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Invariant measures for interval maps without Lyapunov exponents

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Abstract. We construct an invariant measure for a piecewise analytic interval map whose Lyapunov exponent is not defined. Moreover, for a set of full measure, the pointwise Lyapunov exponent is not defined. This map has a Lorenz-like singularity and non-flat critical points.

Key words: interval map, Lyapunov exponents, ergodic measures
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1. *Introduction*

Lyapunov exponents play an important role in the study of the ergodic behavior of dynamical systems. In particular, in the seminal work of Pesin (referred to as the 'Pesin theory'), the existence and positivity of Lyapunov exponents were used to study the dynamics of non-uniformly hyperbolic systems, see for example [**KH95**, Supplement]. Using these ideas, Ledrappier [**Led81**] studied ergodic properties of absolutely continuous invariant measures for regular maps of the interval under the assumption that the Lyapunov exponent exists and is positive. Recently, Dobbs [**Dob14**, **Dob15**] developed the Pesin theory for non-invertible interval maps with Lorenz-like singularities and non-flat critical points. Lima [**Lim20**] constructed a symbolic extension for these maps that code the measures with positive Lyapunov exponents.

In the case of continuously differentiable interval maps, Przytycki proved that ergodic invariant measures have non-negative Lyapunov exponent or they are supported on a strictly attracting periodic orbit of the system. Moreover, there exists a set of full measure for which the pointwise Lyapunov exponent exists and is non-negative, see [**Prz93**] and [**RL20**, Appendix A].

In this paper, we show that the result above cannot be extended to continuous piecewise differentiable interval maps with a finite number of non-flat critical points and Lorenz-like singularities. In particular, we construct a measure for a unimodal map with a Lorenz-like singularity and two non-flat critical points for which the Lyapunov exponent does not exist. Moreover, for this map, the pointwise Lyapunov exponent does not exist for a set of full measure. Thus, our example shows that the techniques developed by Dobbs [**Dob14**, **Dob15**] and Lima [**Lim20**] cannot be extended to all maps with critical points and Lorenz-like singularities.

Maps with Lorenz-like singularities are of interest as they appear in the study of the Lorenz attractor, see [**GW79**, **LT99**] and references therein. Apart from these motivations, these types of maps are of interest on their own because the presence of these types of singularities create expansion and hence enforce the chaotic behavior of the system, see [**ALV09**, **Dob14**, **LM13**] and references therein.

Additionally, the unimodal map that we consider has Fibonacci recurrence of the turning point (or just Fibonacci recurrence). Maps with Fibonacci recurrence first appeared in the work of Hofbauer and Keller [**HK90**] as possible interval maps having a wild attractor. Lyubich and Milnor [**LM93**] proved that unimodal maps with a quadratic critical point and Fibonacci recurrence not only have any Cantor attractor but also have a finite absolutely continuous invariant measure, see also [**KN95**]. Finally, Bruin *et al* [**BKNvS96**] proved that a C^2-unimodal interval map with a critical point of order big enough and with Fibonacci recurrence has a wild Cantor attractor. However, in the work of Branner and Hubbard [**BH92**], in the case of complex cubic polynomials, and the work of Yoccoz, in the case of complex quadratic polynomials, Fibonacci recurrence appeared as the worst pattern of recurrence, see for example [**Hub93**, **Mil00**]. Maps with Fibonacci recurrence also play an important role in the renormalization theory, see for example [**GS18**, **LS12**, **Sma07**] and references therein.

1.1. *Statement of results.* To state our main result, we need to recall some definitions. A continuous map $f\colon [-1,1]\to[-1,1]$ is *unimodal* if there is $c\in(-1,1)$ such that $f|_{[-1,c)}$ is increasing and $f|_{(c,1]}$ is decreasing. We call c the *turning point of f*. For every $A\subset[-1,1]$ and every $x\in[-1,1]$, we denote the *distance from x to A* by

$$\operatorname{dist}(x,A) := \inf\{|x-y|\colon y\in A\}.$$

We will use f' to denote the derivative of f. We will say that the point $c\in[-1,1]$ is a *Lorenz-like singularity* if there exists ℓ^+ and ℓ^- in $(0,1)$, $L>0$, and $\delta>0$ such that the following holds: for every $x\in(c,c+\delta)$,

$$\frac{1}{L|x-c|^{\ell^+}} \le |f'(x)| \le \frac{L}{|x-c|^{\ell^+}}, \tag{1.1}$$

and for every $x\in(c-\delta,c)$,

$$\frac{1}{L|x-c|^{\ell^-}} \le |f'(x)| \le \frac{L}{|x-c|^{\ell^-}}. \tag{1.2}$$

We call ℓ^+ and ℓ^- the *right and left order of c*, respectively. For an interval map f, a point $\hat{c}\in[-1,1]$ is called a *critical point* if $f'(\hat{c})=0$. We will say that a critical point $\hat{c}$ is

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non-flat if there exist $\alpha^+ > 0$, $\alpha^- > 0$, $M > 0$, and $\delta > 0$ such that the following holds: for every $x \in (\hat{c}, \hat{c} + \delta)$,

$$\left| \log \frac{|f'(x)|}{|x - \hat{c}|^{\alpha^+}} \right| \leq M, \tag{1.3}$$

and for every $x \in (\hat{c} - \delta, \hat{c})$,

$$\left| \log \frac{|f'(x)|}{|x - \hat{c}|^{\alpha^-}} \right| \leq M. \tag{1.4}$$

We call α^+ and α^- the *right and left order of* $\hat{c}$, respectively. Let us denote by $\mathrm{Crit}(f)$ the set of critical points of f. If f is a unimodal map with turning point c, we will use the notation $\mathcal{S}(f) := \mathrm{Crit}(f) \cup \{c\}$. Let us denote by C^ω the class of analytic maps. Here we will say that f is a C^ω*-unimodal map* if it is of class C^ω outside $\mathcal{S}(f)$.

We denote the orbit of $x \in [-1, 1]$ under f by

$$\mathcal{O}_f(x) := \{f^n(x) : n \geq 0\}.$$

For a probability measure μ on $[-1, 1]$ that is invariant by f, we define the *pushforward of* μ *by* f as

$$f_*\mu := \mu \circ f^{-1}.$$

Denote by

$$\chi_\mu(f) := \int \log |f'| \, d\mu,$$

its *Lyapunov exponent*, if the integral exists. Similarly, for every $x \in [-1, 1]$ such that $\mathcal{O}_f(x) \cap \mathcal{S}(f) = \emptyset$, denote by

$$\chi_f(x) := \lim_{n \to \infty} \frac{1}{n} \log |(f^n)'(x)|,$$

the *pointwise Lyapunov exponent of* f *at* x, if the limit exists.

Let $\lambda_F \in (0, 2]$ be so that the map $T_{\lambda_F} : [-1, 1] \to [-1, 1]$, defined by

$$T_{\lambda_F}(x) := \lambda_F(1 - |x|) - 1, \tag{1.5}$$

for every $x \in [-1, 1]$, has Fibonacci recurrence and let μ_P be the unique measure that is ergodic, invariant by T_{λ_F}, and supported on $\overline{\mathcal{O}_{T_{\lambda_F}}(0)}$, see §2.2.

THEOREM 1. *Let $h : [-1, 1] \to [-1, 1]$ be a homeomorphism of class C^ω on $[-1, 1] \setminus \{0\}$ with a unique non-flat critical point at 0, and put $\tilde{\mu}_P := h_*\mu_P$. Then the C^ω-unimodal map $f := h \circ T_{\lambda_F} \circ h^{-1}$ has a Lorenz-like singularity at $\tilde{c} := h(0)$ and is so that the following hold.*

(1) *$\chi_{\tilde{\mu}_P}(f)$ is not defined.*
(2) *For $x \in \overline{\mathcal{O}_f(\tilde{c})}$, the pointwise Lyapunov exponent of f at x does not exist if $\mathcal{O}_f(x) \cap \mathcal{S}(f) = \emptyset$, and it is not defined if $\mathcal{O}_f(x) \cap \mathcal{S}(f) \neq \emptyset$.*

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(3) $\log(\mathrm{dist}(\cdot, \mathcal{S}(f))) \notin L^1(\tilde{\mu}_P)$.

(4) *f has exponential recurrence of the Lorenz-like singularity orbit, and thus,*

$$\limsup_{n\to\infty} \frac{-\log|f^n(\tilde{c}) - \tilde{c}|}{n} \in (0, +\infty).$$

The map f in Theorem 1 has a Lorenz-like singularity at $\tilde{c}$ and two non-flat critical points, given by the preimages by f of the Lorenz-like singularity $\tilde{c}$, see Proposition 1.1. Because h is a homeomorphism of class C^ω on $[-1, 1] \setminus \{0\}$, these are non-flat critical points of inflection type.

Dobbs constructed an example of a unimodal map with a flat critical point and singularities at the boundary, for which the Lyapunov exponent of an invariant measure does not exist, see [**Dob14**, Proposition 43]. For interval maps with infinite Lyapunov exponent, see [**Ped20**, Theorem A] and references therein.

The negation of item (3) in Theorem 1 is considered in several works as a regularity condition to study ergodic invariant measures. In [**Lim20**], Lima studied measures satisfying this condition for interval maps with critical points and discontinuities, he called measures satisfying this condition *f-adapted*. By the Birkhoff ergodic theorem, if $\log(\mathrm{dist}(\cdot, \mathcal{S}(f))) \in L^1(\mu)$, then for an ergodic invariant measure μ, we have

$$\lim_{n\to\infty} \frac{1}{n} \log(\mathrm{dist}(f^n(x), \mathcal{S}(f))) = 0,$$

μ-almost everywhere (a.e.). Ledrappier called measures satisfying this last condition *non-degenerated*. For interval maps with a finite number of critical points, see [**Led81**]. The measure $\tilde{\mu}_P$ does not satisfy the non-degenerated condition. For more results related to this condition see [**Lim20**] and references therein.

For continuously differentiable interval maps with a finite number of critical points, every ergodic invariant measure that is not supported on an attracting periodic point satisfies $\lim_{n\to\infty}(1/n)\log(\mathrm{dist}(f^n(x), \mathcal{S}(f))) = 0$, a.e., see [**Prz93**] and [**RL20**, Appendix]. Item (3) in Theorem 1 tells us that we cannot extend this to piecewise differentiable maps with a finite number of critical points and Lorenz-like singularities.

Item (4) in Theorem 1 stresses important information relative to the recurrence of the Lorenz-like singularity. This item represents a crucial difference between smooth interval maps, and the case of interval maps with critical points and Lorenz-like singularities. In the smooth case, certain conditions on the growth of the derivative restrict the recurrence to the critical set, see for example [**CE83**, **GS14**, **Tsu93**] and references therein. Using the terminology in [**DPU96**], item (4) shows that rule I is sharp for the map f in Theorem 1. Finally, we have that the map f in Theorem 1 satisfies Tsuji's weak regularity condition, owing to the interaction between the critical points and the Lorenz-like singularity.

