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SIMPLICIAL QUADRATIC FORMS

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0. Introduction. Simplicial quadratic forms (cf. Definition 1.4), and various equivalent forms, have occasionally been studied in geometry [**8**], and in number theory [**9**], [**10**], in connection with the extremal properties of integral quadratic forms. Our investigations, which employ simple techniques from graph theory and geometry, partly continue both those of Coxeter [**5**], who introduced the graphs described in Section 1, and Vinberg [**20**], [**21**], who described an algorithm for determining a fundamental region for a discrete group acting on spherical, Euclidean, or hyperbolic space. After a preliminary discussion of reflexible forms and the Caley-Klein model for $(n - 1)$-space (1.2), we define a simplicial form and its graph. Having enumerated them completely, we turn in Section 2 to their equivalence, which is related to a geometric dissection. The unit group for each simplicial form can then be determined from Theorem 3.7.

I wish to thank Professor H. S. M. Coxeter for many helpful ideas, and Professor G. Maxwell and the referee for suggesting numerous improvements.

1. Simplicial forms. Let G be a Coxeter group with presentation

(1.1) $\langle R_1, \dots, R_n | (R_iR_j)^{p_{ij}} = I, \ 1 \leqq i, j \leqq n \rangle$

where each $p_{ii} = 1$ and $p_{ij} = \infty$ if there is no corresponding relation. It is known that G may be realized as a subgroup of $GL(\mathbf{R}^n)$ [**2**, Chapter v, § 4]. Indeed, fix a basis $\{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ for $\mathbf{R}^n$ and define on $\mathbf{R}^n$ a symmetric bilinear form $(\cdot, \cdot)$ with $(\mathbf{c}_i, \mathbf{c}_j) = -2 \cos \pi/p_{ij}$. Take $R_i = R(\mathbf{c}_i)$, where for any $\mathbf{u} \in \mathbf{R}^n$ with $\delta = (\mathbf{u}, \mathbf{u}) \neq 0$, we define

$$R(\mathbf{u}) : \mathbf{w} \rightarrow \mathbf{w} - 2\delta^{-1}(\mathbf{w}, \mathbf{u})\mathbf{u}, \ (\mathbf{w} \in \mathbf{R}^n).$$

The reflection $R(\mathbf{u})$ is an involutory automorphism of $(\cdot, \cdot)$.

Naturally, G is called *crystallographic* if it leaves invariant some n-dimensional lattice $L \subset \mathbf{R}^n$. In [**14**, Proposition 1.3], Maxwell shows that L contains a *root lattice* $Q(B)$ generated by a *basic system* $B = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ of roots for G: that is, for $1 \leqq i, j \leqq n$, there are $b_i > 0$ with $\mathbf{e}_i = b_i\mathbf{c}_i$ and $c_{ji} = b_j(\mathbf{c}_j, \mathbf{c}_i)/b_i \in \mathbf{Z}$. Thus, the Cartan matrix $C = [c_{ji}]$ is integral. With respect to the basis B, $R_i : \mathbf{e}_j \rightarrow \mathbf{e}_j - c_{ji}\mathbf{e}_i$, so that $Q(B)$ is also G-invariant.

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If we take $u = \sum u^i \mathbf{e}_i$ and $v = \sum v^j \mathbf{e}_j$ we may define a bilinear form

$$(\mathbf{u}, \mathbf{v}) = \sum_{i=1}^{n} \sum_{j=1}^{n} u^i a_{ij} v^j,$$

where $a_{ij} = (\mathbf{e}_i, \mathbf{e}_j)$. Since $a_{ii} = 2b_i^2$, $c_{ij} = 2a_{ij}/a_{jj}$ and $a_{ii}/a_{jj} = c_{ij}/c_{ji}$, we may suitably rescale L so as to obtain all a_{ii}, $2a_{ij} \in \mathbf{Z}$. Hence, we may assume below that $f(\mathbf{u}) = (\mathbf{u}, \mathbf{u})$ is an integral quadratic form. Since each R_i is a unit (or unimodular automorphism) of f, the group $G = G(f)$ is a subgroup of $\mathcal{O}(f)$, the group of all units for f. The symmetry of the lattice $Q(B)$ under $R_1, \ldots, R_n$ defines f as a reflexible form [**5**, p. 403], a basis dependent notion.

For forms of suitable signature, G may be considered as acting on X^{n-1}, one of $(n - 1)$-dimensional spherical space $\mathbf{S}^{n-1}$, Euclidean space $\mathbf{E}^{n-1}$, or hyperbolic space $\mathbf{H}^{n-1}$, as modelled below using the dual space $\check{\mathbf{R}}^n$. Indeed, we set $X^{n-1} = K/\mathbf{R}^+$ for a suitable cone

$$K \subseteq M = \{\mathbf{x} \in \check{\mathbf{R}}^n | \mathbf{x} \neq \check{\mathbf{0}}\},$$

where $\mathbf{R}^+$ denotes the positive reals. An *ordinary point* of X^{n-1} is thus a ray $\mathbf{R}^+\mathbf{x}$ along some vector $\mathbf{x} \in K$. For notational convenience we represent $\mathbf{R}^+\mathbf{x}$ by $\mathbf{x}$, and assume when necessary that coordinates are positive homogeneous; we also use $(\cdot\,,\,\cdot)$ to denote both the pairing $\mathbf{x}(\mathbf{u}) = (\mathbf{x}, \mathbf{u})$ (for $\mathbf{x} \in \check{\mathbf{R}}^n$, $\mathbf{u} \in \mathbf{R}^n$) and, in the non-degenerate cases, the adjoint form defined naturally on $\check{\mathbf{R}}^n$.

1.2. *The Cayley-Klein Model for* X^{n-1} [**6**, § 12.1, § 14.2].

(a) For $\mathbf{S}^{n-1}$, $(\cdot\,,\cdot)$ is positive definite and $K = M$.

(b) For $\mathbf{E}^{n-1}$, $(\cdot\,,\,\cdot)$ is positive semidefinite with radical spanned by some non-zero $\mathbf{m} \in \mathbf{R}^n$, and $K = \{\mathbf{x} \in M | (\mathbf{x}, \mathbf{m}) > 0\}$. We adjoin to $\mathbf{E}^{n-1}$ all points at infinity (for which $(\mathbf{x}, \mathbf{m}) = 0$).

(c) For $\mathbf{H}^{n-1}$, $(\cdot\,,\,\cdot)$ is non-degenerate with negative inertial index 1 and K is one component of $\{\mathbf{x} \in M | (\mathbf{x}, \mathbf{x}) < 0\}$. We adjoin to $\mathbf{H}^{n-1}$ both Ω, the set of points at infinity $((\mathbf{x}, \mathbf{x}) = 0)$, and all ultra-infinite points $((\mathbf{x}, \mathbf{x}) > 0)$.

