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# ESTIMATING THE SIZE OF THE $(H, G)$-COINCIDENCES SET IN REPRESENTATION SPHERES

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**Abstract**

Let $W$ be a real vector space and let $V$ be an orthogonal representation of a group $G$ such that $V^G = \{0\}$ (for the set of fixed points of $G$). Let $S(V)$ be the sphere of $V$ and suppose that $f : S(V) \to W$ is a continuous map. We estimate the size of the $(H, G)$-coincidences set if $G$ is a cyclic group of prime power order $\mathbb{Z}_{p^k}$ or a $p$-torus $\mathbb{Z}_p^k$.

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## 1. Introduction

Let $G$ be a finite group which acts on a space $X$ and let $f : X \to Y$ be a continuous map from $X$ into another space $Y$. If $H$ is a subgroup of $G$, then $H$ acts on the right on each orbit $Gx$ of $G$ as follows: if $y \in Gx$ and $y = gx$, with $g \in G$, then $h \cdot y = gh^{-1}x$. A point $x \in X$ is said to be an $(H, G)$-*coincidence point* of $f$ (as introduced by Gonçalves *et al.* in [6]) if $f$ sends every orbit of the action of $H$ on the $G$-orbit of $x$ to a single point. Of course, if $H$ is the trivial subgroup, then every point of $X$ is an $(H, G)$-coincidence. If $H = G$, this is the usual definition of a $G$-coincidence point, that is, $f(x) = f(gx)$ for all $g \in G$. Let us denote by $A(f, H, G)$ the set of all $(H, G)$-coincidence points. Borsuk–Ulam theorems estimate the size of the set $A(f, H, G)$. For the case when the target space $Y$ is a CW-complex, this problem was considered by Gonçalves *et al.* [6] (for the subgroup $H = \mathbb{Z}_p$ of a finite group $G$, $X$ a homotopy sphere and $Y$ a CW-complex) and Gonçalves *et al.* [7] (for the subgroup $H = \mathbb{Z}_p$ of a finite group $G$, $X$ under certain (co)homological assumptions and $Y$ a CW-complex). In [5], by considering the target space $Y = M$ a manifold and $H$ a proper nontrivial subgroup of $G$, we proved a formulation of the Borsuk–Ulam theorem for manifolds in terms of $(H, G)$-coincidences which has applications to the famous topological Tverberg problem (see for example, [1]).

Let $W$ be a real vector space and let $V$ be an orthogonal representation of a group $G$ with $V^G = \{0\}$. Let $S(V)$ be the sphere of $V$ and suppose that $f : S(V) \to W$ is a continuous map. We estimate the size of $A(f, H, G)$ if $G$ is a cyclic group of prime power order $\mathbb{Z}_{p^k}$ or a $p$-torus $\mathbb{Z}_p^k$ (Theorems 3.1, 3.2 and 3.5).

## 2. Bourgin–Yang versions of the Borsuk–Ulam theorem for $\mathbb{Z}_{p^k}$ and $\mathbb{Z}_p^k$

Let $G = \mathbb{Z}_{p^k}$ be a cyclic group of prime power order, $k \geq 1$. Given two powers $p^m$, $p^n$ of $p$ with $1 \leq m \leq n \leq k-1$, we set

$$\mathcal{A}_{m,n} := \{G/H \mid H \subset G, p^m \leq |H| \leq p^n\},$$

where $|H|$ is the cardinality of $H$. We write $\mathcal{A}_X$ for a set of all the $G$-orbits of a space $X$ (up to a homeomorphism and thus up to an isomorphism of finite $G$-sets).

Let $V$ be an orthogonal representation of $G = \mathbb{Z}_{p^k}$, $p$ prime, $k \geq 1$, such that $V^G = \{0\}$ (for the set of fixed points of $G$). For $G = \mathbb{Z}_{p^k}$, with $p$ odd, every nontrivial irreducible orthogonal representation is even dimensional and admits a complex structure [10], so $V$ also admits such a structure. We write $d(V) = \dim_{\mathbb{C}} V = \frac{1}{2} \dim_{\mathbb{R}} V$, an integral numerical invariant of $V$.

The following Bourgin–Yang versions of the Borsuk–Ulam theorem for complex orthogonal representations of $G = \mathbb{Z}_{p^k}$, $p$ prime, $k \geq 1$ and for real orthogonal representations of $G = \mathbb{Z}_{2^k}$, $k \geq 1$ are from [8].