1.2. *Example.* Now we will provide an example of a map f as in Theorem 1. Fix ℓ^+ and ℓ^- in $(0, 1)$. Put $\alpha^+ := 1/(1-\ell^+)$ and $\alpha^- := 1/(1-\ell^-)$. Define

$$h_{\alpha^+,\alpha^-} : [-1, 1] \longrightarrow [-1, 1]$$

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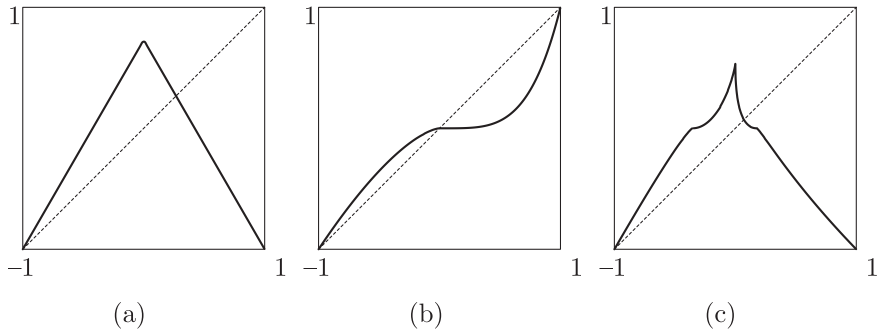

FIGURE 1. Graphs of the functions $T_{\lambda_F}(x)$: $h_\alpha(x)$ for $\alpha^+ = 2$ (a); $\alpha^- = 1.2$ (b); and (c) $f(x)$.

as

$$h_{\alpha^+,\alpha^-}(x) = \begin{cases} |x|^{\alpha^+} & \text{if } x \geq 0, \\ -|x|^{\alpha^-} & \text{if } x < 0. \end{cases} \tag{1.6}$$

So

$$h^{-1}_{\alpha^+,\alpha^-}(x) = \begin{cases} |x|^{1/\alpha^+} & \text{if } x \geq 0, \\ -|x|^{1/\alpha^-} & \text{if } x < 0. \end{cases} \tag{1.7}$$

Then by (1.5), (1.6), (1.7), and the chain rule, we have

$$f'(x) = \lambda_F \frac{h'_{\alpha^+,\alpha^-}(T_{\lambda_F}(h^{-1}_{\alpha^+,\alpha^-}(x)))}{h'_{\alpha^+,\alpha^-}(h^{-1}_{\alpha^+,\alpha^-}(x))}$$

for every $x \in [-1, 1] \setminus \{h_{\alpha^+,\alpha^-}(0)\}$. The function $h'_{\alpha^+,\alpha^-}(T_{\lambda_F}(h^{-1}_{\alpha^+,\alpha^-}(x)))$ is bounded for x close enough to 0, see §5. Then, by (1.6) and (1.7), there exists $L > 0$ such that for every $x \in (h(0), h(\delta))$,

$$\frac{1}{L|x|^{\ell^+}} \leq |f'(x)| \leq \frac{L}{|x|^{\ell^+}},$$

and for every $x \in (h(0), h(-\delta))$,

$$\frac{1}{L|x|^{\ell^-}} \leq |f'(x)| \leq \frac{L}{|x|^{\ell^-}}.$$

Thus, $h(0)$ is a Lorenz-like singularity of f, see Figure 1. Also, by (1.6) and (1.7), if δ is small enough so that $T^{-1}_{\lambda_F}(0) \cap (-\delta, \delta) = \emptyset$, the two critical points of f are non-flat. The one to the left of $h(0)$ has right-order α^+ and left-order α^-, and the one to the right of $h(0)$ has right-order α^- and left-order α^+.

1.3. *Strategy and organization.* We now describe the strategy of the proof of Theorem 1 and the organization of the paper.

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In §2, we review some general theory and results concerning the kneading sequence for unimodal maps and unimodal maps with Fibonacci recurrence. In particular, in §2.1, we will describe the relationship between the kneading map and the kneading sequence, and in §2.2, we define the Fibonacci recurrence. These two elements will be of importance to describe the combinatorics of the critical orbit.

In §3.1, we make a detailed description of the set $\overline{O_{T_{\lambda_F}}(0)}$, and following [**LM93**], we construct a partition of it that will allow us to estimate close return times to the turning point and lower bounds for the distances of these close returns. In §3.2, we estimate how fast the orbit of the turning point return to itself in terms of the return time (see Lemma 3.5). This estimation is of importance because it gives us an exact estimation of the growth of the geometry near the turning point for our map f.

In §4, we describe the unique ergodic invariant measure μ_P supported on $\overline{O_{T_{\lambda_F}}(0)}$, restricted to the partition constructed in §3.1. We need this estimation to prove part (1) in Theorem 1.

In §5, we prove the following proposition that will give us a key bound on the derivative of f in terms of h^{-1}. Without loss of generality, we will assume that h preserves orientation.

PROPOSITION 1.1. *Let h and f be as in Theorem 1. Then f has a Lorenz-like singularity at $\tilde{c}$. Moreover, there exist $\alpha^+ > 1$, $\alpha^- > 1$, $K > 0$, and $\delta > 0$ such that the following property holds: for every $x \in (\tilde{c}, h(\delta))$,*

$$K^{-1}|h^{-1}(x)|^{-\alpha^+} \leq |f'(x)| \leq K|h^{-1}(x)|^{-\alpha^+}, \tag{1.8}$$

and for every $x \in (h(-\delta), \tilde{c})$,

$$K^{-1}|h^{-1}(x)|^{-\alpha^-} \leq |f'(x)| \leq K|h^{-1}(x)|^{-\alpha^-}. \tag{1.9}$$

In §6, we prove the following proposition that implies items (1) and (3) in Theorem 1. Let

$$\log^+ |f'| := \max\{0, \log |f'|\} \quad \text{and} \quad \log^- |f'| := \{0, -\log |f'|\},$$

on $[-1, 1] \setminus \{\tilde{c}\}$.

PROPOSITION 1.2. *Let h and f be as in Theorem 1. Then:*

(i) $\int \log^+ |f'|\, d\tilde{\mu}_P = +\infty$;
(ii) $\int \log^- |f'|\, d\tilde{\mu}_P = -\infty$; *and*
(iii) $\int |\log(\mathrm{dist}(\cdot, \mathcal{S}(f)))|\, \tilde{\mu}_P = +\infty$.

To prove the first part of Proposition 1.2, we use the fact that around the Lorenz-like singularity, the geometry of f grows at the same rate as the measure decreases. This implies that in a sequence of disjoint intervals that converges to the critical point, the integral of $\log |f'|$ is bounded from below by a positive constant. For the second part, we use the fact that the two preimages of the turning point of f are critical points and both belong to the set $\overline{O_f(\tilde{c})}$. The third part of the proposition is a consequence of the estimation that we get in the proof of the first part.

In §7, we prove the following proposition that, along with the fact that f is transitive on $\overline{O_f(\tilde{c})}$, will imply item (2) in Theorem 1. Recall that for $x \in \overline{O_f(\tilde{c})}$ such that $\tilde{c} \in O_f(x)$, we have that the pointwise Lyapunov exponent is not defined, because for n large enough, $\log |(f^n)'(x)|$ is not defined.

PROPOSITION 1.3. *Let h and f be as in Theorem 1. Then, there exists a positive number α such that for every $x \in \overline{O_f(\tilde{c})}$ with $\tilde{c} \notin O_f(x)$, we have that*

$$\liminf_{n\to\infty} \frac{1}{n} \log |(f^n)'(x)| \le \left(1 - \frac{\alpha}{\varphi}\right) \log \lambda < \log \lambda \le \limsup_{n\to\infty} \frac{1}{n} \log |(f^n)'(x)|, \quad (1.10)$$

where $\varphi := (1+\sqrt{5})/2$.

To prove Proposition 1.3, we use the fact that f restricted to the set $\overline{O_f(\tilde{c})}$ is minimal, so the orbit of every point accumulates points far from the turning point. In that case, the derivative is bounded so the limit of that subsequence must be the same as the one in T_{λ_F}. However, if we look at a subsequence that accumulates at the Lorenz-like singularity, the growth of the derivative is exponential with respect to the return time, so the limit of this subsequence will be bounded away from zero.

In §8, we will prove the following proposition that implies item (4) in Theorem 1. Because f is topologically conjugated to the Fibonacci tent map, we know that $h(0)$ is recurrent and that the recurrence times are given by the Fibonacci numbers. Then, to have an estimate on the recurrence of the turning point, it is enough to estimate the decay of the distances $|f^{S(k)}(\tilde{c}) - \tilde{c}|$, where

$$S(0) = 1, \; S(1) = 2, \; S(2) = 3, \; S(3) = 5, \ldots,$$

are the Fibonacci numbers.

PROPOSITION 1.4. *There exist Θ, α', α'' positive numbers, such that*

$$\lambda^{-S(k)\alpha''}\Theta^{-1} \le |f^{S(k)}(\tilde{c}) - \tilde{c}| \le \lambda^{-S(k)\alpha'}\Theta, \quad (1.11)$$

for every $k \ge 1$.

To prove Proposition 1.4, we estimate the diameter of certain symmetric intervals, whose closures are disjoint from the Lorenz-like singularity, and whose lengths approximate the left and right distance of the closest returns to the Lorenz-like singularity. To do this, we use the mean value theorem, the fact that h has a non-flat critical point at 0, and Lemma 3.5 that give us an estimate on the diameter of the preimage of these intervals. The reason to use these intervals is because when we try to make a direct estimation of the distance $|f^{S(k)}(\tilde{c}) - \tilde{c}|$, we do not have control on how close to zero is the derivative of h.

2. *Preliminaries*

Throughout the rest of this work, we will denote by I the closed interval $[-1, 1] \subset \mathbb{R}$. We use $\mathbb{N}$ to denote the set of integers that are greater than or equal to 1 and put $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

We endow I with the distance induced by the absolute value $|\cdot|$ on $\mathbb{R}$. For $x \in \mathbb{R}$ and $r > 0$, we denote by $B(x, r)$ the open ball of I with center at x and radius r. For an interval $J \subset I$, we denote by $|J|$ its length.

For real numbers a, b, we put $[a, b] := [\min\{a, b\}, \max\{a, b\}]$ in the same way $(a, b) := (\min\{a, b\}, \max\{a, b\})$.