Remarks. Any linear subspace $U \subseteq \mathbf{R}^n$ defines a (possibly empty) ordinary subspace $(U \cap K)/\mathbf{R}^+$ of X^{n-1}, to which we adjoin those non-ordinary points lying in U. If U contains no ordinary points, it is said to be *at infinity* (*ultra-infinite*) if it contains at least one (no) points at infinity. Henceforth, we shall conveniently (and loosely) use U itself to refer to the subspace of X^{n-1}.

The isometries of X^{n-1} are defined by those automorphisms of $(\cdot\,,\cdot)$ which preserve K (under the contragredient action on $\check{\mathbf{R}}^n$). In particular for each $\mathbf{u} \in \mathbf{R}^n$ with $(\mathbf{u}, \mathbf{u}) > 0$, there is a geometric reflection $R(\mathbf{u})$ with mirror

$$H(\mathbf{u}) = \{\mathbf{x} \in X^{n-1} | (\mathbf{x}, \mathbf{u}) = 0\}.$$

If the coordinates of $\mathbf{u}$ are taken as positive homogeneous, then the half-space

$$H^-(\mathbf{u}) = \{\mathbf{x} \in X^{n-1} | (\mathbf{x}, \mathbf{u}) \leqq 0\}$$

is well defined. We require the following analytic results [**6**, § 6.7, § 10.7, § 12.1]:

1.3. (a) *For* $(\mathbf{u}, \mathbf{u}), (\mathbf{v}, \mathbf{v}) > 0$, $(\mathbf{u}, \mathbf{v} \in \mathbf{R}^n)$, $U = H(\mathbf{u}) \cap H(\mathbf{v})$ *is ordinary, at infinity, or ultra-infinite according as* $|t| < 1$, $= 1$, *or* > 1, *where*

$$t = -(\mathbf{u}, \mathbf{v})[(\mathbf{u}, \mathbf{u})(\mathbf{v}, \mathbf{v})]^{-1/2}.$$

When U *is ordinary, the dihedral angle containing* $H^-(\mathbf{u}) \cap H^-(\mathbf{v})$ *is* arc cos t.

(b) *If the vector* $\mathbf{x}$ *representing an ordinary point is fixed, the distance from* $\mathbf{x}$ *to* $H(\mathbf{u})$ *increases monotonically with* $(\mathbf{x}, \mathbf{u})^2/(\mathbf{u}, \mathbf{u})$.

Returning now to our reflexible form f, we have each $a_{ii} > 0$, so that the hyperplanes $H_i = H(\mathbf{e}_i)$ enclose a simplex

$$\Lambda = \{\mathbf{x} | (\mathbf{x}, \mathbf{e}_i) \leqq 0, 1 \leqq i \leqq n\}$$

in which $\mathbf{x}$ may denote a non-ordinary point. With respect to the dual basis $\{\check{\mathbf{e}}^1, \ldots, \check{\mathbf{e}}^n\}$, the vertex V_i opposite the wall H_i has coordinates $(0, \ldots, -1, \ldots, 0)$. The reflections $R_1, \ldots, R_n$ in the walls generate a group $G(f)$ of isometries on X^{n-1}. Since $G(f)$ consists of units, it acts discontinuously on X^{n-1} and has some polytope as a fundamental region (a general proof is similar to that in the Euclidean case: [**1**, p. 313]).

Definition 1.4. The reflexible form f is *simplicial* if the simplex Λ is a fundamental region of finite volume for $G(f)$. The case $\mathbf{H}^1$ is excluded.

Remark. For the hyperbolic line, there are infinitely many possibilities for the indefinite binary form f, for which we have no convenient notation (cf. [**15**]). Henceforth, f will denote a simplicial form, and will be named after the corresponding space X^{n-1}. Any form obtained from f by setting various $u^i = 0$ is said to be *derived* from f.

LEMMA 1.5. (a) *The form derived from* f *by setting* $u^i = 0$ *is positive definite* (*perhaps semi-definite in the hyperbolic case*). *All further derived forms are definite.*

(b) *Each derived binary form is one of those listed in Figure* 1.

Proof. Since Λ has finite volume, each vertex V_i is ordinary (or perhaps on Ω in $\mathbf{H}^{n-1}$). The geometry of the bundle centred at V_i is thus spherical or Euclidean [**6**, p. 197], so that the form derived by setting $u^i = 0$ is a spherical or Euclidean simplicial form. Moreover, for $i \neq j$, the edge V_iV_j is ordinary, so that f is positive definite on the space $W = \{\mathbf{u} \in \mathbf{R}^n | u^i = u^j = 0\}$ (cf. [**6**, § 4.5, § 10.8]).

Now each $a_{ii} > 0$; since

$$\cos(\pi/p_{ij}) = (\mathbf{c}_i, \mathbf{c}_j) = -a_{ij}(a_{ii}a_{jj})^{-1/2},$$

$a_{ij} \leqq 0$ for $i \neq j$. By (a), $f(u^i\mathbf{e}_i + u^j\mathbf{e}_j)$ is positive (or semi) definite, so that $a_{ii}a_{jj} - a_{ij}^2 > 0$ (or $= 0$). Since also a_{ii}, a_{jj}, $2a_{ij}/a_{ii} \in \mathbf{Z}$, the only possibilities are those displayed in Figure 1, for positive integers a and b.

(a) $a(u^i)^2 + b(u^j)^2$ $(p_{ij} = 2)$

(b) $a[(u^i)^2 - qu^iu^j + q(u^j)^2]$ $\left\{\begin{matrix} q = 1,2,3,4 \\ p_{ij} = 3,4,6,\infty \end{matrix}\right\}$

(c) $a(u^i - u^j)^2$ $(p_{ij} = \infty)$

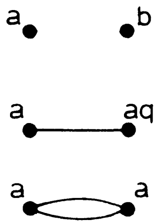

FIGURE 1. Binary Simplicial Forms.

