

TWISTED LOCAL-PRODUCT MANIFOLDS

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This note is concerned with “twisted” analogues of the LP-structures (i.e. local-product structures) and grids of (4). To obtain these twisted structures, we modify the concepts of LP-structure and grid by removing the *ordering* of the local foliations involved in the definitions. The effect of this change is that global foliations need no longer exist, since the locally defined foils may now fit together to form self-intersecting immersed manifolds.

To each twisted LP-manifold M we assign a homomorphism $h: \pi_1(M) \rightarrow S_r$, into the symmetric group S_r of degree r , where r is the number of foliations on M . Using h , we construct a covering $M^* \rightarrow M$ of M by an untwisted LP-manifold M^* .

We extend the structure theorem for grids (4) to the twisted case, and apply this to the special case of complete flat manifolds. Nontrivial twisted grids exist on the flat Klein bottle, on the complete flat Möbius band, and on all but one (the torus $S^1 \times S^1 \times S^1$) of the ten compact flat 3-manifolds.

For simplicity, we restrict attention to twisted local-product structures of type (k, \dots, k) . The definitions and theorems can be extended in an obvious way to include structures of type (k_1, \dots, k_r) . Since we are ultimately interested in grids, we work with smooth (i.e. C^∞) manifolds and maps.

1. Twisted local-products

Let E and F be finite-dimensional real linear spaces, and for any positive integers r and s , let E^r and F^s denote the direct products $E \times \dots \times E$ (r factors) and $F \times \dots \times F$ (s factors) respectively. If U and V are open subsets of E^r and F^s , then for all $i = 1, \dots, r$ and all $j = 1, \dots, s$ we put $U_i = p_i(U)$ and $V_j = q_j(V)$, where $p_i: E^r \rightarrow E$ and $q_j: F^s \rightarrow F$ are the i th and j th projections.

A smooth map $f: U \rightarrow V$ is called a *twisted LP-map* if there exists a map $\Pi: \{1, \dots, r\} \rightarrow \{1, \dots, s\}$ and for each $i = 1, \dots, r$ a smooth map $f_i: U_i \rightarrow V_{j_i}$, where $j_i = \Pi(i)$, such that $f_i(p_i(u)) = q_{j_i}(f(u))$ for all $u \in U$.

A *twisted LP-transformation* in E^r is an invertible twisted LP-map $f: U \rightarrow V$, such that U and V are open subsets of E^r and f^{-1} is a twisted LP-map. Trivially, the set of all twisted LP-transformations in E^r forms a pseudogroup. Thus a *twisted LP-structure* on a smooth m -manifold M is a maximal atlas \mathcal{M} of smooth charts $\xi_a: U_a \rightarrow U'_a$ on M , where U_a is an open subset of E^r and U'_a is an open subset of M , indexed by some set A , such that for all $a, b \in A$, the map

$$f_{ab}: \xi_a^{-1}(U'_a \cap U'_b) \rightarrow \xi_b^{-1}(U'_a \cap U'_b),$$

given by $f_{ab}(u) = \xi_b^{-1}(\xi_a(u))$, is a twisted LP-transformation. If $\dim E = k$, then $m = \dim M = rk$, and we say that (M, \mathcal{M}) is a *twisted LP-manifold* of type $k^{(r)}$.

Similarly, using LP-maps from open subsets of E^r to open subsets of F^s , we can define the notion of LP-map from one LP-manifold modelled on E^r to another modelled on F^s , which completes the construction of the twisted LP-category.

2. Untwisted coverings

In this section, (M, \mathcal{M}) will denote a twisted LP-manifold of type $k^{(r)}$ modelled on E^r . First we construct, for each $x \in M$, a homomorphism $h_p: \pi_1(M, p) \rightarrow S_r$ into the symmetric group S_r of degree r , as follows.

Let $\lambda: I \rightarrow M$ be a loop at x in M , where $I = [0, 1]$. Thus $\lambda(0) = \lambda(1) = x$. Then there is a sequence (ξ_1, \dots, ξ_n) of twisted LP-charts such that $\{U'_1, \dots, U'_n\}$ covers $\lambda(I)$, where $\xi_\alpha: U_{(\alpha)} \rightarrow U'_{(\alpha)}$, and $\lambda \left(\left[\frac{\alpha-1}{n}, \frac{\alpha}{n} \right] \right) \subset U'_{(\alpha)}$ ($\alpha = 1, \dots, n$). By

definition, there exists a permutation $\Lambda_{(\alpha)}$ of the sequence $(1, \dots, r)$ for each coordinate transformation $\xi_{\alpha+1}^{-1}\xi_\alpha$, with $\xi_{n+1} = \xi_1$. Trivially, the permutation $\Lambda = \Lambda_1 \circ \dots \circ \Lambda_n$ is independent of the choice of the charts ξ_α , and is also independent of λ up to homotopy rel. end-points. It is also immediate from the construction that if λ_1 and λ_2 are loops at x , and Λ_1, Λ_2 are the corresponding permutations, then the permutation $\Lambda_1 \circ \Lambda_2$ corresponds to the loop $\lambda_1\lambda_2$.

