

# Mathematical Proceedings of the Cambridge Philosophical Society

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VOL. 175

JULY 2023

PART 1

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*Math. Proc. Camb. Phil. Soc.* (2023), **175**, 1–11 © The Author(s), 2022. Published by Cambridge University Press on behalf of Cambridge Philosophical Society. This is an Open Access article, distributed under the terms of the Creative Commons Attribution licence (<https://creativecommons.org/licenses/by/4.0/>), which permits unrestricted re-use, distribution, and reproduction in any medium, provided the original work is properly cited. 1

doi: [10.1017/S0305004122000469](https://doi.org/10.1017/S0305004122000469)

First published online 28 November 2022

## Most numbers are not normal

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*(Received 25 January 2021; revised 07 November 2022; accepted 31 October 2022)*

### *Abstract*

We show, from a topological viewpoint, that most numbers are not normal in a strong sense. More precisely, the set of numbers  $x \in (0, 1]$  with the following property is comeager: for all integers  $b \geq 2$  and  $k \geq 1$ , the sequence of vectors made by the frequencies of all possible strings of length  $k$  in the  $b$ -adic representation of  $x$  has a maximal subset of accumulation points, and each of them is the limit of a subsequence with an index set of nonzero asymptotic density. This extends and provides a streamlined proof of the main result given by Olsen (2004) in this Journal. We provide analogues in the context of analytic P-ideals and regular matrices.

2020 Mathematics Subject Classification: 11K16 (Primary); 40A35, 11A63 (Secondary)

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### 1. Introduction

A real number  $x \in (0, 1]$  is normal if, informally, for each base  $b \geq 2$ , its  $b$ -adic expansion contains every finite string with the expected uniform limit frequency (the precise definition is given in the next few lines). It is well known that most numbers  $x$  are normal from a

measure theoretic viewpoint, see e.g. [5] for history and generalisations. However, it has been recently shown that certain subsets of nonnormal numbers may have full Hausdorff dimension, see e.g. [1, 4]. The aim of this work is to show that, from a topological viewpoint, most numbers are not normal in a strong sense. This provides another nonanalogue between measure and category, cf. [25].

For each  $x \in (0, 1]$ , denote its unique nonterminating  $b$ -adic expansion by

$$x = \sum_{n \geq 1} \frac{d_{b,n}(x)}{b^n}, \tag{1}$$

with each digit  $d_{b,n}(x) \in \{0, 1, \dots, b - 1\}$ , where  $b \geq 2$  is a given integer. Then, for each string  $s = s_1 \cdots s_k$  with digits  $s_j \in \{0, 1, \dots, b - 1\}$  and each  $n \geq 1$ , write  $\pi_{b,s,n}(x)$  for the proportion of strings  $s$  in the  $b$ -adic expansion of  $x$  which start at some position  $\leq n$ , i.e.,

$$\pi_{b,s,n}(x) := \frac{\#\{i \in \{1, \dots, n\} : d_{b,i+j-1}(x) = s_j \text{ for all } j = 1, \dots, k\}}{n}.$$

In addition, let  $S_b^k$  be the set of all possible strings  $s = s_1 \cdots s_k$  in base  $b$  of length  $k$ , hence  $\#S_b^k = b^k$ , and denote by  $\pi_{b,n}^k(x)$  the vector  $(\pi_{b,s,n}(x) : s \in S_b^k)$ . Of course,  $\pi_{b,n}^k(x)$  belongs to the  $(b^k - 1)$ -dimensional simplex for each  $n$ . However, the components of  $\pi_{b,n}^k(x)$  satisfy an additional requirement: if  $k \geq 2$  and  $s = s_1 \cdots s_{k-1}$  is a string in  $S_b^{k-1}$ , then

$$\pi_{b,s,n}(x) = \sum_{s_k} \pi_{b,ss_k,n}(x) = \sum_{s_0} \pi_{b,s_0s,n}(x) + O(1/n) \quad \text{as } n \rightarrow \infty,$$

where  $s_0s$  and  $ss_k$  stand for the concatenated strings (indeed, the above identity is obtained by a double counting of the occurrences of the string  $s$  as the occurrences of all possible strings  $ss_k$ ; or, equivalently, as the occurrences of all possible strings  $s_0s$ , with the caveat of counting them correctly at the two extreme positions, hence with an error of at most 1). It follows that the set  $L_b^k(x)$  of accumulation points of the sequence of vectors  $(\pi_{b,n}^k(x) : n \geq 1)$  is contained in  $\Delta_b^k$ , where

$$\Delta_b^k := \left\{ (p_s)_{s \in S_b^k} \in \mathbf{R}^{b^k} : \sum_s p_s = 1, p_s \geq 0 \text{ for all } s \in S_b^k, \right. \\ \left. \text{and } \sum_{s_0} p_{s_0s} = \sum_{s_k} p_{ss_k} \text{ for all } s \in S_b^{k-1} \right\}.$$

Then  $x$  is said to be *normal* if

$$\forall b \geq 2, \forall k \geq 1, \forall s \in S_b^k, \quad \lim_{n \rightarrow \infty} \pi_{b,s,n}(x) = 1/b^k.$$

Hence, if  $x$  is normal, then  $L_b^k(x) = \{(1/b^k, \dots, 1/b^k)\}$ . Olsen proved in [23] that the subset of nonnormal numbers with maximal set of accumulation points is topologically large:

**THEOREM 1.1.** *The set  $\{x \in (0, 1] : L_b^k(x) = \Delta_b^k \text{ for all } b \geq 2, k \geq 1\}$  is comeager.*

First, we strenghten Theorem 1.1 by showing that the set of accumulation points  $L_b^k(x)$  can be replaced by the much smaller subset of accumulation points  $\eta$  such that every neighbourhood of  $\eta$  contains “sufficiently many” elements of the sequence, where “sufficiently many” is meant with respect to a suitable ideal  $\mathcal{I}$  of subsets of the positive integers  $\mathbf{N}$ ; see Theorem 2.1. Hence, Theorem 1.1 corresponds to the case where  $\mathcal{I}$  is the family of finite sets.