THEOREM 2.1 [8, Theorem 3.6]. *Let $V$, $W$ be two complex orthogonal representations of the cyclic group $G = \mathbb{Z}_{p^k}$, $p > 2$ prime, $k \geq 1$, such that $V^G = W^G = \{0\}$. Let $f : S(V) \xrightarrow{G} W$ be an equivariant map and $Z_f := f^{-1}(0) = \{v \in S(V) \mid f(v) = 0\}$. Suppose $\mathcal{A}_{S(V)} \subset \mathcal{A}_{m,n}$ and $\mathcal{A}_{S(W)} \subset \mathcal{A}_{m,n}$. Then*

$$\dim Z_f \geq 2\Big(\Big\lceil \frac{(d(V)-1)m}{n} \Big\rceil - d(W)\Big).$$

THEOREM 2.2 [8, Theorem 3.9]. *Let $V$, $W$ be two real orthogonal representations of the cyclic group $G = \mathbb{Z}_{2^k}$, $k \geq 1$, such that $V^G = W^G = \{0\}$. Let $f : S(V) \xrightarrow{G} W$ be an equivariant map and $Z_f = f^{-1}(0)$. Suppose that $\mathcal{A}_{S(V)} \subset \mathcal{A}_{m,n}$ and $\mathcal{A}_{S(W)} \subset \mathcal{A}_{m,n}$. Then*

$$\dim(Z_f) \geq \Big\lceil \frac{(d(V)-1)m}{n} \Big\rceil - d(W).$$

The next result is the classical version of the Bourgin–Yang theorem for a $p$-torus $\mathbb{Z}_p^k = \mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$ ($k$ times).

THEOREM 2.3 [9, Theorem 2.1]. *Let $V$ and $W$ be two orthogonal representations of the group $G = \mathbb{Z}_p^k$ such that $V^G = W^G = \{0\}$. Let $f : S(V) \to W$ be a continuous map. Then*

$$\dim Z_f \geq \dim_{\mathbb{R}} V - \dim_{\mathbb{R}} W - 1.$$

For further recent extensions of the Bourgin–Yang theorem, see [2, 3].

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## 3. Estimating the size of the $(H,G)$-coincidences set

Let $W'$ be a real vector space and $f : S(V) \to W'$ a continuous map. In this section, we estimate the size of the set $A(f, \mathbb{Z}_{p^i}, \mathbb{Z}_{p^k})$ under various assumptions.

THEOREM 3.1. *Let $V$ be a complex orthogonal representation of the cyclic group $G = \mathbb{Z}_{p^k}$, $p \geq 3$ prime and $k \geq 1$, such that $V^G = \{0\}$ and let $W'$ be a real vector space. Let $f : S(V) \to W'$ be a continuous map.*

(1) *If $\mathcal{A}_{S(V)} \subset \mathcal{A}_{1,p^{k-1}}$, then for all $i$ with $1 \leq i \leq k$,*

$$\dim A(f, \mathbb{Z}_{p^i}, \mathbb{Z}_{p^k}) \geq 2\left\lceil \frac{d(V)-1}{p^{k-1}} \right\rceil - (p^k - p^{k-i})\, dW'.$$

(2) *If $\mathcal{A}_{S(V)} \subset \mathcal{A}_{1,p^{i-1}}$ for some $i$ with $1 \leq i \leq k$, then*

$$\dim A(f, \mathbb{Z}_{p^i}, \mathbb{Z}_{p^k}) \geq 2\left\lceil \frac{d(V)-1}{p^{i-1}} \right\rceil - (p^k - p^{k-i})\, dW'.$$

PROOF. Let $i$ be fixed with $1 \leq i \leq k$. Consider the real vector space $\oplus_{j=1}^{p^k} W'$, which is the direct sum of $p^k$ copies of $W'$. The space $\oplus_{j=1}^{p^k} W'$ admits an action of the cyclic group $G = \mathbb{Z}_{p^k}$, given by

$$g(w_1, w_2, \ldots, w_{p^k}) = (w_2, \ldots, w_{p^k}, w_1)$$

for a fixed generator $g \in G$ and for each $(w_1, \ldots, w_{p^k}) \in \oplus_{j=1}^{p^k} W'$.

