KERNELS OF INVERSE SEMIGROUP HOMOMORPHISMS

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The aim of this note is to give an analogue, for an inverse semigroup S, of the theorem for a group G which says that if $\mathscr{G}$ is the set of normal subgroups of G, then the map $N \to (N) = \{(a,b) \in G \times G : ab^{-1} \in N\}$ for $N \in \mathscr{G}$ is a 1:1 order preserving map of $\mathscr{G}$ onto $\Lambda(G)$, the lattice of congruences on G. It will be shown that if E is the semilattice of idempotents of S, $P = \{E_\alpha : \alpha \in J\}$ is a normal partition of E, and $\mathscr{K}$ is a certain collection of self conjugate inverse subsemigroups of S, then the map $K \to (K) = \{(a,b) \in S \times S$: $a^{-1}a$, $b^{-1}b \in E_\alpha$ for some $\alpha \in J$ and $ab^{-1} \in K\}$ for $K \in \mathscr{K}$ is a 1:1 map of $\mathscr{K}$ onto the set of congruences on S which induce P.

0. Introduction

The reader is assumed to be familiar with standard semigroup notation and the elementary properties of inverse semigroups [1]. Throughout, S will denote an inverse semigroup with E as its semilattice of idempotents. The assumption includes a familiarity with $C(E)$, the centralizer of E, self conjugate inverse subsemigroups of S, and the closure $H\omega$ of a subset H of S.

Preston [3] has shown that if $f: S \to T$ is a homomorphism of S onto T and af is idempotent in T, then $af = a^{-1}f = (aa^{-1})f = (a^{-1}a)f$. Thus T is inverse, $(bf)^{-1} = b^{-1}f$, and if f separates idempotents, then $f \circ f^{-1} \subset \mathscr{H}$. Also, Preston has given a complete description of all congruences on S in terms of kernel normal systems of S. In [4], a characterization has been given of the smallest and largest congruences which induce a given partition P of the set E of idempotents.

In [2], Howie has given two characterizations of μ, the largest idempotent separating congruence, neither of which depend on kernel normal systems. Recall that these descriptions are given by

$$\mu = \{(a,b) \in S \times S : a^{-1}ea = b^{-1}eb \text{ for each } e \in E\}, \text{ or equivalently}$$

$$\mu = \{(a,b) \in S \times S : a^{-1}a = b^{-1}b \text{ and } ab^{-1} \in C(E)\}.$$

This note will give a description of the congruences on S similar to Howie's second characterization of μ. In so doing, the closure operator ω will be used to show just which inverse subsemigroups of S can be the kernels of homomorphisms.

1. Kernels of groups and idempotent separating homomorphisms

The two lemmas in this section could, at least in part, be deduced respectively from [5, see also 1, Theorem 7.12; 1, Theorem 7.54]. Full proofs will be given here, however, for completeness.

Let $f: S \to G$ be a homomorphism of S onto a group G. Let $M = \mathrm{Kernel} f = \{a \in S: af = 1_G\}$. Let $\mathscr{K} = \{K \subset S: M \subset K$ and K is a closed $(K\omega = K)$ inverse subsemigroup of $S\}$.

LEMMA 1.1. *The map $K \to Kf$ for $K \in \mathscr{K}$ is a* 1:1 *order preserving map of $\mathscr{K}$ onto the set of subgroups of G. Further, K is self conjugate in S if and only if Kf is self conjugate* (normal) *in G.*

PROOF. It is easy to see that if $K \in \mathscr{K}$, then Kf is a subgroup of G, and that $K \to Kf$ is order preserving.

Suppose then that H is a subgroup of G, $K = Hf^{-1}$, and $y \in K\omega$. Then $k \leqq y$ for some $k \in K$. From $k^{-1}k = k^{-1}y$ follows that $1_G = (kf)^{-1}(yf)$. Thus $kf = yf$ and hence $y \in K$. Thus $K \in \mathscr{K}$ and so $K \to Kf$ is an onto map.

Assume that $K, L \in \mathscr{K}$ and $Kf = Lf$. Let $k \in K$ and let $kf = mf$ with $m \in L$. Then $(m^{-1}m)f = (m^{-1}k)f$ and so $m^{-1}k \in M \subset L$. Thus $mm^{-1}k \in mL \subset L$. But $mm^{-1}k \leqq k$ and so $k \in L\omega = L$. Similarly, $L \subset K$ and hence $K = L$. Since the map $K \to Kf$ is 1:1 it is a simple matter to compute that K is self conjugate in S if and only if Kf is normal in G.