2.1. *The kneading sequence.* Following [**dMvS93**], we will introduce the kneading invariant of a unimodal map and related properties. Let $f: I \to I$ be a unimodal with turning point $c \in (0, 1)$. We will use the notation $c_i := f^i(c)$ for $i \geq 1$. Suppose $f(-1) = f(1) = -1$ and $c_2 < c < c_1$. Let $\Sigma := \{0, 1, c\}^{\mathbb{N}_0}$ be the space of sequences $\underline{x} = (x_0, x_1, x_2, \ldots)$. In Σ, we consider the topology generated by the cylinders

$$[a_0 a_1 \cdots a_{n-1}]_k := \{\underline{x} \in \Sigma : x_{k+i} = a_i \text{ for all } i = 0, 1, \ldots, n-1\}.$$

With this topology, Σ is a compact space. Let us define

$$\begin{aligned} \underline{i}: I &\longrightarrow \Sigma \\ x &\longmapsto (i_0(x), i_1(x), \ldots) \end{aligned}$$

where

$$i_n(x) = \begin{cases} 0 & \text{if } f^n(x) \in [-1, c), \\ 1 & \text{if } f^n(x) \in (c, 1], \\ c & \text{if } f^n(x) = c. \end{cases}$$

The sequence $\underline{i}(x)$ is called *the itinerary of x under f*. Given $n \in \mathbb{N}$ and $x \in I$, there exists $\delta > 0$ such that $i_n(y) \in \{0, 1\}$ and is constant for every $y \in (x, x + \delta)$. Observe that this value is not the same as $i_n(x)$ if x is the turning point. It follows that

$$\underline{i}(x^+) := \lim_{y \downarrow x} \underline{i}(y) \quad \text{and} \quad \underline{i}(x^-) := \lim_{y \uparrow x} \underline{i}(y)$$

always exist. Notice that $\underline{i}(x^-)$ and $\underline{i}(x^+)$ belong to $\{0, 1\}^{\mathbb{N}_0}$. The sequence $e_1, e_2, e_3, \ldots$ defined by $e_j := i_j(c_0^+)$ is called the *kneading invariant* of *f*. A sequence $\underline{a} \in \{0, 1\}^{\mathbb{N}}$ is *admissible* if there exists a unimodal map $f: I \to I$ with kneading invariant $\underline{a}$.

We say that $Q: \mathbb{N} \to \mathbb{N}_0$ defines a *kneading map* if $Q(k) < k$ for all $k \in \mathbb{N}$ and

$$(Q(j))_{k<j<\infty} \geq (Q(Q(Q(k)) + j - k))_{k<j<\infty}$$

for all k with $Q(k) > 0$ ($\geq$ is the lexicographical order). A kneading map leads to an admissible kneading sequence in the following way: define the sequence $S: \mathbb{N}_0 \to \mathbb{N}$ by $S(0) = 1$ and $S(k) = S(k-1) + S(Q(k))$ for $k \geq 1$. The kneading sequence $\{e_j\}_{j\geq 1}$ associated to Q is given by $e_1 = 1$ and the relation

$$e_{S(k-1)+1} e_{S(k-1)+2} \ldots e_{S(k)-1} e_{S(k)} = e_1 e_2 \ldots e_{S(Q(k))-1}(1 - e_{S(Q(k))}), \tag{2.1}$$

for $k \geq 1$. The length of each string in (2.1) is $S(Q(k))$, and thus at the kth step of the process, we can construct $S(Q(k))$ symbols of the sequence. Because for every $k \geq 1$ we have $Q(k) < k$, we get that $Q(1) = 0$. So, for $k = 1$, each string in (2.1) has the 1 symbol. Then

$$e_2 = e_{S(0)+1} = 1 - e_{S(0)} = 0.$$

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Hence,

$$c_2 < c < c_1.$$

2.2. *The Fibonacci tent map.* We will say that a unimodal map f has *Fibonacci recurrence* or it is a *Fibonacci unimodal map* if the kneading map associated to it is given by $Q(1) = 0$ and $Q(k) = k - 2$ for $k > 1$. So the sequence $\{S(n)\}_{n\geq 0}$ is given by the Fibonacci numbers

$$S(0) = 1,\ S(1) = 2,\ S(2) = 3,\ S(3) = 5, \ldots .$$

For a Fibonacci unimodal map f, we have that

$$|c_{S(0)} - c| > |c_{S(1)} - c| > \cdots |c_{S(n)} - c| > |c_{S(n+1)} - c| > \cdots , \tag{2.2}$$

and

$$|c_3 - c| < |c_4 - c|. \tag{2.3}$$

See [**LM93**, Lemma 2.1] and references therein. The set $\overline{O_f(c)}$ is a Cantor set and the restriction of f to this set is minimal and uniquely ergodic, see [**Bru03**, Proposition 1] or [**CRL10**, Proposition 4] and references therein. The kneading invariant for a Fibonacci unimodal map starts like

$$100111011001010011100\ldots$$

Let us consider the tent family $T_S\colon I \to I$ defined by $T_s(x) = s(1 - |x|) - 1$ for every $x \in I$ and every $s \in (0, 2]$. This family is *full*, and thus for every kneading map Q, there is a parameter $s \in (0, 2]$ so that the kneading map of T_s is Q, see [**MT88**] and [**dMvS93**, Ch. 2]. So there exists $\lambda_F \in (0, 2]$ such that the kneading map associated to T_{λ_F} is given by $Q(k) = \max\{0, k - 2\}$.

From now on, we use the notation $T := T_{\lambda_F}$, $\lambda := \lambda_F$, $c := 0$, and $c_i := T^i(c)$.

3. *The set $\overline{O_T(c)}$*

3.1. *The combinatorics of the set $\overline{O_T(c)}$.* In this section, we will give an explicit description of the set $\overline{O_{T(c)}}$ following [**LM93**].

Put $S(-2) = 0$ and $S(-1) = 1$. From (2.1), we obtain that for every $k \geq 0$, the points $c_{S(k)}$ and $c_{S(k+2)}$ are on opposite sides of c. Because

$$c_{S(1)} = c_2 < c < c_1 = c_{S(0)},$$

we conclude that for $k \equiv 0 \pmod 4$, $c_{S(k)}$ is to the right of c and if $k \equiv 2 \pmod 4$, $c_{S(k)}$ is to the left of c. Because we also know that $c_{S(1)}$ is to the left of c, we can conclude that for $k \equiv 1 \pmod 4$, $c_{S(k)}$ is to the left of c, and for $k \equiv 3 \pmod 4$, $c_{S(k)}$ is to the right of c. From this, we can conclude that if k is even, the points $c_{S(k)}$ and $c_{S(k+1)}$ are on opposite sides of c, and therefore,

$$[c_{S(k+1)}, c_{S(k)}] \supseteq [c_{S(k+2)}, c_{S(k)}].$$

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In the case where k is odd, $c_{S(k)}$ and $c_{S(k+1)}$ are on the same side with respect to c, and therefore,

$$[c_{S(k+1)}, c_{S(k)}] \subseteq [c_{S(k+2)}, c_{S(k)}].$$

For each $k \geq 0$, let I_k be the smallest closed interval containing all of the points $c_{S(l)}$ for every $l \geq k$. For each $n \geq 0$, define $I_k^n := T^n(I_k)$. By the above discussion,

$$I_k = \begin{cases} [c_{S(k)}, c_{S(k+1)}] & \text{if } k \text{ is even,} \\ [c_{S(k)}, c_{S(k+2)}] & \text{if } k \text{ is odd.} \end{cases} \tag{3.1}$$

LEMMA 3.1. *For every $k \geq 1$, we have that T^j is injective on $[c_1, c_{S(k)+1}]$. In particular, $I_k^{j+1} = [c_{j+1}, c_{S(k)+1+j}]$ for every $j \in \{1, \ldots, S(k-1) - 1\}$.*

Proof. Because $|c - c_{S(k)}| > |c - c_{S(m)}|$ and $|T([c_{S(k)}, c])| = \lambda |c - c_{S(k)}|$, for every $0 \leq k < m$, we get that $c_{S(k)+1} < c_{S(m)+1} < c_1$, in particular, $I_k^1 = [c_1, c_{S(k)+1}]$. In the case $k \geq 1$, by (2.1) with k replaced by $k+1$, for every $j \in \{1, \ldots, S(k-1) - 1\}$, we have that $c_{S(k)+j}$ and c_j are on the same side with respect to c. Thus, $c \notin [c_{S(k)+j}, c_j] = T^{j-1}[c_{S(k)+1}, c_1]$, and then the map T^j is injective on $[c_1, c_{S(k)+1}]$. In particular, for $1 < j \leq S(k-1)$,

$$\begin{aligned} I_k^j &= T^{j-1}([c_1, c_{S(k)+1}]) \\ &= [c_j, c_{S(k)+j}] \end{aligned} \tag{3.2}$$

□

Note that for $k \geq 1$, by Lemma 3.1, with $j = S(k-1) - 1$,

$$I_k^{S(k-1)} = [c_{S(k-1)}, c_{S(k)+S(k-1)}] = [c_{S(k-1)}, c_{S(k+1)}]. \tag{3.3}$$

Then, by (2.1), $c \in I_k^{S(k-1)}$ and $c \notin I_k^n$ for every $0 < n < S(k-1)$.

LEMMA 3.2. *For all $k \geq 0$, we have that*

$$|c_i - c| > |c_{S(k-1)} - c|,$$

for all $0 < i < S(k)$, with $i \neq S(k-1)$.

Proof. We will use induction on k. The cases $k = 0$ and 1 are vacuously true, and the cases $k = 2$ and 3 are true by the definition of Fibonacci map and (2.3). Suppose now that it is true for k. We will prove that is true for $k+1$.

Case 1: Because

$$|c_{S(k-1)} - c| > |c_{S(k)} - c|,$$

we have that

$$|c_i - c| > |c_{S(k)} - c|$$

for all $0 < i < S(k)$.

Case 2: Because

$$c_{S(k-1)+1} < c_{S(k)+1} < c_1$$

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and T^i is injective on $[c_{S(k-1)+1}, c_1]$ for $0 < i < S(k-2)$ by Lemma 3.1, we have that $c_{S(k)+i} \in (c_{S(k-1)+i}, c_i)$, for $0 < i < S(k-2)$. By the induction hypothesis,

$$|c_i - c| > |c_{S(k-1)} - c| > |c_{S(k)} - c|$$

and

$$|c_{S(k-1)+i} - c| > |c_{S(k-1)} - c| > |c_{S(k)} - c|$$

for $0 < i < S(k-2)$. By (2.1), c_i and $c_{S(k-1)+i}$ lie on the same side of c for $0 < i < S(k-2)$. The above implies that

$$|c_{S(k)+i} - c| > |c_{S(k)} - c|$$

for all $0 < i < S(k-2)$.