Then, for certain ideals  $\mathcal{I}$  (including the case of the family of asymptotic density zero sets), we even strenghten the latter result by showing that each accumulation point  $\eta$  can be chosen to be the limit of a subsequence with “sufficiently many” indexes (as we will see in the next Section, these additional requirements are not equivalent); see Theorem 2.3. The precise definitions, together with the main results, follow in Section 2.

2. Main results

An ideal  $\mathcal{I} \subseteq \mathcal{P}(\mathbf{N})$  is a family closed under finite union and subsets. It is also assumed that  $\mathcal{I}$  contains the family of finite sets  $\text{Fin}$  and it is different from  $\mathcal{P}(\mathbf{N})$ . Every subset of  $\mathcal{P}(\mathbf{N})$  is endowed with the relative Cantor-space topology. In particular, we may speak about  $G_\delta$ -subsets of  $\mathcal{P}(\mathbf{N})$ ,  $F_\sigma$ -ideals, meager ideals, analytic ideals, etc. In addition, we say that  $\mathcal{I}$  is a P-ideal if it is  $\sigma$ -directed modulo finite sets, i.e., for each sequence  $(S_n)$  of sets in  $\mathcal{I}$  there exists  $S \in \mathcal{I}$  such that  $S_n \setminus S$  is finite for all  $n \in \mathbf{N}$ . Lastly, we denote by  $\mathcal{Z}$  the ideal of asymptotic density zero sets, i.e.,

$$\mathcal{Z} = \{S \subseteq \mathbf{N} : d^*(S) = 0\}, \tag{2}$$

where  $d^*(S) := \limsup_n \frac{1}{n} \#(S \cap [1, n])$  stands for the upper asymptotic density of  $S$ , see e.g. [20]. We refer to [14] for a recent survey on ideals and associated filters.

Let  $x = (x_n)$  be a sequence taking values in a topological vector space  $X$ . Then we say that  $\eta \in X$  is an  $\mathcal{I}$ -cluster point of  $x$  if  $\{n \in \mathbf{N} : x_n \in U\} \notin \mathcal{I}$  for all open neighbourhoods  $U$  of  $\eta$ . Note that  $\text{Fin}$ -cluster points are the ordinary accumulation points. Usually  $\mathcal{Z}$ -cluster points are referred to as *statistical cluster points*, see e.g. [13]. It is worth noting that  $\mathcal{I}$ -cluster points have been studied much before under a different name. Indeed, as it follows by [19, theorem 4.2] and [16, lemma 2.2], they correspond to classical “cluster points” of a filter (depending on  $x$ ) on the underlying space, cf. [7, definition 2, p.69].

With these premises, for each  $x \in (0, 1]$  and for all integers  $b \geq 2$  and  $k \geq 1$ , let  $\Gamma_b^k(x, \mathcal{I})$  be the set of  $\mathcal{I}$ -cluster points of the sequence  $(\pi_{b,n}^k(x) : n \geq 1)$ .

**THEOREM 2.1.** *The set  $\{x \in (0, 1] : \Gamma_b^k(x, \mathcal{I}) = \Delta_b^k \text{ for all } b \geq 2, k \geq 1\}$  is comeager, provided that  $\mathcal{I}$  is a meager ideal.*

The class of meager ideals is really broad. Indeed, it contains  $\text{Fin}$ ,  $\mathcal{Z}$ , the summable ideal  $\{S \subseteq \mathbf{N} : \sum_{n \in S} 1/n < \infty\}$ , the ideal generated by the upper Banach density, the analytic P-ideals, the Fubini sum  $\text{Fin} \times \text{Fin}$ , the random graph ideal, etc.; cf. e.g. [3, 14]. Note that  $\Gamma_b^k(x, \mathcal{I}) = L_b^k(x)$  if  $\mathcal{I} = \text{Fin}$ . Therefore Theorem 2.1 significantly strenghtens Theorem 1.1.

*Remark 2.2.* It is not difficult to see that Theorem 2.1 does not hold without any restriction on  $\mathcal{I}$ . Indeed, if  $\mathcal{I}$  is a maximal ideal (i.e., the complement of a free ultrafilter on  $\mathbf{N}$ ), then for each  $x \in (0, 1]$  and all integers  $b \geq 2, k \geq 1$ , we have that the sequence  $(\pi_{b,n}^k(x) : n \geq 1)$  is bounded, hence it is  $\mathcal{I}$ -convergent so that  $\Gamma_b^k(x, \mathcal{I})$  is a singleton.

On a similar direction, if  $x = (x_n)$  is a sequence taking values in a topological vector space  $X$ , then  $\eta \in X$  is an  $\mathcal{I}$ -limit point of  $x$  if there exists a subsequence  $(x_{n_k})$  such that  $\lim_k x_{n_k} = \eta$  and  $\mathbf{N} \setminus \{n_1, n_2, \dots\} \in \mathcal{I}$ . Usually  $\mathcal{Z}$ -limit points are referred to as *statistical limit points*, see e.g. [13]. Similarly, for each  $x \in (0, 1]$  and for all integers  $b \geq 2$  and  $k \geq 1$ , let  $\Lambda_b^k(x, \mathcal{I})$  be the set of  $\mathcal{I}$ -limit points of the sequence  $(\pi_{b,n}^k(x) : n \geq 1)$ . The analogue of Theorem 2.1 for  $\mathcal{I}$ -limit points follows.

**THEOREM 2.3.** *The set  $\{x \in (0, 1] : \Delta_b^k(x, \mathcal{I}) = \Delta_b^k \text{ for all } b \geq 2, k \geq 1\}$  is comeager, provided that  $\mathcal{I}$  is an analytic P-ideal or an  $F_\sigma$ -ideal.*

It is known that every  $\mathcal{I}$ -limit point is always an  $\mathcal{I}$ -cluster point, however they can be highly different, as it is shown in [2, theorem 3.1]. This implies that Theorem 2.3 provides a further improvement on Theorem 2.1 for the subfamily of analytic P-ideals.