Now let $\mathscr{C} = \{K \subset S: E \subset K \subset C(E)$ and K is a self conjugate inverse subsemigroup of $S\}$.

LEMMA 1.2. *The map $K \to (K) = \{(a,b) \in S \times S$: $a^{-1}a = b^{-1}b$ and $ab^{-1} \in K\}$ for $K \in \mathscr{C}$ is a* 1:1 *order preserving map of $\mathscr{C}$ onto the set of idempotent separating congruences on S.*

PROOF. The relation (K) for $K \in \mathscr{C}$ is obviously reflexive on S, and easily symmetric. Suppose then that $(a,b),(b,c) \in (K)$. Then $a^{-1}a = b^{-1}b = c^{-1}c$ and ab^{-1}, $bc^{-1} \in K$. Thus $a^{-1}a = c^{-1}c$ and $ac^{-1} = aa^{-1}ac^{-1} = (ab^{-1})(bc^{-1}) \in KK \subset K$, i.e., $(a,c) \in (K)$. Assume now that $(a,b),(x,y) \in (K)$. Then $(ax^{-1})(ax) = x^{-1}a^{-1}ax = x^{-1}b^{-1}bx = y^{-1}b^{-1}by$ (since $(x,y) \in (K) \subset \mu$) $= (by)^{-1}(by)$, and further $(ax)(by)^{-1} = axy^{-1}b^{-1}bb^{-1} = (axy^{-1}a^{-1})(ab^{-1}) \in aKa^{-1}K \subset KK \subset K$. Thus $(K) \in \Lambda(S)$, the set of all congruences on S.

Suppose that $\rho \in \Lambda(S)$, ρ separates idempotents, and let $K = \{a \in S: a\rho$ is dempotent in $S/\rho\}$. Then K is easily a self conjugate inverse subsemigroup

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of S. Also, if $a \in K$, then $(a, a^{-1}a) \in \rho \subset \mu$ so that $a(a^{-1}a) = a \in C(E)$. Thus $K \in \mathscr{C}$. If $(a, b) \in (K)$, then $a^{-1}a = b^{-1}b$ and $ab^{-1} \in K$. Thus $a\rho = (aa^{-1}a)\rho = (ab^{-1}b)\rho = (ab^{-1}ba^{-1})\rho(b)\rho = (bb^{-1}bb^{-1})\rho(b\rho)$ (since $(a, b) \in \mu) = b\rho$, i.e., $(a, b) \in \rho$. On the other hand, if $(a, b) \in \rho$, then $(a^{-1}a, b^{-1}b) \in \rho \subset \mathscr{H}$ and so $a^{-1}a = b^{-1}b$. Further, $(ab^{-1}, bb^{-1}) \in \rho$ and so $(ab^{-1})\rho$ is idempotent, i.e., $ab^{-1} \in K$. Thus $(K) = \rho$.

Finally, suppose that $K, L \in \mathscr{C}$ with $(K) = (L)$ and $k \in K$. Then $(k, k^{-1}k) \in (K) = (L)$ and hence $k \in L$. Symmetrically, $L \subset K$ and hence $K = L$.

2. Kernels of homomorphisms

In this section, the elements K of $\mathscr{C}$ in Lemma 1.2 will be called full (for $E \subset K$) central (for $K \subset C(E)$) self conjugate inverse subsemigroups of S.

A partition $P = \{E_\alpha : \alpha \in J\}$ is called *normal* provided that for each α, $\beta \in J$ and $a \in S$, there exist $\gamma, \delta \in J$ such that $E_\alpha E_\beta \subset E_\gamma$ and $aE_\alpha a^{-1} \subset E_\delta$ [4, Definition 4.1]. Whenever P is normal, there is a smallest congruence σ on S which induces P [4, Theorem 4.2]. It follows that if T_α is the largest inverse subsemigroup of S with E_α as its set of idempotents [4, Theorem 1.5], then $T_\alpha\sigma^\natural$ is a group $\mathscr{H}$ class of S/σ, say H_α with identity α.

