BULL. AUSTRAL. MATH. SOC. MOS 2092, 2093

VOL. 1 (1969), 231-235.

On the lattice of congruences on a regular semigroup

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A result of Reilly and Scheiblich for inverse semigroups is proved true also for regular semigroups. For any regular semigroup S the relation Θ is defined on the lattice, $\Lambda(S)$, of congruences on S by: $(\rho, \tau) \in \Theta$ if ρ and τ induce the same partition of the idempotents of S. Then Θ is a congruence on $\Lambda(S)$, $\Lambda(S)/\Theta$ is complete and the natural homomorphism of $\Lambda(S)$ onto $\Lambda(S)/\Theta$ is a complete lattice homomorphism.

1. Introduction and summary

Let S be a semigroup, E its set of idempotents, $\Lambda(S)$ its lattice of congruences, and define on $\Lambda(S)$ the relation

$$\Theta = \{(\rho, \sigma) \in \Lambda(S) \times \Lambda(S) : \rho \cap (E \times E) = \sigma \cap (E \times E)\} .$$

Using the work of Munn [4] and Lallement [2], Reilly and Scheiblich [5] have proved the following

THEOREM 3.4 of [5]. *If S is a regular semigroup then*

(i) Θ is a meet compatible equivalence on $\Lambda(S)$;

(ii) each Θ-class is a complete modular sublattice of $\Lambda(S)$.

For inverse semigroups they prove considerably more in Theorem 5.1 [5]. We consider whether or not Theorem 5.1 [5] holds true also for regular semigroups. We answer this in the affirmative by proving the following

Received 21 March 1969. Received by J. Austral. Math. Soc. 6 December 1968. Communicated by G.B. Preston. This research was carried out at Monash University while the author held a Commonwealth Postgraduate Award. The author wishes to thank his supervisor, Professor G.B. Preston, for an inspiring discussion and further invaluable help.

https://doi.org/10.1017/S0004972700041472 Published online by Cambridge University Press

theorem, which reduces to Theorem 5.1 [5] when S is an inverse semigroup.

MAIN THEOREM. *Let S be a regular semigroup, E the set of idempotents of S, and $\Lambda(S)$ the lattice of congruences on S. Define on $\Lambda(S)$ the relation*

$$\Theta = \{(\rho,\sigma) \in \Lambda(S) \times \Lambda(S) : \rho \cap (E\times E) = \sigma \cap (E\times E)\} .$$

Then

(i) *Θ is a congruence on $\Lambda(S)$;*

(ii) *each Θ-class is a complete modular sublattice of $\Lambda(S)$;*

(iii) *the quotient lattice $\Lambda(S)/\Theta$ is complete and the natural homomorphism $\Theta^\natural$ of $\Lambda(S)$ onto $\Lambda(S)/\Theta$ is a complete lattice homomorphism.*

We use wherever possible, and often without comment, the notations and conventions of Clifford and Preston [1]. For any equivalence E on S, we shall often denote the equivalence $E \cap (E\times E)$ by $E|E$. We shall use the following well-known result, which one may readily verify.

RESULT 1. *Let L be a complete lattice and Θ a congruence on L satisfying the following condition.*

(A) *If $\{a_i : i \in I\}$ and $\{b_i : i \in I\}$ (I is some index set) are any two subsets of L such that $(a_i,b_i) \in \Theta$ for each $i \in I$, then $\left(\bigvee_{i\in I} a_i , \bigvee_{i\in I} b_i\right) \in \Theta$ and $\left(\bigwedge_{i\in I} a_i , \bigwedge_{i\in I} b_i\right) \in \Theta$.*

Then the quotient lattice L/Θ is a complete lattice and the natural homomorphism $\Theta^\natural$ of L onto L/Θ is a complete lattice homomorphism.

2. K(σ)-regularity

This section gives a routine generalization of Lallement's Theorem 3.6 [3] (see also Theorem 2.3 [2]).