Each derived binary form is conveniently represented as shown in Figure 1 by a graph having a node labelled "a_{ii}" for each u^i [**5**, p. 415]. If two nodes are joined by λ branches where $\lambda = 0$, 1, or 2, then

$$a_{ij} = -\lambda \max \{a_{ii}/2, a_{jj}/2\}.$$

The form f is likewise represented by a form graph with n labelled nodes; the common label "1" is omitted though understood. For instance, graph (i) in Figure 4 represents the positive form

$$f = x^2 - 2xy + 2y^2 - 2yz + 2z^2.$$

Listed next to each form in Figure 1 is p_{ij}, the period of R_iR_j (a rotation about the facet $H_i \cap H_j$ of Λ). Since Λ is a fundamental region, π/p_{ij} is the dihedral angle $H_i^- \cap H_j^-$; in particular, we easily compute that

1.6(a) Each $p_{ij} \in \{2, 3, 4, 6, \infty\}$, $(1 \leqq i \neq j \leqq n)$.

The semi-definite binary forms in Figure 1, namely form (c) and form (b) with $q = 4$, can occur only with $\mathbf{E}^1$ and $\mathbf{H}^2$ (by 1.5 (a)). In this case, $p_{ij} = \infty$ and $\pi/p_{ij} = 0$, so that H_i is parallel to H_j.

Both the group $G(f)$ and the simplex Λ are represented by another sort of graph, the Coxeter diagram, which has a node for each R_i. Nodes i and j are adjacent when $p_{ij} > 2$, and the corresponding branch is labelled p_{ij}, although the common label "3" is omitted and understood.

The simplicial forms can therefore be enumerated by relabelling the Coxeter diagrams for simplexes of finite volume in X^{n-1} with dihedral angles π/p_{ij} satisfying 1.6(a). A complete list of such simplexes can be extracted from [**7**, p. 297] and [**3**]. Take one of these Coxeter diagrams. Label nodes i and j (and perhaps double the branch) so as to obtain a rational multiple of some form in Figure 1, for which p_{ij} labels the branch joining nodes i and j ($p_{ij} = 2$ indicates no branch). The branch label is deleted. In attempting this for each pair of nodes, an inconsistency can arise only in traversing a circuit, for which the product of successive ratios of node labels yields $2^k 3^l 4^m = 1$ for integers k, l, m. Given condition 1.6(a), a relabelling is therefore possible just when

1.6(b). No circuit in the Coxeter diagram for Λ contains an odd number of branches marked "4" or an odd number marked "6", [**12**, p. 71], [**2**, Chapter v, § 4, Exercise 6].

A suitable integral multiple of the a_{ii}'s yields a simplicial form f; except for doubled branches the form graph is topologically the same as the Coxeter diagram. For example, the Coxeter group [4,3] yields the two simplicial forms whose graphs are displayed in Figure 4 (i), (ii).

PROPOSITION 1.7. *Let f be a simplicial form.*

(a) *If f is Euclidean or hyperbolic, then its graph is connected.*

(b) *If f is spherical, its graph may not be connected; $G(f)$ is the direct product of the Coxeter groups defined by the connected components of the graph.*

Proof. Part (a) follows easily from 1.5(a). Since the involutions R_i and R_j commute if and only if $p_{ij} = 2$ (just when the corresponding nodes are non-adjacent), we obtain (b).

The connected spherical, Euclidean, and hyperbolic simplicial forms are listed in Tables 1, 2 and 3 respectively. Only primitive forms, for which g.c.d.$(a_{11}, \dots, a_{nn}) = 1$, are listed (any multiple of f has the same unit group). The forms are further classified according to the determinant $\Delta(f) = \det M(f)$, where $M(f) = [a_{ij}]$. In the Euclidean case $\Delta(f) = 0$; otherwise the computation is often simplified by expanding $\Delta(f)$ along the row corresponding to a univalent node i in the graph:

$$\Delta(f) = a_{ii}\Delta(f_i) - a_{ij}^2\Delta(f_{ij}).$$

Here $f_i(f_{ij})$ denotes the form derived by deleting node i (and node j) [**5**, p. 426].

Remarks. In [**19**], Vinberg classified all linear groups generated by reflections having the presentation 1.1. Indeed, for any reflexible form the group $G(f)$ acts discontinuously on the interior of some convex cone K. In [**14**], Maxwell sets

$$L^* = \{\mathbf{u} | \mathbf{u} - (\mathbf{u})T \in L, \text{ for all } T \in G\}.$$

When f is non-degenerate, $P(B) = Q(B)^*$ is a (weight) lattice and any G-invariant lattice L satisfies $Q(B) \subseteq L \subseteq P(B)$ and $L^* = P(B)$, for some basic system B. These lattices are thus in one to one correspondence with subgroups of $P(B)/Q(B)$, whose structure is determined easily by computing the elementary divisors of the Cartan matrix C (cf. [**13**, p. 169], [**5**, § 10-14]). We note finally that most reflexible forms have unit groups with more complicated fundamental regions than those considered here: [**11**, pp. 546–569], [**16**], [**17**], [**20**] and [**21**].

2. Geometrical aspects of equivalence. For a fixed basic system $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ suppose that two bilinear forms $(\cdot, \cdot)$ and $[\cdot, \cdot]$ define the simplicial forms f and g respectively. These two forms are equivalent ($f \cong g$) if for some unimodular matrix P, $f(\mathbf{u}) = g(\mathbf{u}P)$, ($\mathbf{u} \in \mathbf{R}^n$). Thus, the unit group $\mathcal{O}(f)$ is isomorphic to $\mathcal{O}(g) = P^{-1}\mathcal{O}(f)P$. Also, $M(f) = PM(g)P^t$, $\Delta(f) = \Delta(g)$, and f and g have the same signature. If we use $[\cdot, \cdot]$ to construct the metric for X^{n-1}, we note that the associated simplex Φ has walls $H(\mathbf{e}_i)$, whereas the simplex Λ for f has walls $H(\mathbf{p}_i)$, where $\mathbf{p}_i$ is the ith row of P.

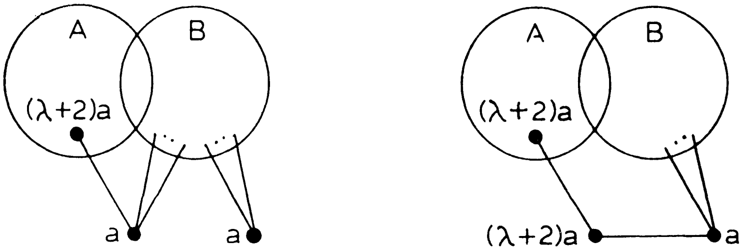

FIGURE 2. Two Graphs for Lemma 2.1.

Suppose that the form f has the left graph in Figure 2. Two special nodes r and s are labelled "a" and joined by λ branches where $\lambda = 0, 1$, or 2; except for a possible third node q (labelled $(\lambda + 2)a$ and adjacent to r but not s), each other node either lies in A and is adjacent to neither r nor s, or lies in B and is adjacent to both r and s by the same number of branches. Either of A and B may be empty. Such a graph may conveniently be called (r, s)-*symmetric* and is easily spotted despite its awkward description.