It is remarkable that there exist  $F_\sigma$ -ideals which are not P-ideals, see e.g. [11, section 1.11]. Also, the family of analytic P-ideals is well understood and has been characterised with the aid of lower semicontinuous submeasures, cf. Section 3. The results in [6] suggest that the study of the interplay between the theory of analytic P-ideals and their representability may have some relevant yet unexploited potential for the study of the geometry of Banach spaces.

Finally, recalling that the ideal  $\mathcal{Z}$  defined in (2) is an analytic P-ideal, an immediate consequence of Theorem 2.3 (as pointed out in the abstract) follows:

**COROLLARY 2.4.** *The set of  $x \in (0, 1]$  such that, for all  $b \geq 2$  and  $k \geq 1$ , every vector in  $\Delta_b^k$  is a statistical limit point of the sequence  $(\pi_{b,n}^k(x) : n \geq 1)$  is comeager.*

It would also be interesting to investigate to what extent the same results for nonnormal points belonging to self-similar fractals (as studied, e.g., by Olsen and West in [24] in the context of iterated function systems) are valid.

We leave as an open question for the interested reader to check whether Theorem 2.3 can be extended for all  $F_{\sigma\delta}$ -ideals including, in particular, the ideal  $\mathcal{I}$  generated by the upper Banach density (which is known to not be a P-ideal, see e.g. [12, p.299]).

### 3. Proofs of the main results

*Proof of Theorem 2.1.* Let  $\mathcal{I}$  be a meager ideal on  $\mathbf{N}$ . It follows by Talagrand’s characterisation of meager ideals [28, theorem 21] that it is possible to define a partition  $\{I_1, I_2, \dots\}$  of  $\mathbf{N}$  into nonempty finite subsets such that  $S \notin \mathcal{I}$  whenever  $I_n \subseteq S$  for infinitely many  $n$ . Moreover, we can assume without loss of generality that  $\max I_n < \min I_{n+1}$  for all  $n \in \mathbf{N}$ .

The claimed set can be rewritten as  $\bigcap_{b \geq 2} \bigcap_{k \geq 1} X_b^k$ , where  $X_b^k := \{x \in (0, 1] : \Gamma_b^k(x, \mathcal{I}) = \Delta_b^k\}$ . Since the family of meager subsets of  $(0,1]$  is a  $\sigma$ -ideal, it is enough to show that the complement of each  $X_b^k$  is meager. To this aim, fix  $b \geq 2$  and  $k \geq 1$  and denote by  $\|\cdot\|$  the Euclidean norm on  $\mathbf{R}^{bk}$ . Considering that  $\{\eta_1, \eta_2, \dots\} := \Delta_b^k \cap \mathbf{Q}^{bk}$  is a countable dense subset of  $\Delta_b^k$  and that  $\Gamma_b^k(x, \mathcal{I})$  is a closed subset of  $\Delta_b^k$  by [19, lemma 3.1(iv)], it follows that

$$\begin{aligned} (0, 1] \setminus X_b^k &= \bigcup_{t \geq 1} \{x \in (0, 1] : \eta_t \notin \Gamma_b^k(x, \mathcal{I})\} \\ &= \bigcup_{t \geq 1} \{x \in (0, 1] : \exists \varepsilon > 0, \{n \in \mathbf{N} : \|\pi_{b,n}^k(x) - \eta_t\| < \varepsilon\} \in \mathcal{I}\} \\ &\subseteq \bigcup_{t,p,m \geq 1} \{x \in (0, 1] : \forall q \geq p, \exists n \in I_q, \|\pi_{b,n}^k(x) - \eta_t\| \geq 1/m\}. \end{aligned}$$

Denote by  $S_{t,p,m}$  the set in the latter union. Thus it is sufficient to show that each  $S_{t,p,m}$  is nowhere dense. To this aim, fix  $t, p, m \in \mathbf{N}$  and a nonempty relatively open set  $G \subseteq (0, 1]$ . We claim there exists a nonempty open set  $U$  contained in  $G$  and disjoint from  $S_{t,p,m}$ . Since  $G$  is nonempty and open in  $(0, 1]$ , there exists a string  $\tilde{s} = s_1 \cdots s_j \in S_b^j$  such that  $x \in G$  whenever  $d_{b,i}(x) = s_i$  for all  $i = 1, \dots, j$ . Now, pick  $x^* \in (0, 1]$  such that  $\lim_n \pi_{b,n}^k(x^*) = \eta_t$ ,

which exists by [22, theorem 1]. In addition, we can assume without loss of generality that  $d_{b,i}(x^*) = s_i$  for all  $i = 1, \dots, j$ . Since  $\pi_{b,n}^k(x^*)$  is convergent to  $\eta_t$ , there exists  $q \geq p + j$  such that  $\|\pi_{b,n}^k(x^*) - \eta_t\| < 1/m$  for all  $n \geq \min I_q$ . Define  $V := \{x \in (0, 1] : d_{b,i}(x) = d_{b,i}(x^*) \text{ for all } i = 1, \dots, \max I_q + k\}$  and note that  $V \subseteq G$  because  $d_{b,i}(x) = s_i$  for all  $i \leq j$  and  $x \in V$ , and  $V \cap S_{t,m,p} = \emptyset$  because, for each  $x \in V$ , the required property is not satisfied for this choice of  $q$  since  $\pi_{b,n}^k(x) = \pi_{b,n}^k(x^*)$  for all  $n \leq \max I_q$ . Clearly,  $V$  has nonempty interior, hence it is possible to choose such  $U \subseteq V$ .

This proves that each  $S_{t,m,p}$  is nowhere dense, concluding the proof.