Case 3: Because $T^{S(k-2)-1}$ is injective on $[c_{S(k-1)+1}, c_1]$, we get that $c_{S(k)+S(k-2)} \in (c_{S(k)}, c_{S(k-2)})$. Also, by (2.1), $c_{S(k)+S(k-2)}$ and $c_{S(k-2)}$ lie on the same side of c, and opposite to $c_{S(k)}$. Then,

$$|c_{S(k)+S(k-2)} - c| < |c_{S(k-2)} - c|.$$

Hence,

$$c_{S(k-2)+1} < c_{S(k)+S(k-2)+1} < c_1.$$

So by Lemma 3.1, $c_{S(k)+S(k-2)+i} \in (c_{S(k-2)+i}, c_i)$ for $0 < i < S(k-3)$. By the induction hypothesis,

$$|c_i - c| > |c_{S(k)} - c| \quad \text{and} \quad |c_{S(k-2)+i} - c| > |c_{S(k)} - c|$$

for $0 < i < S(k-3)$. Because, by (2.1), $c_{S(k-2)+i}$ and c_i lie on the same side of c for $0 < i < S(k-3)$, we get

$$|c_{S(k)+S(k-2)+i} - c| > |c_{S(k)} - c|$$

for all $0 < i < S(k-3)$.

Case 4: It remains to prove that

$$|c_{S(k)+S(k-2)} - c| > |c_{S(k)} - c|.$$

Suppose by contradiction that

$$|c_{S(k)+S(k-2)} - c| < |c_{S(k)} - c|.$$

Then, $c_{S(k)+1} < c_{S(k)+S(k-2)+1} < c_1$. Because $T^{S(k-3)-1}$ is injective on $[c_{S(k)+1}, c_1]$, we get that $c_{S(k+1)} \in (c_{S(k-3)}, c_{S(k)+S(k-3)})$. Noting that by (2.1), $c_{S(k-3)}$ and $c_{S(k)+S(k-3)}$ are on the same side with respect to c, we have either

$$|c_{S(k+1)} - c| > |c_{S(k)+S(k-3)} - c| > |c_{S(k)} - c|$$

or

$$|c_{S(k+1)} - c| > |c_{S(k-3)} - c| > |c_{S(k)} - c|,$$

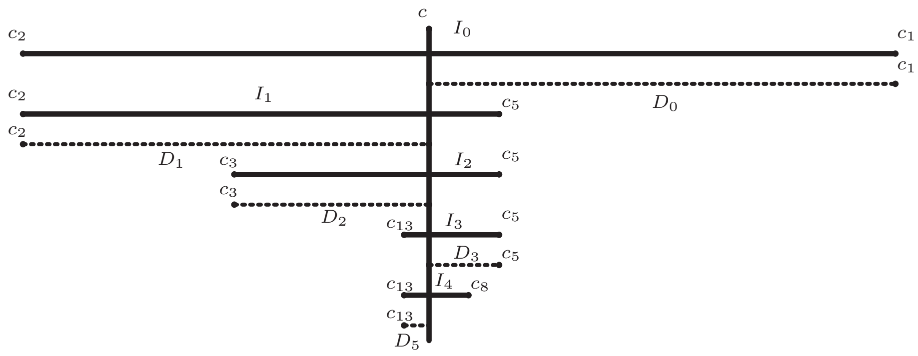

FIGURE 2. First five I_k (solid line) and D_k (dashed line) intervals.

a contradiction. So we must have

$$|c_{S(k)+S(k-2)} - c| > |c_{S(k)} - c|,$$

and this concludes the proof. □

Let us denote

$$J_k := I_{k+1}^{S(k-1)} = [c_{S(k-1)}, c_{S(k+1)+S(k-1)}],$$

and put

$$D_k := [c, c_{S(k)}]$$

for every $k \geq 1$. For every $n \geq 0$, we use the notation:

$$J_k^n := T^n(J_k) = I_{k+1}^{S(k-1)+n}. \tag{3.4}$$

Note that by definition, $D_{k'} \subset [c_{S(k)}, c_{S(k+2)}]$ for every $k' \geq k \geq 1$. See Figure 2.

LEMMA 3.3. *For all* $0 < k < k'$, *we have* $J_{k'} \subset D_{k'-1} \subset I_k$ *and* $J_k \cap J_{k'} = \emptyset$.

Proof. First we will prove that J_{k+1} is contained in D_k and $c \notin J_{k+1}$ for every $k \geq 0$. Fix $k \geq 0$. By (2.1) with k replaced by $k+3$, we have that $c_{S(k)}$ and $c_{S(k+2)+S(k)}$ are on the same side of c. Because

$$|c_{S(k+2)} - c| < |c_{S(k+1)} - c|,$$

we have

$$c_{S(k+1)+1} < c_{S(k+2)+1} < c_1.$$

By Lemma 3.1, $T^{S(k)-1}$ is injective on I_{k+1}^1. Then, $c_{S(k+2)+S(k)} \in (c_{S(k+2)}, c_{S(k)})$. By (2.1) with k replaced by $k+3$, we thus conclude $c_{S(k+2)+S(k)} \in (c_{S(k)}, c)$. Then,

$$J_{k+1} = [c_{S(k)}, c_{S(k)+S(k+2)}] \subset [c, c_{S(k)}] = D_k \subset [c_{S(k)}, c_{S(k+2)}] \subseteq I_k$$

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and $c \notin J_{k+1}$. Because, by definition, for every $k' > k$, we have $D_{k'-1} \subset I_{k'-1} \subset I_k$, we get

$$J_{k'} \subset D_{k'-1} \subset I_k.$$

Now we will prove that J_{k+1} and I_{k+1} are disjoint. If k is even, then $I_{k+1} = [c_{S(k+1)}, c_{S(k+3)}]$. Because $c_{S(k)}$ and $c_{S(k+1)}$ lie on opposite sides with respect to c, we have that $c_{S(k)}$, $c_{S(k)+S(k+2)}$, and $c_{S(k+3)}$ lie on the same side of c. By (3.2) with k replaced by $k+4$, we get

$$|c_{S(k)} - c| > |c_{S(k)+S_{k+2}} - c| > |c_{S(k+3)} - c|.$$

So, $J_{k+1} \cap I_{k+1} = \emptyset$. Now, if k is odd, $I_{k+1} = [c_{S(k+1)}, c_{S(k+2)}]$ and $c_{S(k)}$, $c_{s(k+1)}$, and $c_{S(k)+S(k+2)}$ lie on the same side of c. Suppose that $|c_{S(k+1)} - c| > |c_{S(k)+S(k+2)} - c|$, then

$$[c_{S(k+1)}, c_{S(k)+S(k+2)}] \subset [c_{S(k)}, c_{S(k)+S(k+2)}].$$

Because $T^{S(k-1)}$ is injective on $[c_{S(k)}, c_{S(k)+S(k+2)}]$, then $T^{S(k-1)}$ is injective on $[c_{S(k+1)}, c_{S(k)+S(k+2)}]$. So we get

$$T^{S(k-1)}([c_{S(k+1)}, c_{S(k)+S(k+2)}]) = [c_{S(k+1)+S(k-1)}, c_{S(k+3)}].$$

Because $S(k+1) + S(k-1) < S(k+4)$, by Lemma 3.2 with k replaced by $k+4$, we get

$$|c_{S(k+1)+S(k-1)} - c| > |c_{S(k+3)} - c|.$$

However, $T^{S(k-1)}(J_{k+1}) = [c_{S(k+1)}, c_{S(k+3)}]$, then

$$T^{S(k-1)}(c_{S(k+1)}) = c_{S(k+1)+S(k-1)} \in (c_{S(k+1)}, c_{S(k+3)}),$$

and by (2.1) with k replaced by $k+1$, we have that $c_{S(k+1)+S(k-1)}$ and $c_{S(k-1)}$ are on the same side of c. Because $k-1 \equiv k+3 \pmod 4$, we have that $c_{S(k-1)}$ and $c_{S(k+3)}$ are on the same side of c, so $c_{S(k+1)+S(k-1)} \in (c, c_{S(k+3)})$. Thus,

$$|c_{S(k+1)+S(k-1)} - c| < |c_{S(k+3)} - c|,$$

a contradiction. So we must have $c_{S(k)+S(k+2)} \in (c_{S(k)}, c_{S(k+1)})$ and

$$J_{k+1} \ \cap \ I_{k+1} \ = \ \emptyset. \tag{3.5}$$

This conclude the proof of the lemma. □

Taking $k' = k+1$ in Lemma 3.3, we get $J_{k+1} \subset I_k$, and then

$$J_{k+1} \cup I_{k+1} \subset I_k \subseteq I_k^{S(k-1)}, \tag{3.6}$$

for every $k \geq 1$.

Definition 3.4. For $k \geq 0$, let M_k be the $S(k)$-fold union

$$M_k = \bigcup_{0 \leq n < S(k-1)} I_k^n \cup \bigcup_{0 \leq n < S(k-2)} J_k^n.$$

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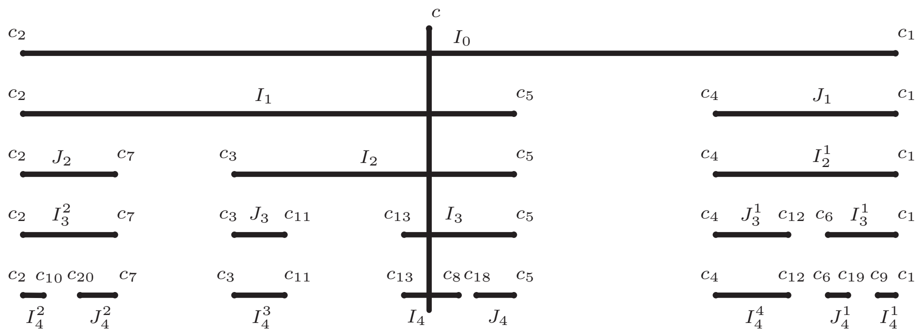

FIGURE 3. First five levels of M_k.

Some examples of M_k are

$$\begin{aligned}
M_0 &= I_0 \\
&= [c_1, c_2], \\
M_1 &= I_1 \cup J_1 \\
&= [c_2, c_5] \cup [c_4, c_1], \\
M_2 &= I_2 \cup I_2^1 \cup J_2 \\
&= [c_3, c_5] \cup [c_4, c_1] \cup [c_2, c_7], \\
M_3 &= I_3 \cup I_3^1 \cup I_3^2 \cup J_3 \cup J_3^1 \\
&= [c_{13}, c_5] \cup [c_6, c_1] \cup [c_2, c_7] \cup [c_3, c_{11}] \cup [c_4, c_{12}], \\
M_4 &= I_4 \cup I_4^1 \cup I_4^2 \cup I_4^3 \cup I_4^4 \cup J_4 \cup J_4^1 \cup J_4^2 \\
&= [c_{13}, c_8] \cup [c_9, c_1] \cup [c_2, c_{10}] \cup [c_3, c_{11}] \cup [c_4, c_{12}] \\
&\quad \cup [c_{14}, c_5] \cup [c_6, c_{19}] \cup [c_{20}, c_7],
\end{aligned}$$

and so on (see Figure 3).