LEMMA 2.1. (a) *Suppose that the form f has the (r, s)-symmetric graph displayed on the left of Figure 2. Then $f \cong g$ where the form g has the right graph in Figure 2 (only node r has changed).*

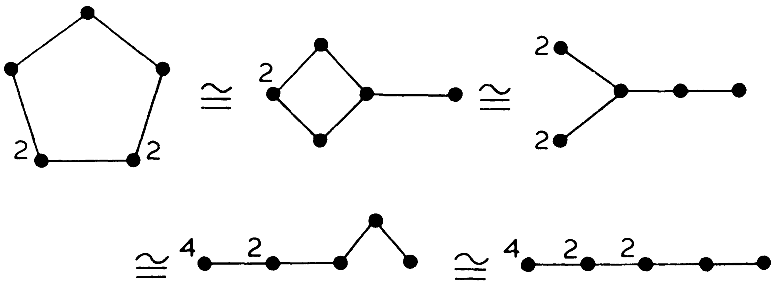

FIGURE 3. Five Equivalent Reflexible Forms.

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(b) *Furthermore, if f and g are simplicial forms with simplexes Λ and Φ respectively, then $\Phi \subseteq \Lambda$.*

Proof. Let the unimodular matrix P have rows $\mathbf{p}_i = \mathbf{e}_i$ $(i \neq r)$ and $\mathbf{p}_r = \mathbf{e}_r + \mathbf{e}_s$. It is easily verified that $M(f) = PM(g)P^t$, and $f \cong g$. The simplexes Φ and Λ differ only in their rth walls; from 1.3(a), $H^-(\mathbf{p}_r) \cap H^-(\mathbf{p}_s)$ is twice the dihedral angle $H^-(\mathbf{e}_r) \cap H^-(\mathbf{e}_s)$ of Φ. Hence, $\Phi \subseteq \Lambda$.

By inspection, we find that every equivalence of simplicial forms is determined by one or more applications of Lemma 2.1; a typical case is illustrated in Figure 3, in which only the first and last forms define simplexes of finite volume. (Clearly, Lemma 2.1(a) pertains to any reflexible form defined by such a graph.) On the other hand, two forms are inequivalent if they differ in rank, signature or determinant, or if just one has a fractional coefficient a_{ij}. Each of these tests fails for the following examples:

(a) Of the three forms whose graphs are shown in Figure 4 (iii), only the last represents 2 (mod 3).

(b) The matrices for the two forms in Figure 4 (iv) have different ranks (mod 2).

An explanation of the layout of the Tables is given in Section 4. We summarize by noting that:

2.2. In each class of equivalent simplicial forms, a unique form g has a simplex with smallest volume. (We shall say that g is *minimal.*)

Again suppose that $f \cong g$ with $\Phi \subseteq \Lambda$. Reflect Φ in its walls, and do the same for the resulting transforms. If some transform $(\Phi)T$, $T \in G(g)$, meets int (Λ), then $(\Phi)T \subseteq \Lambda$: otherwise some $H(\mathbf{p}_i)$ meets int $((\Phi)T)$, so that $H(\mathbf{p}_iT^t)$ meets int (Φ), contradicting the fact that $R(\mathbf{p}_iT^t) \in G(g)$. If Λ has finite volume, then some m distinct copies of Φ pack Λ and $G(f)$ is isomorphic to a subgroup of index m in $G(g)$. In example 2.4 below we use the following facts to determine m (cf. [**18**] for a related geometric approach).

2.3. (a) $(\Phi)T \subseteq \Lambda$, $T \in G(g)$, if and only if all entries in the matrix PT^{-1} are non-negative. (Consider the contragredient action of T on each $V_i = (0, \ldots, -1, \ldots, 0)$.)

(b) The reflection $T^{-1}R_iT$ reflects $(\Phi)T$ through its ith wall into $(\Phi)R_iT$, which by (a) lies in Λ just when the matrix $PT^{-1}R_i \geqq 0$.

Example 2.4. Consider the form g in Figure 4 (v). With the nodes indexed 1 to 4 along the top and node 5 hanging below, P has rows $\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_5$, $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_4$, $\mathbf{e}_5$. By 2.3 (a), $\Phi \subseteq \Lambda$. Since R_i takes $\mathbf{e}_j$ to $\mathbf{e}_j - c_{ji}\mathbf{e}_i$, we obtain the following chart:

Step	T	Rows of PT^{-1}				
(i)	I	$\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_5$,	$\mathbf{e}_2$,	$\mathbf{e}_3$,	$\mathbf{e}_4$,	$\mathbf{e}_5$
(ii)	R_1	$\mathbf{e}_2 + \mathbf{e}_5$	, $\mathbf{e}_1 + \mathbf{e}_2$,	$\mathbf{e}_3$,	$\mathbf{e}_4$,	$\mathbf{e}_5$
(iii)	R_2R_1	$\mathbf{e}_5$,	$\mathbf{e}_1$	, $\mathbf{e}_2 + \mathbf{e}_3$,	$\mathbf{e}_4$,	$\mathbf{e}_2 + \mathbf{e}_5$
(iv)	$R_3R_2R_1$	$\mathbf{e}_5$,	$\mathbf{e}_1$,	$\mathbf{e}_2$	, $\mathbf{e}_3 + \mathbf{e}_4$,	$\mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_5$
(v)	$R_4R_3R_2R_1$	$\mathbf{e}_5$,	$\mathbf{e}_1$,	$\mathbf{e}_2$,	$\mathbf{e}_3$	, $\mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4 + \mathbf{e}_5$

At each step, all coefficients of the $\mathbf{e}_i$ must be non-negative, so there is only one R_i which advances the algorithm ($T = R_i \ldots$). Since the procedure stops after five steps, $m = 5$. Note that since $G(f) \subset G(g)$, each $\mathbf{e}_i$ occurs by itself in some row of the table.

The computation also describes how the m copies of Φ fit together within Λ. Considerable effort is avoided when the graph of f has a chain of m nodes labelled $b, b, \ldots, b, a, a$ in order, with the last two joined by λ branches and with $b = (\lambda + 2)a$. If also these last two are (r, s)-symmetric with respect to a subgraph B disjoint from the chain, then we may apply Lemma 2.1 to obtain an equivalent form g. Moreover, the above procedure yields the index m. Numerous examples can be spotted in the Tables (cf. [**18**, Corollary 5]).