Before we proceed to the proof of Theorem 2.3, we need to recall the classical Solecki’s characterisation of analytic P-ideals. A lower semicontinuous submeasure (in short, lscsm) is a monotone subadditive function  $\varphi : \mathcal{P}(\mathbf{N}) \rightarrow [0, \infty]$  such that  $\varphi(\emptyset) = 0$ ,  $\varphi(\{n\}) < \infty$ , and  $\varphi(A) = \lim_m \varphi(A \cap [1, m])$  for all  $A \subseteq \mathbf{N}$  and  $n \in \mathbf{N}$ . It follows by [26, theorem 3.1] that an ideal  $\mathcal{I}$  is an analytic P-ideal if and only if there exists a lscsm  $\varphi$  such that

$$\mathcal{I} = \{A \subseteq \mathbf{N} : \|A\|_\varphi = 0\}, \quad \|\mathbf{N}\|_\varphi = 1, \quad \text{and} \quad \varphi(\mathbf{N}) < \infty. \tag{3}$$

Here,  $\|A\|_\varphi := \lim_n \varphi(A \setminus [1, n])$  for all  $A \subseteq \mathbf{N}$ . Note that  $\|A\|_\varphi = \|B\|_\varphi$  whenever the symmetric difference  $A \Delta B$  is finite, cf. [11, lemma 1.3.3(b)]. Easy examples of lscsms are  $\varphi(A) := \#A$  or  $\varphi(A) := \sup_n (1/n)\#(A \cap [1, n])$  for all  $A \subseteq \mathbf{N}$  which lead, respectively, to the ideals  $\text{Fin}$  and  $\mathcal{Z}$  through the representation (3).

*Proof of Theorem 2.3.* First, let us suppose that  $\mathcal{I}$  is an  $F_\sigma$ -ideal. We obtain by [2, theorem 2.3] that  $\Lambda_b^k(x, \mathcal{I}) = \Gamma_b^k(x, \mathcal{I})$  for each  $b \geq 2$ ,  $k \geq 1$ , and  $x \in (0, 1]$ . Therefore the claim follows by Theorem 2.1.

Then, we assume hereafter that  $\mathcal{I}$  is an analytic P-ideal generated by a lscsm  $\varphi$  as in (3). Fix integers  $b \geq 2$  and  $k \geq 1$ , and define the function

$$u : (0, 1] \times \Delta_b^k \longrightarrow \mathbf{R} : (x, \eta) \longmapsto \lim_{t \rightarrow \infty} \|\{n \in \mathbf{N} : \|\pi_{b,n}^k(x) - \eta\| \leq 1/t\}\|_\varphi,$$

where  $\|\cdot\|$  stands for the Euclidean norm on  $\mathbf{R}^{b^k}$ . It follows by [2, lemma 2.1] that every section  $u(x, \cdot)$  is upper semicontinuous, so that the set

$$\Lambda_b^k(x, \mathcal{I}, q) := \{\eta \in \Delta_b^k : u(x, \eta) \geq q\}$$

is closed for each  $x \in (0, 1]$  and  $q \in \mathbf{R}$ .

At this point, we prove that, for each  $\eta \in \Delta_b^k$ , the set  $X(\eta) := \{x \in (0, 1] : u(x, \eta) \geq 1/2\}$  is comeager. To this aim, fix  $\eta \in \Delta_b^k$  and notice that

$$\begin{aligned} (0, 1] \setminus X(\eta) &= \bigcup_{t \geq 1} \{x \in (0, 1] : \|\{n \in \mathbf{N} : \|\pi_{b,n}^k(x) - \eta\| \leq 1/t\}\|_\varphi < 1/2\} \\ &= \bigcup_{t \geq 1} \{x \in (0, 1] : \lim_{h \rightarrow \infty} \varphi(\{n \geq h : \|\pi_{b,n}^k(x) - \eta\| \leq 1/t\}) < 1/2\} \\ &= \bigcup_{t,h \geq 1} \{x \in (0, 1] : \varphi(\{n \geq h : \|\pi_{b,n}^k(x) - \eta\| \leq 1/t\}) < 1/2\}. \end{aligned}$$

Denoting by  $Y_{t,h}$  the inner set above, it is sufficient to show that each  $Y_{t,h}$  is nowhere dense. Hence, fix  $G \subseteq (0, 1]$ ,  $\tilde{s} \in S_b^j$ , and  $x^* \in (0, 1]$  as in the proof of Theorem 2.1. Considering that  $\|\cdot\|_\varphi$  is invariant under finite sets, it follows that

$$\varphi(\{n \geq j' : \|\pi_{b,n}^k(x^*) - \eta\| \leq 1/t\}) \geq \|\{n \geq j' : \|\pi_{b,n}^k(x^*) - \eta\| \leq 1/t\}\|_\varphi = u(x^*, \eta) = 1,$$

where  $j' := j + h$ . Since  $\varphi$  is lower semicontinuous, there exists an integer  $j'' > j'$  such that

$$\varphi(\{n \in [j', j''] : \|\pi_{b,n}^k(x^*) - \eta\| \leq 1/t\}) \geq 1/2.$$

Define  $V := \{x \in (0, 1] : d_{b,i}(x) = d_{b,i}(x^*) \text{ for all } i = 1, \dots, j''\}$ . Similarly, note that  $V \subseteq G$  because  $d_{b,i}(x) = s_i$  for all  $i \leq j$  and  $x \in V$ , and  $V \cap Y_{t,h} = \emptyset$  because  $\varphi(\{n \geq h : \|\pi_{b,n}^k(x) - \eta\| \leq 1/t\})$  is at least  $\varphi(\{n \in [j', j''] : \|\pi_{b,n}^k(x) - \eta\| \leq 1/t\}) \geq 1/2$  for all  $x \in V$ . Since  $V$  has nonempty interior, it is possible to choose  $U \subseteq V$  with the required property.