Let S be a semigroup and σ any congruence on S. Let K be any one of Green's relations R, L, H, D, J on S (see [1]). To indicate that we are considering one of Green's relations on a different semigroup T say, we shall use, for example, the phrase "K *on* T". Define now $K(\sigma) = \{(a,b) \in S\times S : a\sigma$ and $b\sigma$ are K-related in $S/\sigma\}$. Then clearly

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$K(\sigma)$ is an equivalence relation on S, $K \subseteq K(\sigma)$, and $\sigma \subseteq K(\sigma)$. Also clearly $H(\sigma) = R(\sigma) \cap L(\sigma)$ and $D(\sigma) = R(\sigma) \circ L(\sigma) = L(\sigma) \circ R(\sigma) = L(\sigma) \vee R(\sigma)$.

For any element a in S, let $K(\sigma)_a$ $[K_a]$ denote the $K(\sigma)$-class [K-class] of S containing the element a, and let $K_{a\sigma}$ denote the K-class of S/σ containing the element $a\sigma$. Clearly there is a one-to-one mapping, ψ say, from the set of $K(\sigma)$-classes of S onto the set of K-classes of S/σ such that, for any element a in S, $K(\sigma)_a\psi = K_{a\sigma}$.

Now for $K \neq D$, there is a natural ordering $\leq$ (see [1] or [2]) on the set of K-classes of any semigroup. We define $\leq$ on the set of $K(\sigma)$-classes of S (for $K \neq D$), as follows. For any two elements a, b in S, $K(\sigma)_a \leq K(\sigma)_b$ if and only if $K_{a\sigma} \leq K_{b\sigma}$. Since ψ is a one-to-one mapping, $\leq$ is a partial ordering. Note, for example, that $R(\sigma)_a \leq R(\sigma)_b$ if and only if

$$\{a\sigma^{\natural}\} \cup (a\sigma^{\natural})(S\sigma^{\natural}) \subseteq \{b\sigma^{\natural}\} \cup (b\sigma^{\natural})(S\sigma^{\natural}) .$$

For any two elements a, b in S, the following are clear.

(i) If $K_a = K_b$, then $K(\sigma)_a = K(\sigma)_b$ (since $K \subseteq K(\sigma)$).

(ii) For $K \neq D$, if $K_a \leq K_b$, then $K(\sigma)_a \leq K(\sigma)_b$.

DEFINITION (from [2]) *Let* E *be any equivalence relation on* S. *Then* S *is called* E*-regular if, for any congruence* ρ *on* S, $\rho|E \subseteq E$ *implies* $\rho \subseteq E$.

THEOREM 1. *If* S *is a regular semigroup, then* S *is* $K(\sigma)$*-regular.*

Proof. With minor alterations (involving (i) and (ii) above), Lallement's proof of Theorem 3.6 [3] is sufficient (see also Theorem 2.3 [2]). The alterations consist mainly in replacing, for example, R_a by $R(\sigma)_a$.

3. The lattice of congruences

LEMMA 1. *Let* S *be a regular semigroup. For each element* i *of some index set* I, *let* ρ_i, ρ_i' *be congruences on* S *such that*

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$(\rho_i, \rho_i') \in \Theta$. *Then*

$$\left(\bigvee_{i\in I} \rho_i \,,\, \bigvee_{i\in I} \rho_i'\right) \in \Theta \,.$$

In particular, Θ *is a join compatible equivalence relation.*

Proof. For each $i \in I$, let α_i and β_i be the least and greatest element respectively of $\rho_i\Theta$ and put $\sigma = \bigvee_{i\in I} \alpha_i$. Define $H(\sigma)$ as in Section 2. Clearly $H(\sigma)|E = \sigma|E$. Then for each $i \in I$,

$$\beta_i|E = \alpha_i|E \subseteq \sigma|E = H(\sigma)|E \,,$$

and from Theorem 1, $\beta_i \subseteq H(\sigma)$. Hence $\bigvee_{i\in I} \beta_i \subseteq H(\sigma)$. Therefore