From the definition, for every $k \geq 0$, the $S(k)$ closed intervals

$$I_k, I_k^1, \ldots, I_k^{S(k-1)-1}, J_k, J_k^1, \ldots, J_k^{S(k-2)-1},$$

are pairwise disjoint, each M_k contains the set $\overline{O_T(c)}$ and they form a nested sequence of closed sets $M_1 \supset M_2 \supset M_3 \supset \cdots$ with intersection equal to the Cantor set $\overline{O_T(c)}$. For a proof of these statements, see [**LM93**, Lemma 3.5]. Now by (3.6), we have that for every $1 \leq m < S(k-1)$,

$$I_{k+1}^m \cup J_{k+1}^m \subset I_k^m.$$

Also, by (3.4) for every $0 \leq n < S(k-2)$,

$$I_{k+1}^{S(k-1)+n} = J_k^n.$$

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Because the sets in M_k are disjoint, we get that

$$\bigcup_{A \in M_{k+1}} (A \cap I_k) = I_{k+1} \cup J_{k+1}. \tag{3.7}$$

3.2. *Diameter estimates.* In this section, we will give an estimate on how the distances $|c_{S(k)} - c|$ decrease as $k \to \infty$.

LEMMA 3.5. *The limit*

$$\lim_{k \to \infty} \lambda^{S(k+1)} |D_k|$$

exists and is strictly positive.

Proof. Because

$$T(D_k) = T([c, c_{S(k)}]) = I_k^1,$$

by (3.3), we have that

$$T^{S(k-1)}(D_k) = [c_{S(k-1)}, c_{S(k+1)}].$$

By (2.1), $c_{S(k-1)}$ and $c_{S(k+1)}$ are on opposite sides of c. Then $D_{k-1} \cap D_{k+1} = \{c\}$, so

$$T^{S(k-1)}(D_k) = D_{k-1} \cup D_{k+1}.$$

Because, by Lemma 3.1, $T^{S(k-1)}$ is injective on I_k and $D_k \subset I_k$, we get that

$$|T^{S(k-1)}(D_k)| = \lambda^{S(k-1)} |D_k| = |D_{k-1}| + |D_{k+1}|.$$

For $k \geq 0$, put $\nu_k := |D_k|/|D_{k+1}|$. By the above, we get

$$\lambda^{S(k-1)} = \nu_{k-1} + \frac{1}{\nu_k}.$$

By (2.2), $\nu_k > 1$, so $0 < \nu_k^{-1} < 1$. Because $\lambda > 1$, we have $\lambda^{S(k-1)} \longrightarrow \infty$ as $k \longrightarrow \infty$. Then, $\nu_k \longrightarrow \infty$ as $k \longrightarrow \infty$. So, $\lambda^{S(k-1)} - \nu_{k-1} \longrightarrow 0$ as $k \longrightarrow \infty$. Then, if we define $C_k := \nu_k \lambda^{-S(k)}$, we have $0 < C_k < 1$ and $C_k \nearrow 1$ exponentially fast as $n \longrightarrow \infty$. By definition of ν_k, we have that

$$\frac{|D_0|}{|D_{k+1}|} = \prod_{i=0}^{k} \nu_i = \prod_{i=0}^{k} \lambda^{S(i)} C_i = \lambda^{S(k+2)-S(1)} \prod_{i=0}^{k} C_i.$$

Then,

$$|D_{k+1}| \lambda^{S(k+2)-S(1)} = |D_0| \left[\prod_{i=0}^{k} C_i \right]^{-1}. \tag{3.8}$$

Because $\prod_{i=0}^{k} C_i$ converge to a strictly positive number as $k \longrightarrow \infty$, the proof is complete. □

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4. *The invariant measure*

Let us denote by μ_P the unique ergodic invariant measure of T restricted to $\overline{O_T(c)}$. As in the previous section, put

$$S(-2) = 0,\ S(-1) = 1,\ S(0) = 1,\ S(1) = 2, \ldots$$

and recall that $\varphi := ((1 + \sqrt{5})/2)$. In this section, we will estimate the value of μ_P over the elements of M_k for every $k \geq 1$.

As mentioned in §3, we know that the set $\overline{O_T(c)}$ is contained in M_k for every $k \geq 1$ and the sets

$$I_k, I_k^1, \ldots I_k^{S(k-1)-1}, J_k, J_k^1, \ldots, J_k^{S(k-2)-1}$$

are disjoint. Then,

$$\sum_{i=0}^{S(k-1)-1} \mu_P(I_k^i) + \sum_{j=0}^{S(k-2)-1} \mu_P(J_k^j) = 1. \tag{4.1}$$

Because T restricted to $\overline{O_T(c)}$ is injective, except at the critical point that has two preimages, we have that

$$\mu_P(I_k^i) = \mu_P(I_k^j), \tag{4.2}$$
$$\mu_P(J_k^p) = \mu_P(J_k^q), \tag{4.3}$$

for every $0 \leq i, j < S(k-1)$ and $0 \leq p, q < S(k-2)$. Then we can write (4.1) as

$$S(k-1)\mu_P(I_k) + S(k-2)\mu_P(J_k) = 1. \tag{4.4}$$

Because

$$I_k \sqcup J_k \subset I_{k-1},$$

we have that

$$\mu_P(I_k) + \mu_P(J_k) = \mu_P(I_{k-1}). \tag{4.5}$$

And because

$$J_{k-1} = I_k^{S(k-1)},$$

using (4.2) with k replaced by $k = 1$, we have that

$$\mu_P(J_{k-1}) = \mu_P(I_k). \tag{4.6}$$

Combining (4.5) and (4.6), we can write

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \mu_P(I_k) \\ \mu_P(J_k) \end{bmatrix} = \begin{bmatrix} \mu_P(I_{k-1}) \\ \mu_P(J_{k-1}) \end{bmatrix}. \tag{4.7}$$

LEMMA 4.1. *For every $m \geq 1$, we have*

$$\mu_P(I_m) = \frac{1}{\varphi^m} \quad \textit{and} \quad \mu_P(J_m) = \frac{1}{\varphi^{m+1}}.$$

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Proof. We will use induction to prove the lemma. For $m = 1$. We can apply $k - 2$ times the equation (4.7) to

$$\begin{bmatrix} \mu_P(I_{k-1}) \\ \mu_P(J_{k-1}) \end{bmatrix},$$

and we can write (4.7) as

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{k-1} \begin{bmatrix} \mu_P(I_k) \\ \mu_P(J_k) \end{bmatrix} = \begin{bmatrix} \mu_P(I_1) \\ \mu_P(J_1) \end{bmatrix}. \tag{4.8}$$

Using that

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{k-1} = \begin{bmatrix} S(k-2) & S(k-3) \\ S(k-3) & S(k-4) \end{bmatrix},$$

for $k \geq 2$, we can write

$$\begin{bmatrix} \mu_P(I_1) \\ \mu_P(J_1) \end{bmatrix} = \begin{bmatrix} S(k-2)\mu_P(I_k) + S(k-3)\mu_P(J_k) \\ S(k-3)\mu_P(I_k) + S(k-4)\mu_P(J_k) \end{bmatrix}. \tag{4.9}$$

Multiplying the first equation in (4.9) by $S(k)/S(k-1)$, we get

$$\frac{S(k)}{S(k-1)}\mu_P(I_1) = \frac{S(k-2)}{S(k-1)}S(k)\mu_P(I_k) + \frac{S(k-3)}{S(k-2)}\frac{S(k)}{S(k-1)}S(k-2)\mu_P(J_k). \tag{4.10}$$

Using (4.4), we can write (4.10) as

$$\frac{S(k)}{S(k-1)}\mu_P(I_1) = S(k)\mu_P(I_k)\left[\frac{S(k-2)}{S(k-1)} - \frac{S(k-3)}{S(k-2)}\right] + \frac{S(k-3)}{S(k-2)}\frac{S(k)}{S(k-1)}. \tag{4.11}$$

Because $S(k)/S(k-1) \longrightarrow \varphi$ as $k \longrightarrow \infty$, taking limit on (4.11) over k, we get

$$\varphi\mu_P(I_1) = 1,$$

and then

$$\mu_P(I_1) = \frac{1}{\varphi}.$$

Using (4.4) with k replaced by 1, we get that

$$\mu_P(J_1) = \frac{1}{\varphi^2}.$$

So the lemma holds for $m = 1$.

Suppose now that the result is true for m. By (4.6), we have that

$$\mu_P(I_{m+1}) = \mu_P(J_m) = \frac{1}{\varphi^{m+1}}.$$

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By (4.5), we have that

$$\begin{aligned}\mu_P(J_{m+1}) &= \mu_P(I_m) - \mu_P(I_{m+1})\\ &= \frac{1}{\varphi^m} - \frac{1}{\varphi^{m+1}}\\ &= \frac{1}{\varphi^m}\left(1 - \frac{1}{\varphi}\right)\\ &= \frac{1}{\varphi^m}\frac{1}{\varphi^2}\\ &= \frac{1}{\varphi^{m+2}},\end{aligned}$$

and we get the result. □

5. *Proof of Proposition 1.1*

In this section, we will give the proof of Proposition 1.1. We use the same notation as in the previous section. Let $h\colon [-1, 1] \to [-1, 1]$ be as in Proposition 1.1 and $f = h \circ T \circ h^{-1}$. As in Theorem 1, put $\tilde{c} := h(0)$.