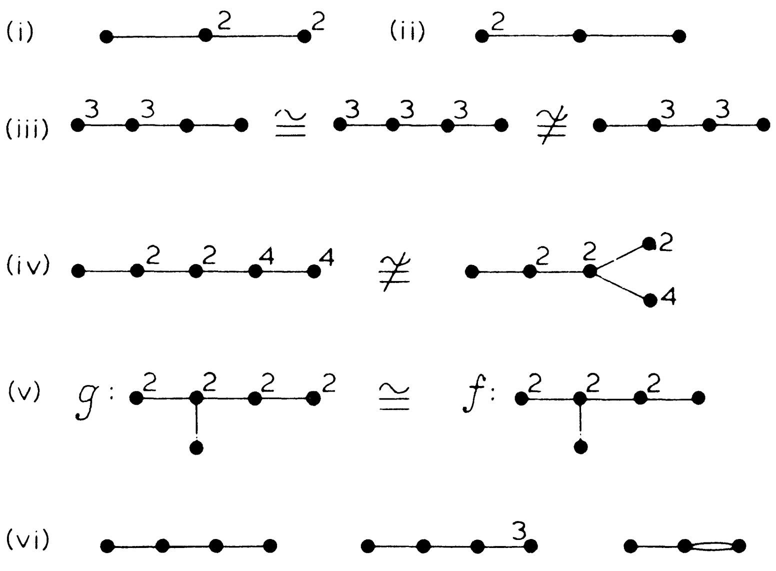

FIGURE 4. Several Simplicial Forms.

The algorithm is awkward only for the equivalent Euclidean forms defined by the fifth and seventh graphs listed in the right half of Table 2. For these forms the simplexes Λ and Φ are similar with corresponding edges

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in the ratio 2 : 1; hence $m = 2^{n-1}$. We remark that in the Tables, when $f \cong g$ with g minimal, then the index $m = [G(g) : G(f)]$ is noted in parentheses next to f.

3. The unit group $\mathscr{O}(f)$. In order to describe $\mathscr{O}(f)$ for the simplicial form f we must first determine F, the group generated by all unimodular reflections $R(\mathbf{u})$. $G(f)$ could be a proper subgroup of F.

LEMMA 3.1. *Suppose $R(\mathbf{u})$ is unimodular for $\mathbf{u} = (u^1, \ldots, u^n)$. Then we may assume that*

(a) $\delta = (\mathbf{u}, \mathbf{u}) > 0$,

(b) $\mathbf{u}$ *is primitive* ($u^1, \ldots, u^n$ *are integers with* g.c.d. 1), *and*

(c) $\delta | 2(\mathbf{u}, \mathbf{e}_j)$, $(1 \leqq j \leqq n)$.

Proof. $R(\mathbf{u})$ maps $\mathbf{e}_j$ to the integral vector

$$\mathbf{e}_j - 2(\mathbf{u}, \mathbf{e}_j)\delta^{-1}(u^1, \ldots, u^n).$$

Thus the u^i's are commensurable, and since $\mathbf{u}$ has positive homogeneous coordinates we obtain (b) and (c).

Let $\mathscr{R}$ be the set of all vectors $\mathbf{u}$ arising in Lemma 3.1. We now state Vinberg's

3.2. *Algorithm for determining a fundamental region Φ for F.*

(a) Let f_n be the form derived from f by setting $u^n = 0$ (i.e., delete the nth node in the graph). Suppose that $G(f_n)$, which is generated by $R_1, \ldots, R_{n-1}$, contains all unimodular reflections of f_n.

(b) Form a sequence of vectors $\mathbf{d}_1 = \mathbf{e}_1, \mathbf{d}_2 = \mathbf{e}_2, \ldots, \mathbf{d}_{n-1} = \mathbf{e}_{n-1}, \mathbf{d}_n, \mathbf{d}_{n+1}, \ldots$ where each $\mathbf{d}_j \in \mathscr{R}$ and for $l \geqq n$, $\mathbf{d}_l$ is some vector $\mathbf{u} \in \mathscr{R}$ for which

(i) $(\mathbf{d}_j, \mathbf{u}) \leqq 0$, $(1 \leqq j < l)$,

(ii) $V_n \in H^-(\mathbf{u})$, and

(iii) the distance from V_n to $H(\mathbf{u})$ is minimized.

Then (c) $\Phi = \bigcap_{j=1} H^-(\mathbf{d}_j)$ is a fundamental region for F.

(d) If for some m, $\bigcap_{j=1}^{m} H^-(\mathbf{d}_j)$ is a polytope of finite volume, then the algorithm stops at $\mathbf{d}_m$.

Proof. A proof for more general groups generated by reflections may be found in [**20**, pp. 27–29].

PROPOSITION 3.3. *Suppose f is a connected simplicial form with a connected derived form f_n, for which $G(f_n)$ contains all unimodular reflections for f_n. Then either*

(a) *$G(f) = F$, the group generated by all unimodular reflections for f, or*

(b) *$f \cong g$, where g is a connected simplicial form with a simplex strictly contained in that of f.*

Proof. While proceeding with the algorithm 3.2, either (a) or (b) must occur.

(i) Condition 3.2 (b) guarantees that $H^-(\mathbf{d}_i) \cap H^-(\mathbf{d}_j)$ is a strip or

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acute angle containing V_n; conversely, if $H(\mathbf{u})$ and $H(\mathbf{v})$ are walls of some $\Phi_0 = (\Phi)T_0$, where

$$T_0 \in F \quad \text{and} \quad V_n \in \Phi_0 \subseteq H^-(\mathbf{u}) \cap H^-(\mathbf{v}),$$

then $(\mathbf{u}, \mathbf{v}) \leqq 0$ [**21**, pp. 328, 331]. Thus, in using the algorithm, we may suppose without loss of generality that $\Phi \subseteq \Lambda$.

(ii) Consider $\mathbf{d}_n = \mathbf{u} = (u^1, \ldots, u^n) \in \mathscr{R}$. Now $H(\mathbf{u})$ meets int (Λ), the interior of Λ, if and only if some $u^i < 0 < u^j$; and $V_i \in H(\mathbf{u})$ if and only if $u^i = 0$, in which case $R(\mathbf{u})$ defines a unit for the corresponding derived form (by 3.1). This fact along with condition 3.2 (a) implies that $(0, \ldots, 0, -1) = V_n \notin H(\mathbf{u})$. Consequently conditions 3.2 (a), (b) imply that $u^n > 0$ and that

$$p(\mathbf{u}) = a_{nn}(u^n)^2/(\mathbf{u}, \mathbf{u})$$

is minimized for $\mathbf{u}$ (cf. 1.3 (b)). Even when $V_n \in \Omega$ in $\mathbf{H}^{n-1}$, $p(\mathbf{u})$ is the appropriate parameter [**21**, p. 328]. Note that if $p(\mathbf{u}) = p(\mathbf{e}_n) = 1$, we could choose $\mathbf{d}_n = \mathbf{e}_n$ so that by 3.2 (d), $\Phi = \Lambda$ and $F = G(f)$ (conclusion 3.3 (a)). Thus we may assume that $p(\mathbf{u}) < p(\mathbf{e}_n) = 1$; since by (i) we have $\Phi \subseteq \Lambda$, this implies that $H(\mathbf{u})$ meets int (Λ).