Denote by $\Delta(W'^{p^{k-i}})$ the diagonal of $\oplus_{j=1}^{p^k} W' = W'^{p^{k-i}} \oplus \cdots \oplus W'^{p^{k-i}}$. Then

$$\bigoplus_{j=1}^{p^k} W' = \Delta(W'^{p^{k-i}}) \oplus (\Delta(W'^{p^{k-i}}))^{\perp},$$

where $\Delta(W'^{p^{k-i}})^{\perp}$ is the orthogonal complement of $\Delta(W'^{p^{k-i}})$. Now $\Delta(W'^{p^{k-i}})$ is a $G$-subspace of $\oplus_{j=1}^{p^k} W'$ of dimension $p^{k-i} \dim W'$, so $\Delta(W^{p^{k-i}})^{\perp}$ is a $G$-subrepresentation of $\oplus_{j=1}^{p^k} W'$ of dimension $(p^k - p^{k-i}) \dim W'$ for which $(\Delta(W'^{p^{k-i}})^{\perp})^G = \{0\}$.

Denote by $a_1, \ldots, a_r$ a set of representatives of the left lateral classes of $G/\mathbb{Z}_{p^i}$, where $r = p^{k-i}$. Consider the map

$$F : S(V) \to \Delta(W'^{p^{k-i}}) \oplus \Delta(W'^{p^{k-i}})^{\perp}$$

defined by

$$F(x) = (F_0(x), F_1(x), \ldots, F_{p^i-1}(x)),$$

where $F_j(x) = (f(a_1 h^j x), \ldots, f(a_r h^j x))$, $j = 0, 1, \ldots, p^i - 1$, for a fixed generator $h \in \mathbb{Z}_{p^i}$. The linear orthogonal projection along the diagonal $\Delta(W'^{p^{k-i}})$ defines a $G$-equivariant map $\rho : \Delta(W'^{p^{k-i}}) \oplus \Delta(W'^{p^{k-i}})^{\perp} \to \Delta(W'^{p^{k-i}})^{\perp}$. Let us denote by $l$ the

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composition

$$S(V) \xrightarrow{F} \Delta(W'^{p^{k-i}}) \oplus \Delta(W'^{p^{k-i}})^{\perp} \xrightarrow{\rho} \Delta(W'^{p^{k-i}})^{\perp},$$

with

$$Z_l = l^{-1}(0) = (\rho \circ F)^{-1}(0) = F^{-1}(\Delta(W'^{p^{k-i}})) = A(f, \mathbb{Z}_{p^i}, \mathbb{Z}_{p^k}).$$

For a fixed generator $g \in G$, we can take $h = g^{p^{k-i}}, a_1 = e, a_2 = g, \dots, a_r = g^{p^{k-i}-1}$, and then $F$ is a $G$-equivariant map. Moreover,

$$\mathcal{A}_{S(\Delta(W'^{p^{k-i}})^{\perp})} \subset \mathcal{A}_{1,p^{i-1}} \subset \mathcal{A}_{1,p^{k-1}}.$$

To check the validity of the inclusion $\mathcal{A}_{S(\Delta(W'^{p^{k-i}})^{\perp})} \subset \mathcal{A}_{1,p^{i-1}}$, it suffices to prove that the cardinality of the orbit $\mathbb{Z}_{p^k} w$ belongs to the set $\{p^k, p^{k-1}, \dots, p^{k-i+1}\}$ for any $w = (w_1, \dots, w_{p^k}) \in S(\Delta(W'^{p^{k-i}})^{\perp})$. From [4, Ch. 1, Proposition 4.1], the cardinality of the orbit $\mathbb{Z}_{p^k} w$ belongs to the set $\{p^k, p^{k-1}, \dots, p, p^0 = 1\}$. Let $w = (w_1, \dots, w_{p^k})$ be an element in $S(\Delta(W'^{p^{k-i}})^{\perp})$ and suppose that $|\mathbb{Z}_{p^k} w| \in \{p^{k-i}, p^{k-i-1}, \dots, p^0 = 1\}$, that is, $|\mathbb{Z}_{p^k} w| = p^j$ for some $j$ with $0 \le j \le k - i$.

*Assertion.* We have $\mathbb{Z}_{p^k} w = \{w, gw, \dots, g^{p^j-1} w\}$, for a fixed generator $g$ of $\mathbb{Z}_{p^k}$.

In fact, consider a cyclic group G, $g \in G$ a fixed generator and $\{w, gw, \dots, g^{s-1} w\}$ the maximum set of the first $s$ elements of the orbit $Gw$ that are distinct from each other. From this definition, $g^s w \in \{w, gw, \dots, g^{s-1} w\}$. Suppose that

$$g^s w = g^i w \quad \text{for some } i \text{ with } 1 \le i \le s - 1.$$

Then

$$g^{s-i} w = w \quad \text{where } 1 \le s - i \le s - 1.$$

However, this contradicts the definition of the set $\{w, gw, \dots, g^{s-1} w\}$.