Because h has a non-flat critical point at 0, by (1.3) and (1.4), there are $\alpha^+ > 0$, $\alpha^- > 0$, and $\delta > 0$ such that

$$e^{-M}|\hat{x}|^{\alpha^+} \le |h'(\hat{x})| \le e^M|\hat{x}|^{\alpha^+}, \tag{5.1}$$

for every $\hat{x} \in (0, \delta)$ and

$$e^{-M}|\hat{x}|^{\alpha^-} \le |h'(\hat{x})| \le e^M|\hat{x}|^{\alpha^-}, \tag{5.2}$$

for every $\hat{x} \in (-\delta, 0)$. Because $c = 0$ and $c \notin I_k^1 = [c_{c_{S(k)+1}}, c_1]$, we have that there exist positive real numbers W_1 and W_2 such that for every $x \in I_k$,

$$W_1 \le |h'(T(x))| \le W_2. \tag{5.3}$$

Proof of Proposition 1.1. By the chain rule, we have

$$f'(x) = \lambda\frac{h'(T(h^{-1}(x)))}{h'(h^{-1}(x))}, \tag{5.4}$$

for every $x \in (h(-\delta), h(\delta)) \setminus \{\tilde{c}\}$. Let $K := \max\{\lambda^{-1}e^M W_1^{-1}, \lambda e^M W_2\}$. Then by (5.3) and (5.4), we have that for every $x \in (\tilde{c}, h(\delta))$,

$$\frac{1}{K|h^{-1}(x)|^{\alpha^+}} \le |f'(x)| \le \frac{K}{|h^{-1}(x)|^{\alpha^+}}, \tag{5.5}$$

and for every $x \in (h(-\delta), \tilde{c})$,

$$\frac{1}{K|h^{-1}(x)|^{\alpha^-}} \le |f'(x)| \le \frac{K}{|h^{-1}(x)|^{\alpha^-}}. \tag{5.6}$$

Now, from (5.1) and (5.2), there exist $M_1 > 0$ and $M_2 > 0$ such that for every $x \in (0, \delta)$,

$$M_1^{-1}|x|^{\alpha^+ + 1} \le |h(x)| \le M_1|x|^{\alpha^+ + 1},$$

and for every $x \in (-\delta, 0)$,

$$M_2^{-1}|x|^{\alpha^-+1} \leq |h(x)| \leq M_2|x|^{\alpha^-+1}.$$

Because h is a homeomorphism, there exist constants $M_3 > 0$ and $M_4 > 0$ such that for every $x \in (\tilde{c}, h(\delta))$,

$$M_3^{-1}|x-\tilde{c}|^{1/(\alpha^++1)} \leq |h^{-1}(x)| \leq M_3|x-\tilde{c}|^{1/(\alpha^++1)}, \tag{5.7}$$

and for every $x \in (h(-\delta), \tilde{c})$,

$$M_4^{-1}|x-\tilde{c}|^{1/(\alpha^-+1)} \leq |h^{-1}(x)| \leq M_4|x-\tilde{c}|^{1/(\alpha^-+1)}. \tag{5.8}$$

Then, by (5.5)–(5.8), we have that for every $x \in (\tilde{c}, h(\delta))$,

$$\frac{1}{M_3|x-\tilde{c}|^{(\alpha^+)/(\alpha^++1)}} \leq |f'(x)| \frac{M_3}{|x-\tilde{c}|^{(\alpha^+)/(\alpha^++1)}},$$

and for every $x \in (\tilde{c}, h(-\delta))$,

$$\frac{1}{M_4|x-\tilde{c}|^{(\alpha^-)/(\alpha^-+1)}} \leq |f'(x)| \leq \frac{M_4}{|x-\tilde{c}|^{(\alpha^-)/(\alpha^-+1)}}.$$

Thus, f has a Lorenz-like singularity at $\tilde{c}$. $\square$

6. *Proof of Proposition 1.2*

In this section, we will give the proof of Proposition 1.2. We will use the same notation as in the previous sections.

First, take $\alpha := \max\{\alpha^+, \alpha^-\}$, where α^+ and α^- are given in Proposition 1.1. From (1.8) and (1.9), we get that for every $x \in (h(-\delta), h(\delta)) \setminus \{\tilde{c}\}$,

$$\frac{1}{K|h^{-1}(x)|^{\alpha}} \leq |f'(x)|. \tag{6.1}$$

Proof of Proposition 1.2. First we prove (i). Let $0 < \hat{\delta} < \delta$ be such that $|f'| > 1$ on $(h(-\hat{\delta}), h(\hat{\delta})) \setminus \{\tilde{c}\}$ (such $\hat{\delta}$ exists because $\tilde{c}$ is a Lorenz-like singularity). By Lemma 3.5, we have that $c_{S(k)} \longrightarrow c$, as $k \longrightarrow \infty$, then there exists $k \geq 2$ so that $I_k \subset (-\hat{\delta}, \hat{\delta})$. Thus, (6.1) holds on $h(I_k) \setminus \{\tilde{c}\}$ and $|f'| > 1$ on $h(I_k) \setminus \{\tilde{c}\}$. For $n > k$, put $L_n := \lambda^{S(n+1)}|D_n|$. Recall that

$$\log^+ |f'| := \max\{0, \log|f'|\} \quad \text{and} \quad \log^- |f'| := \{0, -\log|f'|\},$$

on $I \setminus \{\tilde{c}\}$. By Lemma 3.3, for every $n > k$, we have $J_n \subset I_k$ and for every $k < n < n'$, we have $J_n \cap J_{n'} = \emptyset$. So, because $\tilde{\mu}_P(\{\tilde{c}\}) = 0$,

$$\int \log^+ |f'|\, d\tilde{\mu}_P \geq \int_{h(I_k)} \log|f'|\, d\tilde{\mu}_P \geq \sum_{n>k} \int_{h(J_n)} \log|f'|\, d\tilde{\mu}_P.$$

Recall that $\varphi := ((1+\sqrt{5})/2)$. Then for each $n > k$ and $x \in J_n$, we have, by Lemma 3.3 and (6.1),

$$|f'(h(x))| \geq K^{-1}\frac{1}{|D_{n-1}|^{\alpha}} = K^{-1}\lambda^{\alpha S(n)} L_{n-1}^{-\alpha}. \tag{6.2}$$

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By the above together with Lemma 4.1 and the fact that $S(n) \geq \frac{1}{3}\varphi^{n+2}$,

$$\begin{aligned}\int_{h(J_n)} \log|f'|\, d\tilde{\mu}_P &\geq \tilde{\mu}_P(h(J_n)) \log\left|K^{-1}\lambda^{\alpha S(n)} L_{n-1}^{-\alpha}\right| \\ &\geq \left(\frac{1}{\varphi}\right)^{n+1} [\alpha S(n) \log(\lambda) + \alpha \log(K^{1/\alpha} L_{n-1})^{-1}] \\ &\geq \frac{\varphi\alpha}{3} \log(\lambda) + \alpha \left(\frac{1}{\varphi}\right)^{n+1} \log(K^{1/\alpha} L_{n-1})^{-1}. \end{aligned} \tag{6.3}$$

By Lemma 3.5, $(1/\varphi)^{n+1} \log(K^{1/\alpha} L_{n-1})^{-1} \longrightarrow 0$ as $n \longrightarrow \infty$. We get that

$$\int \log^+|f'|\, d\tilde{\mu}_P = +\infty. \tag{6.4}$$

Now we prove (ii). Suppose by contradiction that

$$\int \log^-|f'|\, d\tilde{\mu}_P < +\infty. \tag{6.5}$$

By the chain rule,

$$\log|f'(x)| = \log(\lambda) + \log|h'(T(h^{-1}(x)))| - \log|h'(h^{-1}(x))|, \tag{6.6}$$

on $I \setminus \{\tilde{c}\}$. Because h' has a unique critical point, $\log|h'|$ is bounded away from the critical point. In particular, $\log|h'|$ is bounded from above in all I. Then, $-\log|h'|$ is bounded from below in I. In particular, because $\tilde{\mu}_P(\{\tilde{c}\}) = 0$, the integral

$$\int \log|h' \circ h^{-1}|\, d\tilde{\mu}_P$$

is defined. Because the only critical points of f are the points in $f^{-1}(\{\tilde{c}\})$, we have that $\log|f'|$ is bounded away from $\{\tilde{c}\} \cup f^{-1}\{\tilde{c}\}$. Let $\widetilde{V} \subset I \setminus \{\tilde{c}\}$ be a neighborhood of $f^{-1}\{\tilde{c}\}$ such that $\log|f'(x)| < 0$ for $x \in \widetilde{V}$, then by (6.5) and (6.6),

$$-\infty < \int_{\tilde{V}} \log|f'|\, d\tilde{\mu}_P = \log(\lambda)\tilde{\mu}_P(\tilde{V}) + \int_{\tilde{V}} (\log|h' \circ T \circ h^{-1}| - \log|h' \circ h^{-1}|)\, d\tilde{\mu}_P.$$

Because $h^{-1}(\widetilde{V})$ is a neighborhood of $T^{-1}(c)$, the function $-\log|h' \circ h^{-1}|$ is bounded on $\widetilde{V}$. However, because

$$h^{-1} \circ f(x) = T \circ h^{-1}(x) = c,$$

we have $h' \circ T \circ h^{-1}(x) = 0$ if $x \in f^{-1}(\tilde{c})$. Thus, $h' \circ T \circ h^{-1}(x) \neq 0$ for $x \in I \setminus \widetilde{V}$. Then $\log|h' \circ T \circ h^{-1}|$ is bounded in $I \setminus \widetilde{V}$. So,

$$\int_{I \setminus \tilde{V}} \log|h' \circ T \circ h^{-1}|\, d\tilde{\mu}_P > -\infty. \tag{6.7}$$

Now,

$$\begin{aligned}-\infty &< \int_{\tilde{V}} \log|f'|\, d\tilde{\mu}_P \\ &\leq \Big(\log(\lambda) + \max_{x\in\tilde{V}}\{-\log|h' \circ h^{-1}(x)|\}\Big)\tilde{\mu}_P(\tilde{V}) + \int_{\tilde{V}} (\log|h' \circ T \circ h^{-1}|)\, d\tilde{\mu}_P.\end{aligned}$$

So,

$$\int_{\tilde{V}} \log |h' \circ T \circ h^{-1}| \, d\tilde{\mu}_P > -\infty. \tag{6.8}$$

Together with (6.7), this implies that

$$\int \log |h' \circ T \circ h^{-1}| \, d\tilde{\mu}_P,$$

is finite. Because the integral

$$\int - \log |h' \circ h^{-1}| \, d\tilde{\mu}_P,$$

is defined, we have

$$\begin{aligned} \int \log |f'| \, d\tilde{\mu}_P &= \log(\lambda) + \int \log |h' \circ T \circ h^{-1}| \, d\tilde{\mu}_P + \int - \log |h' \circ h^{-1}| \, d\tilde{\mu}_P \\ &= \log(\lambda) + \int \log |h' \circ h^{-1} \circ f| \, d\tilde{\mu}_P + \int - \log |h' \circ h^{-1}| \, d\tilde{\mu}_P, \end{aligned}$$

and because $\tilde{\mu}_P$ is f invariant, we get

$$\int \log |f'| \, d\tilde{\mu}_P = \log(\lambda),$$

which contradicts (6.4). This contradiction completes the proof of part (ii).