(iii) LEMMA 3.4. *For some i, $(\mathbf{u}, \mathbf{e}_i) > 0$.*

Proof. Since f is connected, all cofactors A^{ij} in $\Delta(f) = \det(a_{ij})$ are positive, except that $A^{ii} = 0$ when $V_i \in \Omega$ in $\mathbf{H}^{n-1}$ [**4**, p. 601]. If all $(\mathbf{u}, \mathbf{e}_i) \leqq 0$, then

$$0 \leqq \sum_i \sum_j (\mathbf{u}, \mathbf{e}_i)A^{ij}(\mathbf{u}, \mathbf{e}_j) = (\mathbf{u}, \mathbf{u})\Delta(f),$$

giving a contradiction for $\mathbf{H}^{n-1}$ and $\mathbf{E}^{n-1}$. If for $\mathbf{S}^{n-1}$ each

$$\Delta(f)u^j = \sum_i A^{ij}(\mathbf{u}, \mathbf{e}_j) \leqq 0,$$

then each $u^j \leqq 0$ so that $H(\mathbf{u})$ would not meet int (Λ).

(iv) Let $\mathbf{w} = \mathbf{u} - u^n\mathbf{e}_n$. Thus, for $\delta = (\mathbf{u}, \mathbf{u})$,

$$p(\mathbf{u}) = (f_n/\delta) - 1 + u^n(2(\mathbf{u}, \mathbf{e}_n)/\delta).$$

Now $p(\mathbf{u}) < p(\mathbf{e}_n) = 1$ and $f_n \geqq 0$ by 1.5 (a). By 3.1 (c), 3.2 (b) and 3.4, both u^n and $2(\mathbf{u}, \mathbf{e}_n)/\delta$ are positive integers. Hence both equal 1 and $(\mathbf{w}, \mathbf{w}) = a_{nn}$. Furthermore,

$$(\mathbf{w}, \mathbf{e}_j) = (\mathbf{u}, \mathbf{e}_j) - u^n a_{nj}.$$

By 3.1, a_{nn} divides both $2a_{nj}$ and $2(\mathbf{u}, \mathbf{e}_n) = \delta$, which in turn divides $2(\mathbf{u}, \mathbf{e}_j)$ (by 3.1). Hence, $(\mathbf{w}, \mathbf{w}) = a_{nn}$ divides $2(\mathbf{w}, \mathbf{e}_j)$, $1 \leqq j \leqq n$, and $R(\mathbf{w})$ defines a unit for f_n. But by hypothesis 3.2 (a), $R(\mathbf{w}) \in G(f_n)$ so that $H(\mathbf{w})$ cannot cut the (conical) fundamental region for $G(f_n)$. Thus $u^1, \ldots, u^{n-1}$ cannot differ in sign; but $H(\mathbf{u})$ meets int (Λ) and $u^n > 0$. Hence $u^i \leqq 0$, $(1 \leqq i \leqq n-1)$. In short,

3.5. If $\mathbf{u} \neq \mathbf{e}_n$, $H(\mathbf{u})$ cuts off the nth corner of Λ, so that $\Phi = H_1^- \cap \ldots \cap H_{n-1}^- \cap H^-(\mathbf{u})$ has finite volume and is the fundamental region for F.

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(v) Let P be the unimodular matrix with rows $\mathbf{e}_1, \ldots, \mathbf{e}_{n-1}, \mathbf{u}$. Thus, $PM(f)P^t = M(g)$ for a simplicial form g with simplex Φ (of finite volume). Since f_n is connected, then so is g: otherwise, $(\mathbf{u}, \mathbf{e}_j) = 0$, $1 \leqq j \leqq n - 1$, and

$$\delta = 2u^n(\mathbf{u}, \mathbf{e}_n) = 2\sum_{j=1}^{n} u^j(\mathbf{u}, \mathbf{e}_j) = 2\delta,$$

whence $\delta = 0$ (a contradiction). This concludes the proof of 3.3.

Case (b) of Proposition 3.3 is clearly impossible when f is the minimal form described in 2.2. By applying Proposition 3.3 in an inductive way, we may conclude that $G(f) = F$ for all minimal forms except those in Figure 4 (vi), for which it is also true by special consideration that $F = G(f)$.

Example 3.6. The third graph in Figure 4 (vi) describes the form

$$f(\mathbf{u}) = (u^1)^2 - u^1u^2 + (u^2)^2 - 2u^2u^3 + (u^3)^2.$$

Using the notation of this section we find that $\delta | 2(\mathbf{w}, \mathbf{u})$ for the vectors $\mathbf{w} = (0, 1, 1)$, $(1, 2, 2)$, and $(2, 4, 3)$ arising from $(a_{ij})^{-1}$. Thus, $\delta | u^1$, u^2, and $2u^3$, and since $\mathbf{u} = (u^1, u^2, u^3)$ is primitive, $\delta = 1$ or 2. But some $u^i < 0 < u^j$, so that $\mathbf{u} = \pm(1, -1, -1)$ or $\pm(1, 0, -1)$, for which $2 = \delta \nmid u^i$. Hence, $F = G(f)$.

We conclude with the following description of the unit group $\mathcal{O}(f)$ for f.

THEOREM 3.7. (a) *Each connected simplicial form f is equivalent to a minimal form g, and $\mathcal{O}(f) \cong \mathcal{O}(g)$.*

(b) $\mathcal{O}(g) = B \times [A \cdot G(g)]$, *where*

(i) *in the semidirect product $A \cdot G(g)$, A is the automorphism group for the (labelled) graph for g.*

(ii) *for $\mathbf{E}^{n-1}$ and $\mathbf{H}^{n-1}$, $B = \{\pm I\}$; for $\mathbf{S}^{n-1}$, $B = \{I\}$.*

Proof. Part (a) follows from 2.2, and the semidirect product is described in [**20**, p. 27]. For any automorphism T in the cases $\mathbf{E}^{n-1}$ and $\mathbf{H}^{n-1}$, just one of T or $-T$ preserves the space (cf. 1.2), whence $B = \{\pm I\}$. But the central inversion $-I$ is an isometry of $\mathbf{S}^{n-1}$, so that it must appear in the semidirect product.