Thus for each $x \in M$ there is a homomorphism $h_x: \pi_1(M, x) \rightarrow S_r$, given by $h_x[\lambda] = \Lambda$. Moreover, if $y \in M$ and γ is any path in M from x to y , then $h_y = h_x \circ \gamma^*$, where $\gamma^*: \pi_1(M, y) \rightarrow \pi_1(M, x)$ is the isomorphism given by $\gamma^*[\mu] = [\gamma\mu\gamma^{-1}]$, μ being any loop in M at y . It follows that if M is connected, then the homomorphism h_x is independent of x up to isomorphisms of $\pi_1(M, x)$, and we can write $h: \pi_1(M) \rightarrow S_r$, choosing some fixed $x \in M$ as base-point.

Now let $H: M^* \rightarrow M$ denote the regular covering of M corresponding to the normal subgroup $\ker h$ of $\pi_1(M)$. Thus $\pi_1(M^*)$ is isomorphic to $\ker h$. Since any smooth immersion $f: M' \rightarrow M$, where $\dim M' = \dim M$, induces a twisted LP-structure \mathcal{M}' on M' from the given twisted LP-structure on M , we can give M^* a twisted LP-structure \mathcal{M}^* with respect to which H is a twisted LP-covering. Thus \mathcal{M}^* determines a homomorphism $h^*: \pi_1(M^*) \rightarrow S_r$. However, $h^* = h \circ H^*$ where $H^*: \pi_1(M^*) \rightarrow \pi_1(M)$ is induced by H . Hence h^* is trivial. Now there is an obvious “forgetful” functor Φ from the LP-category (of type (k, \dots, k)) to the twisted LP-category, and $(N, \mathcal{N}) \in \text{im } \Phi$ if and only if the homomorphism $\pi_1(N) \rightarrow S_r$ is trivial. In view of these remarks, it is natural to call (M^*, \mathcal{M}^*) the *smallest untwisted covering* of (M, \mathcal{M}) .

3. Twisted grids

Let (M, \mathcal{M}) be a twisted LP-manifold as before, and let ρ be a smooth Riemannian metric on M . We say that \mathcal{M} and ρ are *compatible*, and call

$\Gamma = (\mathcal{M}, \rho)$ a twisted grid on M , if \mathcal{M}^* and $\rho^* = H^*\rho$ are compatible in the sense of (4), where $H: (M^*, \mathcal{M}^*) \rightarrow (M, \mathcal{M})$ is the smallest untwisted covering of (M, \mathcal{M}) . Thus $\Gamma = (M, \rho)$ is a twisted grid on M if and only if $\Gamma^* = (\mathcal{M}^*, \rho^*)$ is a grid on M^* .

To obtain a structure theorem for complete twisted grids (i.e. twisted grids (\mathcal{M}, ρ) for which ρ is complete), we have only to combine the smallest untwisted covering $H: (M^*, \Gamma^*) \rightarrow (M, \Gamma)$ with the universal covering

$$K: (\tilde{M}^*, \tilde{\Gamma}^*) \rightarrow (M^*, \Gamma^*)$$

and apply the structure theorem for grids. Now we can identify the universal covering $(\tilde{M}, \tilde{\Gamma})$ of (M, Γ) with $(\tilde{M}^*, \tilde{\Gamma}^*)$, and put $J = H \circ K$ for the universal covering map.

Let $\{M_1, \dots, M_r\}$ be the foil-set of (M^*, Γ^*) at some point $x \in M^*$. Then $(\tilde{M}, \tilde{\Gamma})$ is twisted grid-isomorphic to $\left(\prod_{i=1}^r \tilde{M}_i, \Gamma' \right)$, where Γ' is the standard trivially twisted grid on $\prod_{i=1}^r \tilde{M}_i$ with respect to the product metric $\prod_{i=1}^r \rho_i$, and ρ_i is the metric induced on \tilde{M}_i from M^* by the inclusion.