Finally we prove (iii). By Proposition 1.1, f has a Lorenz-like singularity at $\tilde{c}$, then there exist $\delta > 0$, $\ell^+ > 0$, $\ell^- > 0$ and $L > 0$ such that (1.1) holds for every $x \in (\tilde{c}, \tilde{c} + \delta)$ and (1.2) holds for every $x \in (\tilde{c} - \delta, \tilde{c})$. Let $\ell := \max\{\ell^+, \ell^-\}$, and choose $0 < \hat{\delta} \leq \delta$ so that $\log |f'(x)| > 0$ and $\operatorname{dist}(x, \mathcal{S}(f)) = |x - \tilde{c}|$ for every $x \in (\tilde{c} - \hat{\delta}, \tilde{c} + \hat{\delta}) \setminus \{\tilde{c}\}$. Let $m \geq 2$ be so that $I_m \subset (\tilde{c} - \hat{\delta}, \tilde{c} + \hat{\delta})$. Then, for every $x \in I_m \setminus \{\tilde{c}\}$, we have

$$\log |f'(x)| \leq \log(L) - \ell \log(\operatorname{dist}(x, \mathcal{S}(f))).$$

So, for every $n \geq m$

$$\int_{h(J_n)} \log |f'| \, d\tilde{\mu}_P \leq \log(L) \tilde{\mu}_P(J_n) + \ell \int_{h(J_n)} | \log |x - \tilde{c}|| \, d\tilde{\mu}_P(x).$$

By (6.3) and Lemma 3.5, we get that

$$+\infty = \int_{h(I_m)} | \log(\operatorname{dist}(x, \mathcal{S}(f)))| \, d\tilde{\mu}_P(x),$$

so $\log(\operatorname{dist}(x, \mathcal{S}(f))) \notin L^1(\tilde{\mu}_P)$. This concludes the proof of the proposition. □

7. *Proof of Proposition 1.3*

In this section, we will prove Proposition 1.3. We will use the same notation as in the previous sections. Recall that f' is not defined at $\tilde{c}$, so for x in I whose orbit contains $\tilde{c}$, the derivative $(f^n)'$ at x does not exist for large n.

Let α^+ and α^- be the right and left critical orders of 0 as the critical point of h, and let $\alpha := \max\{\alpha^+, \alpha^-\}$. Let $M > 0$ be as in (5.1) and (5.2). Fix $k > 2$ big enough so that (6.1) holds on $h(I_k)$, and (5.1) and (5.2) hold on I_k.

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For every $x \in I$ such that $\tilde{c} \notin \mathcal{O}_f(x)$, we put

$$\chi_f^+(x) := \limsup_{n\to\infty} \frac{1}{n} \log |(f^n)'(x)|, \tag{7.1}$$

and

$$\chi_f^-(x) := \liminf_{n\to\infty} \frac{1}{n} \log |(f^n)'(x)|. \tag{7.2}$$

For every $x \in h(I_k)$, we put $\hat{x} := h^{-1}(x) \in I_k$. The proof of Proposition 1.3 is given after the following lemma.

LEMMA 7.1. *For every $\hat{x} \in I_k \cap \overline{\mathcal{O}_T(c)}$, there exists an increasing sequence of positive integers $\{n_i\}_{i\geq 1}$ such that*

$$T^{n_i}(\hat{x}) \in I_{k+i} \quad \textit{and} \quad T^m(\hat{x}) \notin I_{k+i+1},$$

for all $i \geq 1$ and all $n_i + 1 \leq m < n_{i+1}$. Moreover,

$$S(k+i) - S(k) \leq n_i \leq S(k+i+2) - S(k+2), \tag{7.3}$$

for all $i > 1$.

Proof. We will prove the lemma by induction. Let $\hat{x} \in I_k \cap \overline{\mathcal{O}_T(c)}$. Recall that for any integer $k' \geq 1$, we have that

$$I_{k'}, I_{k'}^1, \ldots I_{k'}^{S(k'-1)-1}, J_{k'}, J_{k'}^1, \ldots, J_{k'}^{S(k'-2)-1}$$

are pairwise disjoint. Now, by (3.7),

$$\hat{x} \in I_{k+1} \quad \text{or} \quad \hat{x} \in J_{k+1}.$$

If $\hat{x} \in J_{k+1}$, for every $1 \leq m < S(k-1)$,

$$T^m(\hat{x}) \in J_{k+1}^m,$$

and thus

$$T^m(\hat{x}) \notin I_{k+1}$$

and

$$T^{S(k-1)}(\hat{x}) \in I_{k+1}.$$

In this case, $n_1 = S(k-1)$ satisfies the desired properties. If $\hat{x} \in I_{k+1}$, for every $1 \leq m < S(k)$,

$$T^m(\hat{x}) \in I_{k+1}^m,$$

and thus

$$T^m(\hat{x}) \notin I_{k+1},$$

and

$$T^{S(k)}(\hat{x}) \in I_{k+1} \quad \text{or} \quad T^{S(k)}(\hat{x}) \in J_{k+1}.$$

In the former case, $n_1 = S(k)$ satisfies the desired properties. In the latter case, we have that for $1 \leq m < S(k) + S(k-1) = S(k+1)$,

$$T^m(\hat{x}) \notin I_{k+1}$$

and

$$T^{S(k+1)}(\hat{x}) \in I_{k+1}.$$

So $n_1 = S(k+1)$ satisfies the desired properties. So we have

$$S(k-1) \leq n_1 \leq S(k+1).$$

Now suppose that for some $i \geq 1$, there is n_i satisfying the conclusions of the lemma. Thus,

$$T^{n_i}(\hat{x}) \in I_{k+i}$$

and

$$S(k+i) - S(k) \leq n_i \leq S(k+i+2) - S(k+2).$$

By (3.7),

$$T^{n_i}(\hat{x}) \in I_{k+i+1} \quad \text{or} \quad T^{n_i}(\hat{x}) \in J_{k+i+1}.$$

If $T^{n_i}(\hat{x}) \in J_{k+i+1}$, for every $1 \leq m < S(k+i-1)$,

$$T^{m+n_i}(\hat{x}) \in J^m_{k+i+1},$$

and thus,

$$T^{m+n_i}(\hat{x}) \notin I_{k+i+1}$$

and

$$T^{S(k+i-1)+n_i}(\hat{x}) \in I_{k+i+1}.$$

In this case, $n_{i+1} = S(k+i-1) + n_i$ satisfies the desired properties. If $T^{n_i}(\hat{x}) \in I_{k+i+1}$, for every $1 \leq m < S(k+i)$,

$$T^{m+n_i}(\hat{x}) \in I^m_{k+i+1},$$

and thus,

$$T^{m+n_i}(\hat{x}) \notin I_{k+i+1},$$

and

$$T^{S(k+i)+n_i}(\hat{x}) \in I_{k+i+1} \quad \text{or} \quad T^{S(k+i)+n_i}(\hat{x}) \in J_{k+i+1}.$$

In the former case, $n_{i+1} = S(k+i) + n_i$ satisfies the desired properties. In the latter case, we have that for $1 \leq m < S(k+i) + S(k+i-1) = S(k+i+1)$,

$$T^{m+n_i}(\hat{x}) \notin I_{k+i+1}$$

and

$$T^{S(k+i+1)+n_i}(\hat{x}) \in I_{k+i+1}.$$

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So $n_{i+1} = S(k+i+1) + n_i$ satisfies the desired properties. So we have

$$S(k+i-1) \leq n_{n_i+1} \leq S(k+i+1) + n_i.$$

Because n_i satisfies (7.3),

$$S(k+i+1) - S(k) \leq n_{i+1} \leq S(k+i+3) - S(k+2).$$

This concludes the proof of the lemma. □

Proof of Proposition 1.3. Let $x \in \overline{O_f(\tilde{c})}$, with $\tilde{c} \notin O_f(x)$. Then, $\hat{x} = h^{-1}(x) \in \overline{O_T(c)}$ and $c \notin O_T(\hat{x})$. By the chain rule, we have that for every $n \geq 1$,

$$(f^n)'(x) = \prod_{i=0}^{n-1} \lambda \frac{h'(T(T^i(\hat{x})))}{h'(T^i(\hat{x}))} = \lambda^n \frac{h'(T^n(\hat{x}))}{h'(\hat{x})}. \tag{7.4}$$

Then,

$$\frac{1}{n} \log |(f^n)'(x)| = \log \lambda + \frac{1}{n} \log |h'(T^n(\hat{x}))| - \frac{1}{n} \log |h'(\hat{x})|, \tag{7.5}$$

for every $n \geq 1$. So, using (7.1) and (7.2), we get

$$\chi_f^+(x) = \log \lambda + \limsup_{n\to\infty} \frac{1}{n} \log |h'(T^n(\hat{x}))|, \tag{7.6}$$

and

$$\chi_f^-(x) = \log \lambda + \liminf_{n\to\infty} \frac{1}{n} \log |h'(T^n(\hat{x}))|. \tag{7.7}$$

Now, because $\hat{x} \in \overline{O_T(c)}$, we have that $\hat{x}$ belongs to one of the following sets:

$$I_k, I_k^1, \ldots, I_k^{S(k-1)-1}, J_k, \ldots, J_k^{S(k-2)-1}.$$

So there exists $0 \leq l_k(x) < S(k)$ such that $T^{l_k(x)}(\hat{x}) \in I_k$. Let $\{n_i\}_{i\geq 1}$ be as in Lemma 7.1 for $T^{l_k(x)}(x)$. Note that for every $i \geq 1$,

$$T^{n_i+l_k(x)+1}(\hat{x}) \in I_{k+i}^1 \subset [c_{S(k)+1}, c_1].$$

Then by (5.3) and (7.6), we have that

$$\log \lambda \leq \chi_f^+(x).$$

Now by (7.3), we have that

$$\frac{1}{S(k+i+2)} \leq \frac{1}{n_i}. \tag{7.8}$$

Also, because for every $i \geq 1$,

$$T^{n_i+l_k(x)}(\hat{x}) \in I_{k+i},$$

by (2.2) and (3.1), we get

$$|T^{n_i+l_k(x)}(\hat{x})| \leq |c_{S(k+i)}| = |D_{k+i}|. \tag{7.9}$$

By (6.1), we have

$$\frac{1}{\lambda K |T^{n_i+l_k(x)}(\hat{x})|^{\alpha}} \leq \frac{|h'(T^{n_i+l_k(x)+1}(\hat{x}))|}{|h'(T^{n_i+l_k(x)}(\hat{x}))|}. \tag{7.10}$$

Combining (5.3), (7.9), and (7.10), we get

$$|h'(T^{n_i+l_k(x)}(\hat{x}))| \leq \lambda K W_2 |D_{k+i}|^{\alpha}. \tag{7.11}$$

Because $|D_{k+i}| \longrightarrow 0$ as $i \longrightarrow \infty$, there exists $i' \geq 1$ such that for every $i \geq i'$,

$$|D_{k+i}| < \left(\frac{1}{\lambda K W_2}\right)^{1/\alpha}.$$

Then for every $i \geq i'$, from (7.11), we get

$$\log |h'(T^{n_i+l_k(x)}(\hat{x}))| \leq \log(\lambda K W_2 |D_{k+i}|^{\alpha}) < 0.$$

By the above and (7.8),

$$\frac{1}{n_i} \log |h'(T^{n_i+l_k(x)}(\hat{x}))| \leq \frac{1}{S(k+i+2)} \log(\lambda K W_2 |D_{k+i}|^{\alpha}).$$

Taking the limit as $i \longrightarrow \infty$,

$$\lim_{i\to\infty} \frac{1}{n_i} \log |h'(T^{n_i+l_k(x)}(\hat{x}))| \leq \lim_{i\to\infty} \frac{1}{S(k+i+2)} \log |D_{k+i}|^{\alpha}.$$

Using Lemma 3.5,

$$\begin{aligned}
\lim_{i\to\infty} \frac{1}{S(k+i+2)} \log |D_{k+i}|^{\alpha} &= \lim_{i\to\infty} \frac{1}{S(k+i+2)} \log \lambda^{-\alpha S(k+i+1)} \\
&= -\alpha \log \lambda \lim_{i\to\infty} \frac{S(k+i+1)}{S(k+i+2)} \\
&= -\frac{\alpha}{\varphi} \log \lambda.
\end{aligned}$$

Then by (7.7),

$$\chi_f^-(x) \leq \left(1 - \frac{\alpha}{\varphi}\right) \log \lambda < \log \lambda.$$

This concludes the proof of the proposition. □

8. *Proof of Proposition 1.4*

In this section, we will give the proof of Proposition 1.4. We will use the same notation as in the previous sections.