$$\left(\bigvee_{i\in I} \alpha_i\right)|E \subseteq \left(\bigvee_{i\in I} \beta_i\right)|E \subseteq H(\sigma)|E = \sigma|E = \left(\bigvee_{i\in I} \alpha_i\right)|E \,,$$

whence $\left(\bigvee_{i\in I} \alpha_i\right)|E = \left(\bigvee_{i\in I} \beta_i\right)|E$. Since, for each $i \in I$, we have $\alpha_i \subseteq \rho_i \subseteq \beta_i$ and $\alpha_i \subseteq \rho_i' \subseteq \beta_i$ we easily obtain that $\left(\bigvee_{i\in I} \rho_i\right)|E = \left(\bigvee_{i\in I} \rho_i'\right)|E = \left(\bigvee_{i\in I} \alpha_i\right)|E$, giving the required result.

Following Reilly and Scheiblich [5], we see that

$$\begin{aligned}\left(\bigcap_{i\in I} \rho_i\right) \cap (E\times E) &= \bigcap_{i\in I} [\rho_i \cap (E\times E)] = \bigcap_{i\in I} [\rho_i' \cap (E\times E)] \\ &= \left(\bigcap_{i\in I} \rho_i'\right) \cap (E\times E) \,.\end{aligned}$$

From this and Lemma 1, it follows that Θ is a congruence on $\Lambda(S)$ satisfying condition *(A)* of Result 1. The proof of the main theorem is now complete.

COROLLARY 1. *For each element* ρ *in* $\Lambda(S)$, *let* $M(\rho)$ *be the greatest element of the* Θ*-class* $\rho\Theta$. *Then for any two congruences* ρ , σ *on* S , *if* $\rho \subseteq \sigma$ *then* $M(\rho) \subseteq M(\sigma)$.

Proof. Now from Lemma 1, $M(\rho) \vee M(\sigma) \in (\rho \vee \sigma)\Theta$, and so $M(\rho) \subseteq M(\rho) \vee M(\sigma) \subseteq M(\rho \vee \sigma) = M(\sigma)$.

REMARK 1. If $m(\rho)$ denotes the least element of $\rho\Theta$, then it is clear that $\rho \subseteq \sigma$ implies $m(\rho) \subseteq m(\sigma)$, since $m(\rho) = [\rho \cap (E\times E)]^*$, the congruence on S generated by the relation $\rho \cap (E\times E)$.

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REMARK 2. For any $\rho \in \Lambda(S)$, the Θ-class $\rho\Theta$ is isomorphic to the lattice of idempotent separating congruences on $S/m(\rho)$ (see the proof of (ii) Theorem 3.4 [5]) and if S is an inverse semigroup then $\Lambda(S)/\Theta$ is (isomorphic to) a sublattice of the lattice of congruences on the semilattice E; for let ρ, σ be any elements of $\Lambda(S)$. Then $(\rho \cap \sigma)|E = (\rho|E) \cap (\sigma|E)$ and $(\rho \vee \sigma)|E = (\rho|E) \vee (\sigma|E)$ (the proof of Theorem 5.1 [5] shows that $(\rho_1 \vee \rho_3)|E = (\rho_1|E) \vee (\rho_3|E)$). Let us call $\{\rho \cap (E \times E) : \rho \in \Lambda(S)\}$ the set of normal congruences on E, and denote it by $N(E)$. Then $N(E)$ is a sublattice of the lattice of congruences on E and is isomorphic to $\Lambda(S)/\Theta$. By considering any Brandt semigroup with three or more idempotents, we see that $N(E)$ is not always equal to the lattice of all congruences on E. We note that the normal congruences on E are characterized as follows (due to Reilly and Scheiblich; see Definition 4.1 [5]): a congruence ζ on E is normal if and only if for every pair $(e, f) \in \zeta$ and every element $a \in S$, we have $(aea^{-1}, afa^{-1}) \in \zeta$.

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https://doi.org/10.1017/S0004972700041472 Published online by Cambridge University Press