4. Remarks on the tables. The graphs for all primitive connected simplicial forms are listed according to geometric type, determinant $\Delta(f)$, and integral equivalence. Within each half of any table, the left column of graphs contains inequivalent minimal forms (cf. 2.2) separated by any equivalent forms in the right column. The index (cf. Section 2) in parentheses beside each equivalent form should not be confused with the node labels.

TABLE 1 - CONNECTED SPHERICAL SIMPLICIAL FORMS

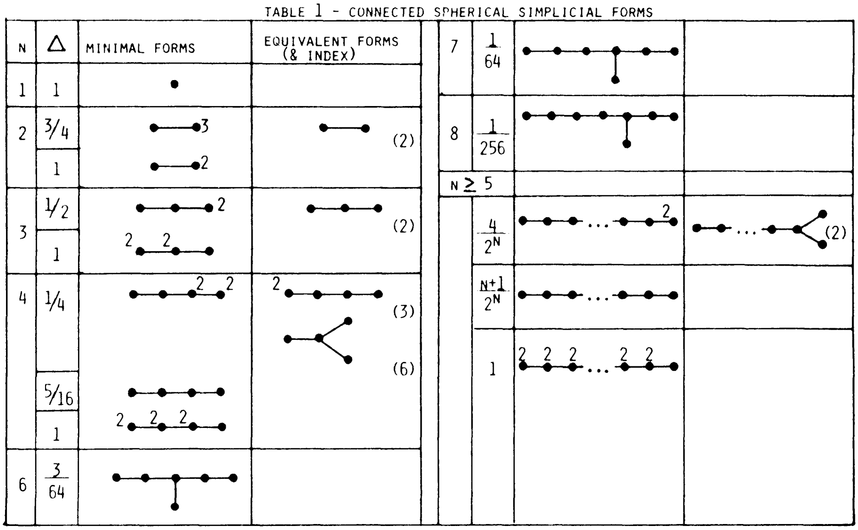

TABLE 2 - EUCLIDEAN SIMPLICIAL FORMS (△ = 0)

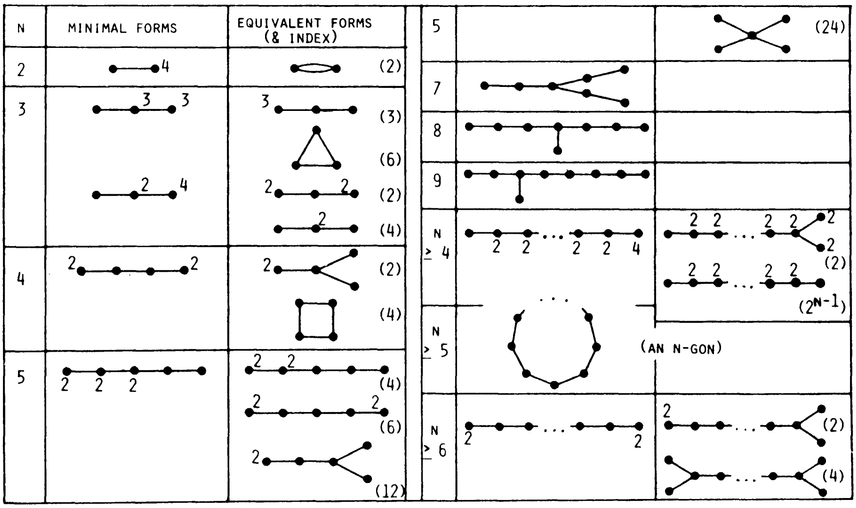

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TABLE 3-A - HYPERBOLIC SIMPLICIAL FORMS

