mathcal{M}[i]) = 0 \text{ for any } i > n \right\}.
\end{aligned}
$$

Assume that $(\mathcal{Z}, \mathcal{Z})$ forms a $\mathcal{P}$-*mutation pair* in the sense of [IYo]; that is, the following conditions are satisfied. (1) $\mathcal{P} \subset \mathcal{Z}$ and $\operatorname{Hom}_{\mathcal{T}}(\mathcal{P}, \mathcal{Z}[1]) = 0 = \operatorname{Hom}_{\mathcal{T}}(\mathcal{Z}, \mathcal{P}[1])$; (2) For any $Z \in \mathcal{Z}$, there exists triangles

$$
\begin{aligned}
Z \longrightarrow P' \longrightarrow Z' \longrightarrow Z[1], \\
Z'' \longrightarrow P'' \longrightarrow Z \longrightarrow Z''[1]
\end{aligned}
$$

with $P', P'' \in \mathcal{P}$ and $Z', Z'' \in \mathcal{Z}$. Then the additive subfactor $\mathcal{Z}/[\mathcal{P}]$ carries a structure of a triangulated category ([IYo]). Recall from [IYa] that a pair $(\mathcal{X}, \mathcal{Y})$ of two additive full subcategories of $\mathcal{T}$ is called a *co-t-structure* if the following three conditions are satisfied.

(a) $\mathcal{X} = {}^{\perp}(\mathcal{Y}[1])$ and $\mathcal{Y} = (\mathcal{X}[-1])^{\perp}$.
(b) $\mathcal{T} = \mathcal{X} * (\mathcal{Y}[1])$.
(c) $\mathcal{Y}$ is closed under [1].

In this case, the subcategory $\mathcal{X} \cap \mathcal{Y}$ is called the *co-heart* of $(\mathcal{X}, \mathcal{Y})$.

Theorem A.1 ([IYa, Theorem 4.7]) *Let the notation be as above. Assume that $(\mathcal{Z}, \mathcal{Z})$ forms a $\mathcal{P}$-mutation pair. If the two pairs $({}^{\perp}\mathcal{S}_{<0}, \mathcal{S}_{\le 0})$ and $(\mathcal{S}_{\ge 0}, \mathcal{S}_{>0}^{\perp})$ both are co-t-structures with co-heart $\mathcal{P}$, then the composition $\mathcal{Z} \subset \mathcal{P} \xrightarrow{\rho} \mathcal{U}$ induces a triangulated equivalence*

$$
\frac{\mathcal{Z}}{[\mathcal{P}]} \xrightarrow{\bar{\rho}} \mathcal{U}.
$$

Theorem A.2 *Let R be a ring, and let $\mathcal{T} = D^b(\operatorname{Mod} R)_{fGP}$ and $\mathcal{P} = \operatorname{Proj} R$. Then we have a triangulated equivalence*

$$
\frac{\mathcal{Z}}{[\mathcal{P}]} \xrightarrow{\bar{\rho}} \mathcal{U},
$$

which is the same as the one given in Theorem 3.4.

Proof We will sketch a proof by checking the conditions in Theorem A.1.

(1) It is true that $\mathcal{P}$ is a presilting subcategory of $\mathcal{T}$ and $\mathcal{P} = \operatorname{add} \mathcal{P}$.

Define a functor $\mathbb{P}\colon D^b(\operatorname{Mod} R)_{fGP} \hookrightarrow K^-(\mathcal{P})$ that sends each X to its projective resolution. It is then fully faithful. In the following, the essential image of $\mathbb{P}$ is also denoted by $\mathcal{T}$.

(2) It is obvious that $\mathcal{S} = \operatorname{thick}_{\mathcal{T}}(\mathcal{P}) = K^b(\mathcal{P})$.

(3) We claim that $\mathcal{Z} = \mathbf{GP}(\operatorname{Mod} R)$. Let $T \in \mathcal{Z} \subset \mathcal{T}$. Suppose that the Gorenstein projective dimension of $T \le g (> 0)$. Then by [V, Theorem 3.4], we have that $\sup H(T) \ge -g$ and $\operatorname{Coker} d_T^{-g}$ is Gorenstein projective. Write T as

$$
\cdots \longrightarrow T^{-g} \longrightarrow T^{-g+1} \longrightarrow \cdots \longrightarrow T^{-1} \longrightarrow T^{0} \longrightarrow T^{1} \longrightarrow \cdots \longrightarrow T^{n} \longrightarrow 0 \longrightarrow \cdots.
$$

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By the definition of $\mathcal{Z}$, we have $\mathrm{Hom}_{\mathcal{T}}(\mathcal{P}[i], T) = 0$ for any $i \le -1$. This implies that $H^j \mathrm{Hom}_R(\mathcal{P}, T) = 0$ for any $j \ge 1$, and hence

$$T \cong \cdots \longrightarrow T^{-g} \longrightarrow T^{-g+1} \longrightarrow \cdots \longrightarrow T^{-1} \longrightarrow T_1^0 \longrightarrow 0 \longrightarrow \cdots$$

in $\mathcal{T}$, where T_1^0 is a summand of T^0.

Since $T \in {}^{\perp}\mathcal{S}_{<0}$, we have the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & T^{-3} \arrow[r] \arrow[d, "="'] & T^{-2} \arrow[r] \arrow[d, "="'] & T^{-1} \arrow[r] \arrow[d, "="'] & T_1^0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & T^{-3} \arrow[r] & T^{-2} \arrow[r] & T^{-1} \arrow[r] & 0 \arrow[r] & \cdots .
\end{tikzcd}$$

It induces that this cochain map is null-homotopic, and hence T is exact in degree ≤ -1.

Since $\mathrm{Hom}_{\mathcal{T}}(T, \mathcal{P}[i]) = 0$ for any $i \ge 1$, we have $H^i \mathrm{Hom}_R(T, \mathcal{P}) = 0$ for any ≥ 1. Then we obtain an exact sequence

$$\cdots \longrightarrow T^{-g} \longrightarrow T^{-g+1} \longrightarrow \cdots \longrightarrow T^{-1} \longrightarrow T_1^0 \longrightarrow C \longrightarrow 0,$$

which remains exact after applying $\mathrm{Hom}_R(\cdot, \mathcal{P})$. Since Coker d_T^{-g} is Gorenstein projective, we have that C is Gorenstein projective by [Ho, Corollary 2.11]. So T is a projective resolution of C, and hence $\mathcal{Z} \subseteq \mathbf{GP}(\mathrm{Mod}\, R)$. The opposite inclusion is clear by the definition of Gorenstein projective modules.

(4) $(\mathcal{Z}, \mathcal{Z})$ forms a $\mathcal{P}$-mutation pair, and the triangulated structure in $\mathcal{Z}/[\mathcal{P}]$ coincides with that in $\underline{\mathbf{GP}}(\mathrm{Mod}\, R)$. Hence we have $\underline{\mathbf{GP}}(\mathrm{Mod}\, R) \cong \mathcal{Z}/[\mathcal{P}]$.

(5) It is easy to show that the two pairs $({}^{\perp}\mathcal{S}_{<0}, \mathcal{S}_{\le 0})$ and $(\mathcal{S}_{\ge 0}, \mathcal{S}_{>0}^{\perp})$ both are co-t-structures with co-heart $\mathcal{P}$. ■

Acknowledgments The authors thank Liran Shaul for pointing out the reference [Y] and the referee for his/her useful suggestions.

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