Finally, let  $E$  be a countable dense subset of  $\Delta_b^k$ . Considering that  $X := \{x \in (0, 1] : E \subseteq \Lambda_b^k(x, \mathcal{I}, 1/2)\}$  is equal to  $\bigcap_{\eta \in E} X(\eta)$ , it follows that the set  $X$  is comeager. However, considering that

$$\Lambda_b^k(x, \mathcal{I}) = \bigcup_{q>0} \Lambda_b^k(x, \mathcal{I}, q)$$

by [2, theorem 2.2] and that  $\Lambda_b^k(x, \mathcal{I}, 1/2)$  is a closed subset such that  $E \subseteq \Lambda_b^k(x, \mathcal{I}, 1/2) \subseteq \Delta_b^k(x, \mathcal{I}) \subseteq \Delta_b^k$  for all  $x \in X$ , we obtain that  $\Lambda_b^k(x, \mathcal{I}, 1/2) = \Lambda_b^k(x, \mathcal{I}) = \Delta_b^k$  for all  $x \in X$ . In particular, the claimed set contains  $X$ , which is comeager. This concludes the proof.

### 4. Applications

#### 4.1. Hausdorff and packing dimensions

We refer to [10, chapter 3] for the definitions of the Hausdorff dimension and the packing dimension.

**PROPOSITION 4.1.** *The sets defined in Theorem 2.1 and Theorem 2.3 have Hausdorff dimension 0 and packing dimension 1.*

*Proof.* Reasoning as in [23], the claimed sets are contained in the corresponding ones with ideal  $\text{Fin}$ , which have Hausdorff dimension 0 by [22, theorem 2.1]. In addition, since all sets are comeager, we conclude that they have packing dimension 1 by [10, corollary 3.10(b)].

#### 4.2. Regular matrices

We extend the main results contained in [15, 27]. To this aim, let  $A = (a_{n,i} : n, i \in \mathbf{N})$  be a *regular matrix*, that is, an infinite real-valued matrix such that, if  $z = (z_n)$  is a  $\mathbf{R}^d$ -valued sequence convergent to  $\eta$ , then  $A_n z := \sum_i a_{n,i} z_i$  exists for all  $n \in \mathbf{N}$  and  $\lim_n A_n z = \eta$ , see e.g. [9, chapter 4]. Then, for each  $x \in (0, 1]$  and integers  $b \geq 2$  and  $k \geq 1$ , let  $\Gamma_b^k(x, \mathcal{I}, A)$  be the set of  $\mathcal{I}$ -cluster points of the sequence of vectors  $(A_n \pi_b^k(x) : n \geq 1)$ , where  $\pi_b^k(x)$  is the sequence  $(\pi_{b,n}^k(x) : n \geq 1)$ .

In particular,  $\Gamma_b^k(x, \mathcal{I}, A) = \Gamma_b^k(x, \mathcal{I})$  if  $A$  is the infinite identity matrix.

**THEOREM 4.2.** *The set  $\{x \in (0, 1] : \Gamma_b^k(x, \mathcal{I}, A) \supseteq \Delta_b^k \text{ for all } b \geq 2, k \geq 1\}$  is comeager, provided that  $\mathcal{I}$  is a meager ideal and  $A$  is a regular matrix.*

*Proof.* Fix a regular matrix  $A = (a_{n,i})$  and a meager ideal  $\mathcal{I}$ . The proof goes along the same lines as the proof of Theorem 2.1, replacing the definition of  $S_{t,m,p}$  with

$$S'_{t,m,p} := \{x \in (0, 1] : \forall q \geq p, \exists n \in I_q, \|A_n \pi_b^k(x) - \eta_t\| \geq 1/m\}.$$

Recall that, thanks to the classical Silverman–Toeplitz characterisation of regular matrices, see e.g. [9, theorem 4.1, II] or [8], we have that  $\sup_n \sum_i |a_{n,i}| < \infty$ . Since  $\lim_n \pi_{b,n}^k(x^*) = \eta_t$ , it follows that there exist sufficiently large integers  $q \geq p + j$  and  $j_A \geq j$  such that, if  $d_{b,i}(x) = d_{b,i}(x^*)$  for all  $i = 1, \dots, j_A + k$ , then

$$\begin{aligned} \|A_n \pi_b^k(x) - \eta_t\| &\leq \|A_n \pi_b^k(x^*) - \eta_t\| + \left\| \sum_i a_{n,i} (\pi_{b,i}^k(x) - \pi_{b,i}^k(x^*)) \right\| \\ &\leq \|A_n \pi_b^k(x^*) - \eta_t\| + \sum_i |a_{n,i}| \|\pi_{b,i}^k(x) - \pi_{b,i}^k(x^*)\| \\ &\leq \|A_n \pi_b^k(x^*) - \eta_t\| + \sum_{i > j_A} |a_{n,i}| < \frac{1}{m} \end{aligned} \tag{4}$$

for all  $n \in I_q$ . We conclude analogously that  $S'_{t,m,p}$  is nowhere dense.

The main result in [27] corresponds to the case  $\mathcal{I} = \text{Fin}$  and  $k = 1$ , although with a different proof; cf. also Example 4.10 below.

At this point, we need an intermediate result which is of independent interest. For each bounded sequence  $x = (x_n)$  with values in  $\mathbf{R}^k$ , let  $\text{K-core}(x)$  be the Knopp core of  $x$ , that is, the convex hull of the set of accumulation points of  $x$ . In other words,  $\text{K-core}(x) = \text{co } L_x$ , where  $\text{co } S$  is the convex hull of  $S \subseteq \mathbf{R}^k$  and  $L_x$  is the set of accumulation points of  $x$ . The ideal version of the Knopp core has been studied in [16, 18]. The classical Knopp theorem states that, if  $k = 2$  and  $A$  is a nonnegative regular matrix, then

$$\text{K-core}(Ax) \subseteq \text{K-core}(x) \tag{5}$$

for all bounded sequences  $x$ , where  $Ax = (A_n x : n \geq 1)$ , see [17, p. 115]; cf. [9, chapter 6] for a textbook exposition. A generalisation in the case  $k = 1$  can be found in [21]. We show, in particular, that a stronger version of Knopp’s theorem holds for every  $k \in \mathbf{N}$ .