Now, if $g^t w \in Gw$, for some $t \in \mathbb{N}$, we have $t = ns + r$ with $0 \le r \le s - 1$. Therefore,

$$g^t w = g^{ns+r} w = g^r (g^{ns}) w = g^r w \in \{w, gw, \dots, g^{s-1} w\},$$

since $g^{ns} w = (g^s \cdots g^s) w = w$ and $0 \le r \le s - 1$.

Thus, for a fixed generator $g$ of $\mathbb{Z}_{p^k}$,

$$\begin{aligned} w = g^{p^j} w &= g^{p^j} (w_1, \dots, w_{p^j}, \dots, w_{(p^{k-j}-1)p^j+1}, \dots, w_{p^k}) \\ &= (w_{p^j+1}, \dots, w_{2p^j}, \dots, w_{(p^{k-j}-1)p^j+1}, \dots, w_{p^k}, w_1, \dots, w_{p^j}) \end{aligned}$$

and so $w \in \Delta(W'^{p^j})$. Since

$$\Delta(W') \subset \Delta(W'^{p}) \subset \cdots \subset \Delta(W'^{p^{k-i-1}}) \subset \Delta(W'^{p^{k-i}})$$

and $j \in \{0, 1, \dots, k - i\}$, we conclude that $w \in \Delta(W'^{p^j}) \subset \Delta(W'^{p^{k-i}})$, which is a contradiction since $\Delta(W'^{p^{k-i}}) \cap S(\Delta(W'^{p^{k-i}})^{\perp}) = \emptyset$.

This proves the assertion and the theorem follows from Theorem 2.1. □

We also have the following estimate for the size of $A(f,\mathbb{Z}_{2^i},\mathbb{Z}_{2^k})$.

THEOREM 3.2. *Let $V$ be a real orthogonal representation of the cyclic group $G=\mathbb{Z}_{2^k}$, $k\ge 1$, such that $V^G=\{0\}$ and let $W'$ be a real vector space. Let $f:S(V)\to W'$ be a continuous map.*

(1) *If $\mathcal{A}_{S(V)}\subset\mathcal{A}_{1,2^{k-1}}$, then for all $i$ with $1\le i\le k$,*

$$\dim A(f,\mathbb{Z}_{2^i},\mathbb{Z}_{2^k})\ge\left\lceil\frac{d(V)-1}{2^{k-1}}\right\rceil-(2^{k-1}-2^{k-i})\,dW'.$$

(2) *If $\mathcal{A}_{S(V)}\subset\mathcal{A}_{1,2^{i-1}}$, then for some $i$ with $1\le i\le k$,*

$$\dim A(f,\mathbb{Z}_{2^i},\mathbb{Z}_{2^k})\ge\left\lceil\frac{d(V)-1}{2^{i-1}}\right\rceil-(2^{k-1}-2^{k-i})\,dW'.$$

PROOF. For $G=\mathbb{Z}_{2^k}$, $k\ge 1$, using the same steps as in the proof of Theorem 3.1 and applying Theorem 2.2 gives the result. □

REMARK 3.3. We observe that Theorems 3.1 and 3.2 have peculiar characteristics that differentiate them from the classic results on $(H,G)$-coincidences. The first is that the action of the group $G$ on the sphere $S(V)$ is not necessarily free. The second is that the theorems provide an estimate for the dimension of the set of $(H,G)$-coincidences of a continuous function $f:S(V)\to W'$, for all subgroups $H=\mathbb{Z}_{p^i}$ of $G=\mathbb{Z}_{p^k}$.

EXAMPLE 3.4. Let $G$ and $W'$ be $\mathbb{Z}_4$ and $\mathbb{R}$, respectively. Let $\pi:S^1\to\mathbb{R}$ be the projection on the first factor and $p:\mathbb{R}\to\mathbb{R}$ be the polynomial function $p(x)=x(x-1)(x+1)$. Consider the action of $\mathbb{Z}_4$ on $S^1$ as the rotation of $\pi/4$. Then $f=p\circ\pi$ is such that $A(f,\mathbb{Z}_2,\mathbb{Z}_4)=\{(1,0),(0,1),(-1,0),(0,-1)\}$ and therefore $\dim A(f,\mathbb{Z}_2,\mathbb{Z}_4)=0$. In this case, we have the equality

$$\dim A(f,\mathbb{Z}_{2^i},\mathbb{Z}_{2^k})=\left\lceil\frac{d(V)-1}{2^{i-1}}\right\rceil-(2^{k-1}-2^{k-i})\,dW',$$

where $V=\mathbb{R}^2$, $k=2$ and $i=1$.