For every $k \geq 1$, put

$$D_k^+ := (|c_{S(k)}|, c) \quad \text{and} \quad D_k^- := (-|c_{S(k)}|, c)$$
$$A_k^+ := D_k^+ \setminus \overline{D_{k+1}^+} \quad \text{and} \quad A_k^- := D_k^- \setminus \overline{D_{k+1}^-}.$$

Observe that

$$|A_k^+| = |D_k| - |D_{k+1}| = |A_k^-|. \tag{8.1}$$

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LEMMA 8.1. *There exist α''_+, α''_-, α'_+, α'_-, K, and Q positive real numbers such that*

$$\lambda^{-S(k)\alpha''_+} Q^{-1} \leq |h(A_k^+)| \leq \lambda^{-S(k)\alpha'_+} Q, \tag{8.2}$$

and

$$\lambda^{-S(k)\alpha''_-} Q^{-1} \leq |h(A_k^-)| \leq \lambda^{-S(k)\alpha'_-} Q, \tag{8.3}$$

for every $k \geq K$.

Proof. By Lemma 3.5, there exists $\beta > 0$ such that

$$\lim_{n\to\infty} \lambda^{S(k+1)}|D_k| = \beta.$$

Let $\varepsilon > 0$ be small enough so that $(\beta - \varepsilon)/(\beta + \varepsilon) \geq 1/2$. Let $M > 0$ be as in (5.1) and (5.2). Fix $K > 0$ big enough so that for every $k \geq K$, (5.1), (5.2) hold on A_k, and the following holds:

$$\lambda^{-S(k+1)}(\beta - \varepsilon) \leq |D_k| \leq \lambda^{-S(k+1)}(\beta + \varepsilon), \tag{8.4}$$

$$\lambda^{-S(k)} \leq \tfrac{1}{4}, \tag{8.5}$$

and

$$\frac{S(k+1)}{S(k)} < \varphi + \varepsilon. \tag{8.6}$$

By (8.4) with k replaced by $k + 1$, we get

$$\lambda^{S(k+2)} \frac{1}{\beta + \varepsilon} \leq \frac{1}{|D_{k+1}|} \leq \lambda^{S(k+2)} \frac{1}{\beta - \varepsilon}. \tag{8.7}$$

Combining (8.4) and (8.7), we get

$$\lambda^{S(k)} \frac{\beta - \varepsilon}{\beta + \varepsilon} \leq \frac{|D_k|}{|D_{k+1}|} \leq \lambda^{S(k)} \frac{\beta + \varepsilon}{\beta - \varepsilon}. \tag{8.8}$$

For $k \geq K$, using the mean value theorem on the function $h\colon A_k^+ \longrightarrow h(A_k^+)$, there exists $\gamma^+ \in A_k^+$ such that

$$\frac{|h(A_k^+)|}{|A_k^+|} = |h'(\gamma^+)|. \tag{8.9}$$

Let α^+ be the right order of 0 as a critical point of h. By (5.1), we have

$$e^{-M}|\gamma^+|^{\alpha^+} \leq |h'(\gamma^+)| \leq e^M|\gamma^+|^{\alpha^+}. \tag{8.10}$$

Because $\gamma^+ \in A_k^+$, we have that

$$|D_{k+1}| \leq |\gamma^+| \leq |D_k|.$$

Then, by (8.4), (8.5), (8.9), and (8.10), we have that

$$e^{-M}|D_{k+1}|^{\alpha^+ + 1}\left(\frac{|D_k|}{|D_{k+1}|} - 1\right) \leq |h(A_k^+)| \leq e^M|D_k|^{\alpha^+ + 1}\left(1 - \frac{|D_{k+1}|}{|D_k|}\right). \tag{8.11}$$

Using (8.8) in (8.11), we obtain

$$e^{-M}(\beta-\varepsilon)^{\alpha^{+}+1}\lambda^{-S(k+2)(\alpha^{+}+1)}\left(\lambda^{S(k)}\frac{\beta-\varepsilon}{\beta+\varepsilon}-1\right) \leq |h(A_k^+)| \leq$$
$$e^{M}(\beta+\varepsilon)^{\alpha^{+}+1}\lambda^{-S(k+1)(\alpha^{+}+1)}\left(1-\lambda^{-S(k)}\frac{\beta-\varepsilon}{\beta+\varepsilon}\right). \tag{8.12}$$

Put

$$Q_1 := e^{-M}\left(\frac{\beta}{3}\right)^{\alpha^{+}+1}\frac{1}{4} \quad \text{and} \quad Q_2 := e^{M}2\beta.$$

By (8.5) and because $(\beta-\varepsilon)/(\beta+\varepsilon) \geq 1/2$, we have that

$$Q_1 \leq e^{-M}(\beta-\varepsilon)^{\alpha^{+}+1}\left(\frac{\beta-\varepsilon}{\beta+\varepsilon}-\lambda^{-S(k)}\right),$$

and

$$e^{M}(\beta+\varepsilon)^{\alpha^{+}+1}\left(1-\lambda^{-S(k)}\frac{\beta-\varepsilon}{\beta+\varepsilon}\right) \leq Q_2,$$

for every $k \geq K$. Then,

$$\lambda^{-S(k+2)(\alpha^{+}+1)}\lambda^{S(k)}Q_1 \leq |h(A_k^+)| \leq \lambda^{-S(k+1)(\alpha^{+}+1)}Q_2. \tag{8.13}$$

Finally, put

$$\alpha'_{+} := \alpha^{+}+1 \quad \text{and} \quad \alpha''_{+} := (\varphi+\varepsilon)^2(\alpha^{+}+1)-1.$$

Because $S(k) = S(k+2) - S(k-1)$, by (8.6), we have

$$-S(k+2)(\alpha^{+}+1)+S(k) = -S(k)\left(\frac{S(k+2)}{S(k)}(\alpha^{+}+1)-1\right)$$
$$\geq -S(k)\alpha''_{+}.$$

Then, taking $Q := \max\{Q_1^{-1}, Q_2\}$, we have

$$\lambda^{-S(k)\alpha''_{+}}Q^{-1} \leq |h(A_k^+)| \leq \lambda^{S(k)\alpha'_{+}}Q.$$

In the same way, we can prove (8.3). □

Proof of Proposition 1.4. Let $\alpha''_{+}, \alpha''_{-}, \alpha'_{+}, \alpha'_{-}, K$, and Q, be as in Lemma 8.1. By (8.1), we have that

$$\sum_{m=0}^{n} |h(A_{k+m}^+)| = |h(D_k^+)| - |h(D_{k+n+1}^+)|, \tag{8.14}$$

for every $n \geq 0$. Then by (8.2) and (8.14), we get

$$Q^{-1}\sum_{m=0}^{n}\lambda^{-S(k+m)\alpha''_{+}} \leq |h(D_k^+)| - |h(D_{k+n+1}^+)| \leq Q\sum_{m=0}^{n}\lambda^{-S(k+m)\alpha'_{+}}, \tag{8.15}$$

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for every $n \geq 0$. Now, for $m \geq 0$, we have that

$$S(k+m) = S(k) + \sum_{j=0}^{m-1} S(k+j-1). \tag{8.16}$$

Put

$$F_{k+m} := \sum_{j=0}^{m-1} S(k+j-1), \quad \text{and} \quad \sigma'(k) := 1 + \sum_{i=0}^{\infty} \lambda^{-\alpha'_+ F_{k+i}}.$$

Then, combining (8.15) and (8.16), we obtain

$$\lambda^{-S(k)\alpha''_+} Q^{-1} \leq |h(D_k^+)| - |h(D_{k+n+1}^+)| \leq \lambda^{-S(k)\alpha'_+} Q\sigma'(k). \tag{8.17}$$

If we put

$$\Theta := Q\bigg(1 + \sum_{i=0}^{\infty} \lambda^{-\alpha''_+ S(i)}\bigg),$$

then for every $k \geq K$ and every $m \geq 0$, we have

$$Q\sigma'(k) \leq \Theta, \quad \text{and} \quad \Theta^{-1} \leq Q^{-1}.$$

Then

$$\lambda^{-S(k)\alpha''_+} \Theta^{-1} \leq |h(D_k^+)| - |h(D_{k+n+1}^+)| \leq \lambda^{-S(k)\alpha'_+} \Theta. \tag{8.18}$$

Because $|D_{k+n+1}| \longrightarrow 0$ as $n \longrightarrow \infty$, and h is continuous, taking the limit in (8.18) as $n \longrightarrow \infty$, we obtain

$$\lambda^{-S(k)\alpha''_+} \Theta^{-1} \leq |h(D_k^+)| \leq \lambda^{-S(k)\alpha'_+} \Theta. \tag{8.19}$$

In the same way, we can prove that

$$\lambda^{-S(k)\alpha''_-} \Theta^{-1} \leq |h(D_k^-)| \leq \lambda^{-S(k)\alpha'_-} \Theta. \tag{8.20}$$

Finally, put $\alpha'' := \max\{\alpha''_-, \alpha''_+\}$ and $\alpha' := \min\{\alpha'_-, \alpha'_+\}$. For any $k \geq K$, we have that

$$|f^{S(k)}(\tilde{c}) - \tilde{c}| = |h(D_k^+)| \quad \text{or} \quad |f^{S(k)}(\tilde{c}) - \tilde{c}| = |h(D_k^-)|.$$

In any case, by (8.19) and (8.20), the result follows. □

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