**PROPOSITION 4.3.** *Let  $x = (x_n)$  be a bounded sequence taking values in  $\mathbf{R}^k$ , and fix a regular matrix  $A$  such that  $\lim_n \sum_i |a_{n,i}| = 1$ . Then inclusion (5) holds.*

*Proof.* Define  $\kappa := \sup_n \|x_n\|$  and let  $\eta$  be an accumulation point of  $Ax$ . It is sufficient to show that  $\eta \in K := \text{K-core}(x)$ . Possibly deleting some rows of  $A$ , we can assume without loss of generality that  $\lim Ax = \eta$ . For each  $m \in \mathbf{N}$ , let  $K_m$  be the closure of  $\text{co}\{x_m, x_{m+1}, \dots\}$ , hence  $K \subseteq K_m$ . Define  $d(a, C) := \min_{b \in C} \|a - b\|$  for all  $a \in \mathbf{R}^k$  and nonempty compact sets  $C \subseteq \mathbf{R}^k$ . In addition, for each  $m \in \mathbf{N}$ , let  $Q_m(a) \in K_m$  be the unique vector such that  $d(a, K_m) = \|a - Q_m(a)\|$ . Similarly, let  $Q(a)$  be the vector in  $K$  which minimizes its distance with  $a$ . Then, notice that, for all  $n, m \in \mathbf{N}$ , we have

$$\begin{aligned} d(A_n x, K) &\leq \inf_{b \in K} \inf_{c \in \mathbf{R}^k} (\|A_n x - c\| + \|c - b\|) \\ &\leq \inf_{c \in K_m} \inf_{b \in K} (\|A_n x - c\| + \|c - b\|) \\ &\leq \inf_{c \in K_m} \|A_n x - c\| + \sup_{y \in K_m} \inf_{b \in K} \|y - b\| \\ &= d(A_n x, K_m) + \sup_{y \in K_m} d(y, K). \end{aligned}$$

Since  $d(\eta, K) = \lim_n d(A_n x, K)$  by the continuity of  $d(\cdot, K)$ , it is sufficient to show that both  $d(A_n x, K_m)$  and  $\sup_{y \in K_m} d(y, K)$  are sufficiently small if  $n$  is sufficiently large and  $m$  is chosen properly.

To this aim, fix  $\varepsilon > 0$  and choose  $m \in \mathbf{N}$  such that  $\sup_{y \in K_m} d(y, K) \leq \varepsilon/2$ . Indeed, it is sufficient to choose  $m \in \mathbf{N}$  such that  $d(x_n, L_x) < \varepsilon/2$  for all  $n \geq m$ : indeed, in the opposite,



the subsequence  $(\mathbf{x}_j)_{j \in J}$ , where  $J := \{n \in \mathbb{N} : d(\mathbf{x}_n, L_x) \geq \varepsilon/2\}$ , would be bounded and without any accumulation point, which is impossible. Now pick  $\mathbf{y} \in K_m$  so that  $\mathbf{y} = \sum_j \lambda_{ij} \mathbf{x}_{ij}$  for some strictly increasing sequence  $(i_j)$  of positive integers such that  $i_1 \geq m$  and some real nonnegative sequence  $(\lambda_{ij})$  with  $\sum_j \lambda_{ij} = 1$ . It follows that

$$d(\mathbf{y}, K) \leq \left\| \mathbf{y} - \sum_j \lambda_{ij} Q(\mathbf{x}_{ij}) \right\| \leq \sum_j \lambda_{ij} \|\mathbf{x}_{ij} - Q(\mathbf{x}_{ij})\| \leq \sum_j \lambda_{ij} d(\mathbf{x}_{ij}, L_x) \leq \frac{\varepsilon}{2}.$$

Suppose for the moment that  $A$  has nonnegative entries. Since  $A$  is regular, we get  $\lim_n \sum_i a_{n,i} = 1$  and  $\lim_n \sum_{i < m} a_{n,i} = 0$  by the Silverman–Toeplitz characterisation, hence  $\lim_n \sum_{i \geq m} a_{n,i} = 1$  and there exists  $n_0 \in \mathbb{N}$  such that  $\sum_{i \geq m} a_{n,i} \geq 1/2$  for all  $n \geq n_0$ . Thus, for each  $n \geq n_0$ , we obtain that  $d(A_n \mathbf{x}, K_m) = \|A_n \mathbf{x} - Q_m(A_n \mathbf{x})\| \leq \alpha_n + \beta_n + \gamma_n$ , where

$$\alpha_n := \left\| A_n \mathbf{x} - \frac{A_n \mathbf{x}}{\sum_i a_{n,i}} \right\|, \quad \beta_n := \left\| \frac{A_n \mathbf{x}}{\sum_i a_{n,i}} - Q_m \left( \frac{A_n \mathbf{x}}{\sum_i a_{n,i}} \right) \right\|$$

and

$$\gamma_n := \left\| Q_m \left( \frac{A_n \mathbf{x}}{\sum_i a_{n,i}} \right) - Q_m(A_n \mathbf{x}) \right\|.$$

Recalling that  $\kappa = \sup_n \|\mathbf{x}_n\|$ , it is easy to see that

$$\gamma_n \leq \alpha_n \leq \kappa \sum_i |a_{n,i}| \cdot \left( 1 - \frac{1}{\sum_i a_{n,i}} \right).$$