If we take $p(x)=x^2(x-1)(x+1)$ and $f=p\circ\pi$, then all points of $S^1$ are $(\mathbb{Z}_2,\mathbb{Z}_4)$-coincidence points of $f$, that is, $A(f,\mathbb{Z}_2,\mathbb{Z}_4)=S^1$ and therefore, $\dim A(f,\mathbb{Z}_2,\mathbb{Z}_4)=1$.

The next result is an $(H,G)$-coincidence version of the Bourgin–Yang theorem for $p$-torus $\mathbb{Z}_p^k$.

THEOREM 3.5. *Let $V$ and $W'$ be two orthogonal representations of the group $G=\mathbb{Z}_p^k$ such that $V^G=W'^G=\{0\}$. Let $f:S(V)\to W'$ be a continuous map. Then*

$$\dim A(f,\mathbb{Z}_p^i,\mathbb{Z}_p^k)\ge\dim_{\mathbb{R}}V+(p^k-p^{k-i})\dim_{\mathbb{R}}W'-1.$$

PROOF. Let $a_1,\dots,a_r$ be a set of representatives of the left lateral classes of $G/\mathbb{Z}_p^i$, where $r=p^{k-i}$. Let $\mathbb{Z}_p^i=\{h_1,\dots,h_{p^i}\}$ be a fixed enumeration of elements of $\mathbb{Z}_p^i$.

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Consider the map

$$F : S(V) \to \Delta(W'^{p^{k-i}}) \oplus \Delta(W'^{p^{k-i}})^{\perp}$$

defined by

$$F(x) = (F_1(x), F_2(x), \dots, F_{p^i}(x)),$$

where $F_j(x) = (f(a_1h_jx), \dots, f(a_rh_jx)), j = 1, \dots, p^i$.

For a fixed enumeration $\mathbb{Z}_p^k = \{g_1, \dots, g_{p^k}\}$ of the elements of $\mathbb{Z}_p^k$, we define a $\mathbb{Z}_p^k$-action on $\Delta(W'^{p^{k-i}}) \oplus \Delta(W'^{p^{k-i}})^{\perp}$ as follows: for each $g_j \in \mathbb{Z}_p^k$ and for each $(y_1, \dots, y_{p^k}) \in \Delta(W'^{p^{k-i}}) \oplus \Delta(W'^{p^{k-i}})^{\perp}$, set

$$g_j\,(y_1, \dots, y_{p^k}) = (y_{\sigma_{g_j}(1)}, \dots, y_{\sigma_{g_j}(p^k)}),$$

where the permutation $\sigma_{g_j}$ is defined by $\sigma_{g_j}(k) = u$, $g_kg_j = g_u$. Then $F$ becomes $\mathbb{Z}_p^k$-equivariant.

The linear orthogonal projection along the diagonal $\Delta(W'^{p^{k-i}})$ defines a $G$-equivariant map

$$\rho : \Delta(W'^{p^{k-i}}) \oplus \Delta(W'^{p^{k-i}})^{\perp} \to \Delta(W'^{p^{k-i}})^{\perp}.$$

Let us denote by $l$ the composition

$$S(V) \xrightarrow{F} \Delta(W'^{p^{k-i}}) \oplus \Delta(W'^{p^{k-i}})^{\perp} \xrightarrow{\rho} \Delta(W'^{p^{k-i}})^{\perp},$$

with $Z_l = l^{-1}(0) = (\rho \circ F)^{-1}(0) = F^{-1}(\Delta(W'^{p^{k-i}})) = A(f, \mathbb{Z}_p^i, \mathbb{Z}_p^k)$. From Theorem 2.3, $\dim Z_l \geq \dim_{\mathbb{R}} V + \dim_{\mathbb{R}} \Delta(W'^{p^{k-i}})^{\perp} - 1$, that is,

$$\dim A(f, \mathbb{Z}_p^i, \mathbb{Z}_p^k) \geq \dim_{\mathbb{R}} V + (p^k - p^{k-i}) \dim_{\mathbb{R}} W' - 1. \qquad \square$$

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