It follows that (M, Γ) is twisted grid-isomorphic to the quotient $(\prod \tilde{M}_i, \Gamma')/G$ of $(\prod \tilde{M}_i, \Gamma')$ by a discrete freely acting group G of twisted grid-automorphisms. We note that the normal subgroup G^* of G of grid-automorphisms of the standard grid on $\prod \tilde{M}_i$ yields the smallest untwisted covering of (M, Γ) . That is, $(\prod \tilde{M}_i, \Gamma')/G^*$ is twisted grid-isomorphic to (M^*, Γ^*) .

Conversely, any such quotient is a twisted grid-manifold. In the next section, we give some simple examples of this construction.

4. Twisted grids on flat manifolds

To illustrate the above discussion, we examine a few simple examples of twisted grids on flat manifolds. These are particularly easy to handle, since the universal Riemannian covering of any complete flat m -manifold is isometric to Euclidean m -space R^m with its standard flat metric.

There are only five complete flat 2-manifolds: the Euclidean plane E itself; the cylinder C ; the torus T ; the Möbius band M ; and the Klein bottle K (see, for example, (5)). All admit untwisted grids, but only K and M possess non-trivial twisted grids (of type $1^{(2)}$). Figure 1 illustrates the twisted grid on M . In this case, $M = R^2/Z$, where Z acts on R^2 as a group of isometries generated by the glide-reflexion $g: R^2 \rightarrow R^2$ given by $g(x, y) = (y + 1, x + 1)$.

The compact flat 3-manifolds have been determined by Hantzsche and Wendt (3) (see also (1) and (5)). There are ten of these, and all but the torus $S^1 \times S^1 \times S^1$ admit nontrivial twisted grids. For example, we can represent one of these flat manifolds as the quotient R^3/G , where G is the freely acting discrete group of isometries generated by the isometries $g_i: R^3 \rightarrow R^3$ ($i = 1, 2, 3$),

where

$$g_1(x, y, z) = (x + 1, y, z),$$

$$g_2(x, y, z) = (x, y + 1, z),$$

$$g_3(x, y, z) = \left(-\frac{1}{2}x + \frac{\sqrt{3}}{2}y, -\frac{\sqrt{3}}{2}x - \frac{1}{2}y, z + \frac{1}{3} \right).$$

The lines L_1, L_2 and L_3 in \mathbf{R}^3 generated by $(1, -1, 0)$, $(1, 1, -2)$ and $(1, 1, 1)$ determine an untwisted grid on \mathbf{R}^3 that is carried down to a nontrivial twisted grid on \mathbf{R}^3/G . In fact, the image of the homomorphism $h: \pi_1(\mathbf{R}^3/G) \rightarrow S_3$ is isomorphic to Z_3 in this case.

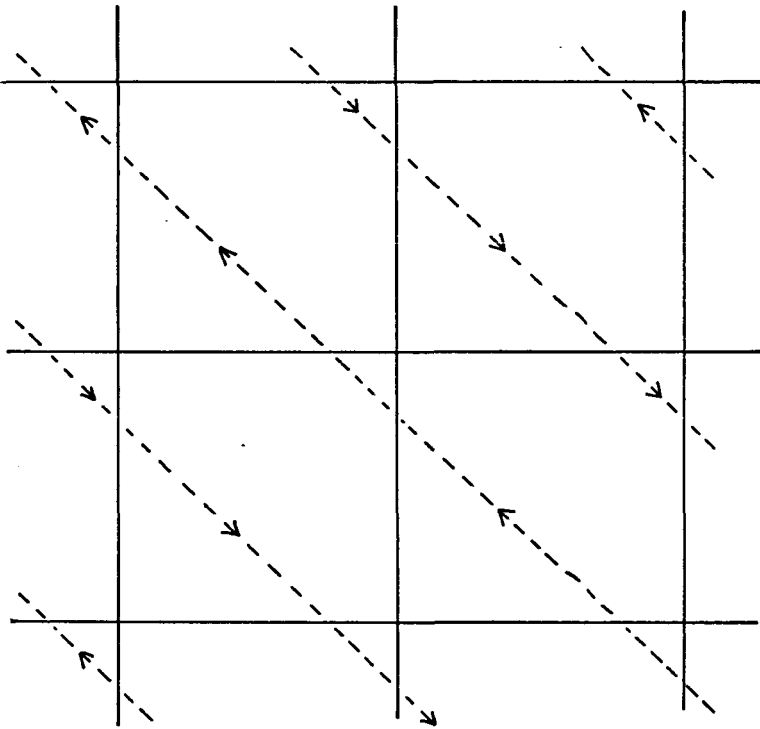


FIG. 1

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