Now let $P = \{E_\alpha : \alpha \in J\}$ be a normal partition of E. Let $\theta(P)$ be the set of congruences on S which induce P and let σ be the smallest element of $\theta(P)$. For each $\alpha \in J$, let T_α be the largest inverse subsemigroup of S with E_α as its set of idempotents. Let $M_\alpha = E_\alpha\omega \cap T_\alpha$ and let $N_\alpha = \{a \in T_\alpha : E_\alpha E_\beta \subset E_\gamma$ implies $aE_\beta a^{-1} \subset E_\gamma\}$. Let $M(P) = \cup \{M_\alpha : \alpha \in J\}$ and let $N(P) = \cup \{N_\alpha : \alpha \in J\}$. Let $\mathscr{K}(P) = \{K \subset S : M(P) \subset K \subset N(P)$, K is a self conjugate inverse subsemigroup of S, and $K_\alpha = K \cap T_\alpha$ is closed in $T_\alpha(K_\alpha = K_\alpha\omega \cap T_\alpha)\}$.

THEOREM 2.1. *The map* $K \to (K) = \{(a, b) \in S \times S : a^{-1}a, b^{-1}b \in E_\alpha$ *for some* $\alpha \in J$ *and* $ab^{-1} \in K\}$ *is a* $1:1$ *order preserving map of* $\mathscr{K}(P)$ *onto* $\theta(P)$. *Furthermore,* $M(P)$, $N(P) \in \mathscr{K}(P)$.

PROOF. Since $\rho \to \rho/\sigma$ $(= \{(a\sigma, b\sigma) : (a, b) \in \rho\})$ for $\rho \in \theta(P)$ is a $1:1$ order preserving map of $\theta(P)$ onto the set of idempotent separating congruences of S/σ, it is enough by Lemma 1.2 to show that $K \to K\sigma^\natural$ for $K \in \mathscr{K}(P)$ is a $1:1$ order preserving map of $\mathscr{K}(P)$ onto the set of full central self conjugate subsemigroups of S/σ.

Since M_α is the smallest closed self conjugate inverse subsemigroup of T_α which contains E_α [2, Lemma 3.4], then $M_\alpha\sigma^\natural = \alpha$ by Lemma 1.1. Thus $M(P)\sigma = E(S/\sigma)$, the set of idempotents of S/σ. Assume now that $K \in \mathscr{K}(P)$, and let $k \in K$, say $k \in K_\alpha \subset N_\alpha$. Let $\beta \in J$ and let $\alpha\beta = \gamma$. Then $(k\sigma)\beta = (k\sigma)\beta(k\sigma)^{-1}(k\sigma) = \gamma(k\sigma) = \beta\alpha(k\sigma) = \beta(k\sigma)$. Thus $K\sigma^\natural$, and also $N(P)\sigma^\natural$, $\subset C(E(S/\sigma))$. Hence $K\sigma^\natural$ is a full central self conjugate inverse subsemigroup of S/σ.

Suppose now that H is a full central self conjugate inverse subsemigroup of S/σ and let $K = H(\sigma^{\natural-1})$. Immediately, K is self conjugate and inverse. Since $(E(S/\sigma))\sigma^{\natural-1} = M(P)$ by Lemma 1.1, $M(P) \subset K$. Now let $k \in K$. Since $k\sigma$ is a group element of S/σ, say $k\sigma \in H_\alpha$, then $k \in T_\alpha$. Also if $\alpha\beta = \gamma$, then $(kE_\beta k^{-1})\sigma = (k\sigma)\beta(k\sigma)^{-1} = \beta\alpha = \gamma$. Thus $kE_\beta k^{-1} \subset E_\gamma$ and so $k \in N_\alpha$. Hence $K \subset N(P)$. Now let $K_\alpha = K \cap T_\alpha$, i.e., $K_\alpha = (H \cap H_\alpha)\sigma^{\natural-1}$. Since $H \cap H_\alpha$ is a subgroup of H_α, then K_α is closed in T_α by Lemma 1.1. This completes the argument that $M(P), N(P) \in \mathscr{K}(P)$ and $K \to K\sigma$ is a map of $\mathscr{K}(P)$ onto the set of full central self conjugate inverse subsemigroups of S/σ.

Finally, if K, $L \in \mathscr{K}(P)$ and $K\sigma^\natural = L\sigma^\natural$, then $K\sigma^\natural \cap H_\alpha = L\sigma^\natural \cap H_\alpha$ for each $\alpha \in J$. Thus $K_\alpha = L_\alpha$ for each α again by Lemma 1.1 and so $K = L$.

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