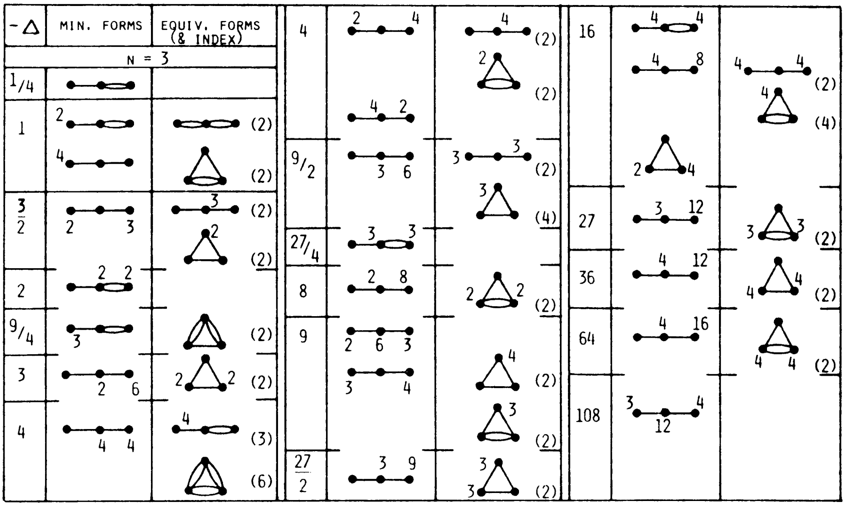

TABLE 3-B - HYPERBOLIC SIMPLICIAL FORMS

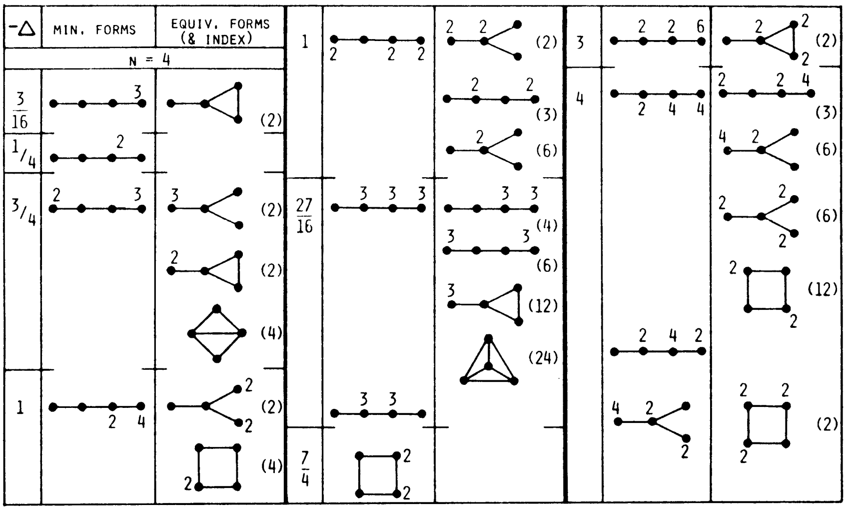

TABLE 3-C - HYPERBOLIC SIMPLICIAL FORMS

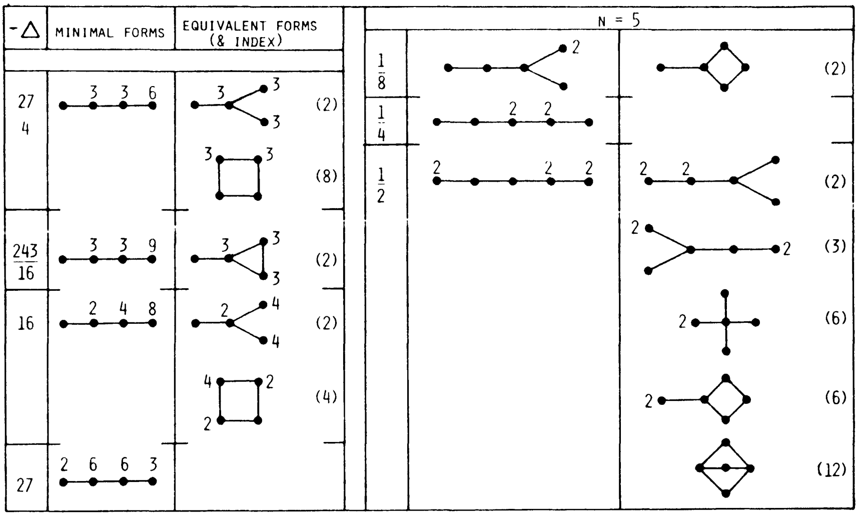

TABLE 3-D - HYPERBOLIC SIMPLICIAL FORMS

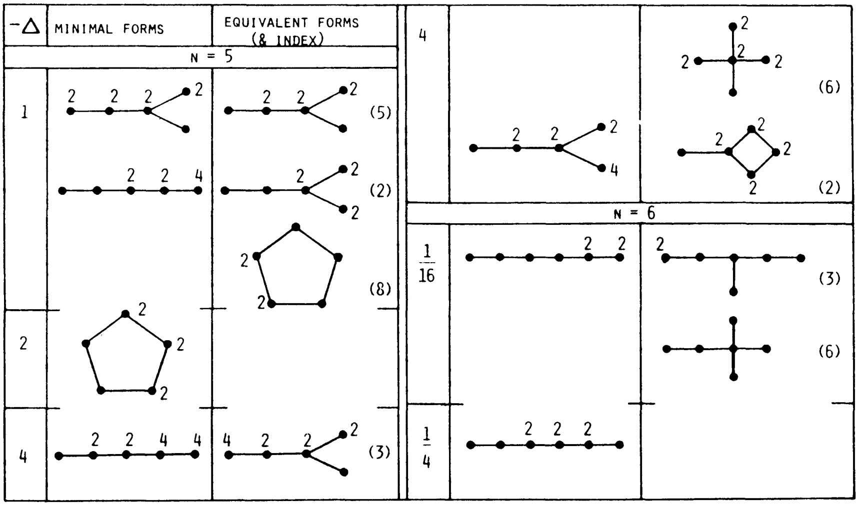

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TABLE 3-E – HYPERBOLIC SIMPLICIAL FORMS

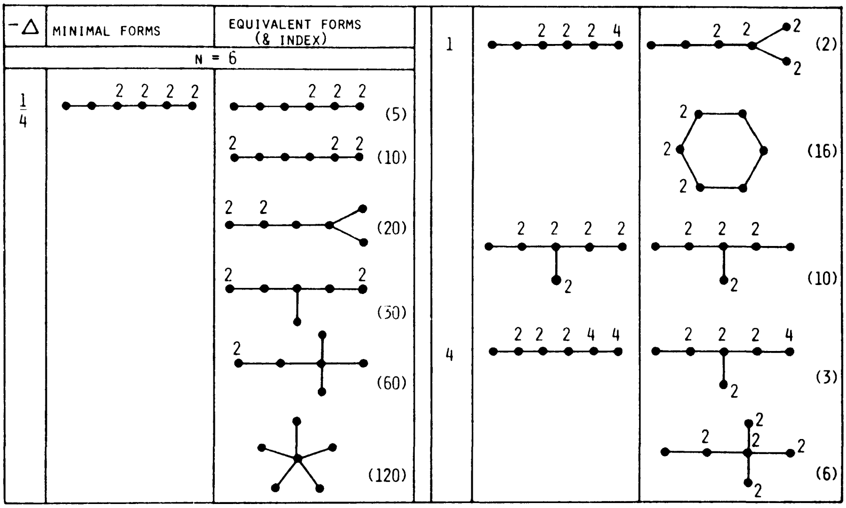
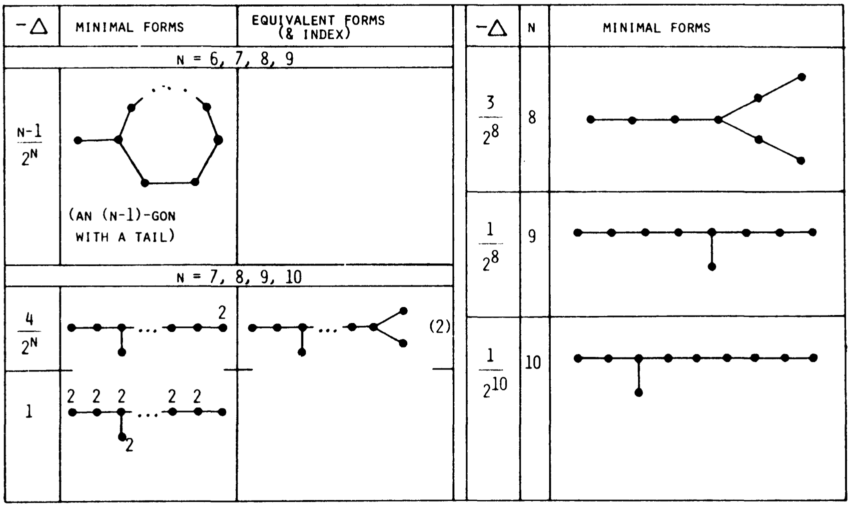


TABLE 3-F – HYPERBOLIC SIMPLICIAL FORMS



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