In addition, setting  $t_n := \sum_{i \geq m} a_{n,i} / \sum_i a_{n,i} \in [0, 1]$  for all  $n \geq n_0$ , we get

$$\begin{aligned} \beta_n &\leq \left\| \frac{\sum_i^* a_{n,i}}{\sum_i a_{n,i}} - \frac{\sum_{i \geq m}^* a_{n,i}}{\sum_{i \geq m} a_{n,i}} \right\| \\ &= \frac{1}{\sum_{i \geq m} a_{n,i} \sum_i a_{n,i}} \left\| \sum_{i \geq m} a_{n,i} (\sum_{i < m}^* + \sum_{i \geq m}^*) - \sum_i a_{n,i} \sum_{i \geq m}^* \right\| \\ &= \frac{1}{\sum_{i \geq m} a_{n,i}} \left\| t_n \sum_{i < m}^* + (1 - t_n) \sum_{i \geq m}^* \right\| \\ &\leq 2\kappa \left( t_n \sum_{i < m} |a_{n,i}| + (1 - t_n) \sum_i |a_{n,i}| \right), \end{aligned} \tag{6}$$

where  $\sum_{i \in I}^*$  stands for  $\sum_{i \in I} a_{n,i} \mathbf{x}_i$ . Note that the hypothesis that the entries of  $A$  are nonnegative has been used only in the first line of (6), so that  $\sum_{i \geq m}^* / \sum_{i \geq m} a_{n,i} \in K_m$ . Since  $\lim_n \sum_{i < m} |a_{n,i}| = 0$ ,  $\lim_n t_n = 1$ , and  $\sup_n \sum_i |a_{n,i}| < \infty$  by the regularity of  $A$ , it follows that all  $\alpha_n, \beta_n, \gamma_n$  are smaller than  $\varepsilon/6$  if  $n$  is sufficiently large. Therefore  $d(A_n \mathbf{x}, K) \leq \varepsilon$  and, since  $\varepsilon$  is arbitrary, we conclude that  $\boldsymbol{\eta} = \lim_n A_n \mathbf{x} \in K$ .

Lastly, suppose that  $A$  is a regular matrix such that  $\lim_n \sum_i |a_{n,i}| = 1$  and let  $B = (b_{n,i})$  be the nonnegative regular matrix defined by  $b_{n,i} = |a_{n,i}|$  for all  $n, i \in \mathbb{N}$ . Considering that

$$d(A_n \mathbf{x}, K_m) \leq \|A_n \mathbf{x} - B_n \mathbf{x}\| + d(B_n \mathbf{x}, K_m) \leq \kappa \sum_i |a_{n,i} - |a_{n,i}|| + \varepsilon,$$

and that  $\lim_n \sum_i |a_{n,i} - |a_{n,i}|| = 0$  because  $\lim_n \sum_i a_{n,i} = \lim_n \sum_i |a_{n,i}| = 1$ , we conclude that  $d(A_n \mathbf{x}, K_m) \leq 2\varepsilon$  whenever  $n$  is sufficiently large. The claim follows as before.



The following corollary is immediate:

**COROLLARY 4.4.** *Let  $\mathbf{x} = (x_n)$  be a bounded sequence taking values in  $\mathbf{R}^k$ , and fix a nonnegative regular matrix  $A$ . Then inclusion (5) holds.*

*Remark 4.5.* Inclusion (5) fails for an arbitrary regular matrix: indeed, let  $A = (a_{n,i})$  be the matrix defined by  $a_{n,2n} = 2$ ,  $a_{n,2n-1} = -1$  for all  $n \in \mathbf{N}$ , and  $a_{n,i} = 0$  otherwise. Set also  $k = 1$  and let  $x$  be the sequence such that  $x_n = (-1)^n$  for all  $n \in \mathbf{N}$ . Then  $A$  is regular and  $\lim Ax = 3 \notin \{-1, 1\} = \mathbf{K}\text{-core}(x)$ .

*Remark 4.6.* Proposition 4.3 keeps holding on a (possibly infinite dimensional) Hilbert space  $X$  with the following proviso: replace the definition of  $\mathbf{K}\text{-core}(x)$  with the closure of  $\text{co } L_x$  (this coincides in the case that  $X = \mathbf{R}^k$ ) and assume that the sequence  $x$  is contained in a compact set (so that  $\mathbf{K}\text{-core}(x)$  is also nonempty).

With these premises, we can strengthen Theorem 4.2 as follows.

**THEOREM 4.7.** *The set  $\{x \in (0, 1] : \Gamma_b^k(x, \mathcal{I}, A) = \Delta_b^k \text{ for all } b \geq 2, k \geq 1\}$  is comeager, provided that  $\mathcal{I}$  is a meager ideal and  $A$  is a regular matrix such that  $\lim_n \sum_i |a_{n,i}| = 1$ .*

*Proof.* Let us suppose that  $A = (a_{n,i})$  is nonnegative regular matrix, i.e.,  $a_{n,i} \geq 0$  for all  $n, i \in \mathbf{N}$ , and fix a meager ideal  $\mathcal{I}$ , a real  $x \in (0, 1]$ , and integers  $b \geq 2, k \geq 1$ . Thanks to Theorem 4.2, it is sufficient to show that every accumulation point of the sequence  $(A_n \pi_b^k(x) : n \geq 1)$  is contained in the convex hull of the set of accumulation points of  $(\pi_{b,n}^k(x) : n \geq 1)$ , which is in turn contained into  $\Delta_b^k$ . This follows by Proposition 4.3.

Since the family of meager sets is a  $\sigma$ -ideal, the following is immediate by Theorem 4.7.

**COROLLARY 4.8.** *Let  $\mathcal{A}$  be a countable family of regular matrices such that  $\lim_n \sum_i |a_{n,i}| = 1$ . Then the set  $\{x \in (0, 1] : \Gamma_b^k(x, \mathcal{I}, A) = \Delta_b^k \text{ for all } b \geq 2, k \geq 1, \text{ and all } A \in \mathcal{A}\}$  is comeager, provided that  $\mathcal{I}$  is a meager ideal.*

It is worth to remark that the main result [15] is obtained as an instance of Corollary 4.8, letting  $\mathcal{A}$  be the set of iterates of the Cesàro matrix (note that they are nonnegative regular matrices), and setting  $k = 1$  and  $\mathcal{I} = \text{Fin}$ . The same holds for the iterates of the Hölder matrix and the logarithmic Riesz matrix as in [24, sections 3 and 4].

Next, we show that the hypothesis  $\lim_n \sum_i |a_{n,i}| = 1$  for the entries of the regular matrix in Theorem 4.7 cannot be removed.

*Example 4.9.* Let  $A = (a_{n,i})$  be the matrix such that  $a_{n,(2n-1)!} = -1$  and  $a_{n,(2n)!} = 2$  for all  $n \in \mathbf{N}$ , and  $a_{n,i} = 0$  otherwise. It is easily seen that  $A$  is regular. Then, set  $b = 2, k = 1$ , and  $\mathcal{I} = \text{Fin}$ . We claim that the set of all  $x \in (0, 1]$  such that 2 is an accumulation point of the sequence  $\pi_{2,1}(x) = (\pi_{2,1,n}(x) : n \geq 1)$  is comeager. Indeed, its complement can be rewritten as  $\bigcup_{m,p} S_{m,p}$ , where

$$S_{m,p} := \{x \in (0, 1] : |A_n \pi_{2,1}(x) - 2| \geq 1/m \text{ for all } n \geq p\}.$$

Let  $x^* \in (0, 1]$  such that  $d_{2,n}(x^*) = 1$  if and only if  $(2i - 1)! \leq n < (2i)!$  for some  $i \in \mathbf{N}$ . Then it is easily seen that  $\lim_n \pi_{2,1,n}(x^*) = 2$ . Along the same lines of the proof of Theorem 4.2, it follows that each  $S_{m,p}$  is meager. We conclude that  $\{x \in (0, 1] : \Gamma_2^1(x, \text{Fin}, A) = \Delta_2^1\}$  is meager,

which proves that the condition  $\lim_n \sum_i |a_{n,i}| = 1$  in the statement of Theorem 4.7 cannot be removed.

In addition, the main result in [27] states that Theorem 4.2, specialised to the case  $\mathcal{I} = \text{Fin}$  and  $k = 1$ , can be further strengthened so that the set  $\{x \in (0, 1] : \Gamma_b^1(x, \text{Fin}, A) \supseteq \Delta_b^1\}$  for all  $b \geq 2$  and all regular  $A$  is comeager. Taking into account the argument in the proof of Theorem 4.7, this would imply that the set

$$\{x \in (0, 1] : \Gamma_b^1(x, \text{Fin}, A) = \Delta_b^1 \text{ for all } b \geq 2 \text{ and all nonnegative regular } A\} \quad (7)$$

should be comeager. However, this is false as it is shown in the next example.

*Example 4.10.* For each  $y \in (0, 1]$ , let  $(e_{y,k} : k \geq 1)$  be the increasing enumeration of the infinite set  $\{n \in \mathbb{N} : d_{2,n}(y) = 1\}$ . Then, let  $\mathcal{A} = \{A_y : y \in (0, 1]\}$  be family of matrices  $A_y = (a_{n,i}^{(y)})$  with entries in  $\{0, 1\}$  so that  $a_{n,i}^{(y)} = 1$  if and only if  $e_{y,n} = i$  for all  $y \in (0, 1]$  and all  $n, i \in \mathbb{N}$ . Then each  $A_y$  is a nonnegative regular matrix. It follows, for each ideal  $\mathcal{I}$ ,

$$\{x \in (0, 1] : \Gamma_2^1(x, \mathcal{I}, A) = \Delta_2^1 \text{ for all } A \in \mathcal{A}\} = \emptyset.$$

Indeed, for each  $x \in (0, 1]$ , the sequence  $\pi_2^1(x) = (\pi_{2,n}^1(x) : n \geq 1)$  has an accumulation point  $\eta \in \Delta_2^1$ . Hence there exists a subsequence  $(\pi_{2,n_k}^1(x) : k \geq 1)$  which is convergent to  $\eta$ . Equivalently,  $\lim A_y \pi_2^1(x) = \eta$ , where  $y \in (0, 1]$  is defined such that  $e_{y,k} = n_k$  for all  $k \in \mathbb{N}$ . Therefore  $\{\eta\} = \Gamma_2^1(x, \mathcal{I}, A_y) \neq \Delta_2^1$ . In particular, the set defined in (7) is empty.

Lastly, the analogues of Theorem 4.2 and Theorem 4.7 hold for  $\mathcal{I}$ -limit points, if  $\mathcal{I}$  is an  $F_\sigma$ -ideal or an analytic P-ideal. Indeed, denoting with  $\Lambda_b^k(x, \mathcal{I}, A)$  the set of  $\mathcal{I}$ -limit points of the sequence  $(A_n \pi_b^k(x) : n \geq 1)$ , we obtain:

**THEOREM 4.11.** *Let  $A$  be a regular matrix and let  $\mathcal{I}$  be an  $F_\sigma$ -ideal or an analytic P-ideal. Then the set  $\{x \in (0, 1] : \Lambda_b^k(x, \mathcal{I}, A) \supseteq \Delta_b^k\}$  for all  $b \geq 2, k \geq 1\}$  is comeager.*

*Moreover, the set  $\{x \in (0, 1] : \Lambda_b^k(x, \mathcal{I}, A) = \Delta_b^k\}$  for all  $b \geq 2, k \geq 1\}$  is comeager if, in addition,  $A$  satisfies  $\lim_n \sum_i |a_{n,i}| = 1$ .*

*Proof.* The first part goes along the same lines of the proof of Theorem 2.3. Here, we replace  $\pi_b^k(x)$  with  $(A_n \pi_b^k(x) : n \geq 1)$  and using the chain of inequalities (4): more precisely, we consider  $j'' \in \mathbb{N}$  such that  $\varphi(\{n \in [j', j''] : \|A_n \pi_b^k(x') - \eta\| \leq 1/2t\}) \geq 1/2$ , and, taking into considering (4), we define  $V := \{x \in (0, 1] : d_{b,i}(x) = d_{b,i}(x^*) \text{ for all } i = 1, \dots, k + j'''\}$ , where  $j'''$  is a sufficiently large integer such that  $\sum_{i > j'''} |a_{n,i}| \leq 1/2t$  for all  $n \in [j', j'']$ .

The second part follows, as in Theorem 4.7, by the fact that every accumulation point of  $(A_n \pi_b^k(x) : n \geq 1)$  belongs to  $\Delta_b^k$ .

*Acknowledgments.* P. Leonetti is grateful to PRIN 2017 (grant 2017CY2NCA) for financial support.

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