J. Austral. Math. Soc. 20 (Series A) (1975), 419–430.

SEMI-STABLE AND STABLE CACTI

K. L. McAVANEY

(Received 21 May 1974)

1. Introduction

Holton (1973) introduced the following concept. A graph G is *semi-stable* if there exists a point v in G for which $\Gamma(G_v) = \Gamma(G)_v$: where $\Gamma(G)$ is the automorphism group of G, G_v is the graph G with v deleted and $\Gamma(G)_v$ is the subgroup of $\Gamma(G)$ that fixes v. We say G is semi-stable at v. A *partial stabilising sequence* in G is a sequence $v_1, v_2, \cdots v_k$ of its points such that $\Gamma(G)_{v_1 v_2 \cdots v_i} = \Gamma(G_{v_1 v_2 \cdots v_i})$ for $i = 1, 2, \cdots, k$. If there exists a partial stabilising sequence in G for which k equals the number of points of G then G is said to be *stable* (Holton (1973a)). Most notation and terminology in what follows is explained in Harary (1969).

It is known (Heffernan (1972), Robertson and Zimmer (1972)) that all trees except the paths P_n with $n > 3$ and the smallest identity tree (T_2 in Figure 6) are semi-stable. We showed in McAvaney, Grant and Holton (1974) that the only unicyclic graphs that are not semi-stable are those in Figure 1. In Section 3 we show that these are the only cacti with a cycle that are not semi-stable. (A cactus is a connected graph in which each line lies on at most one cycle).

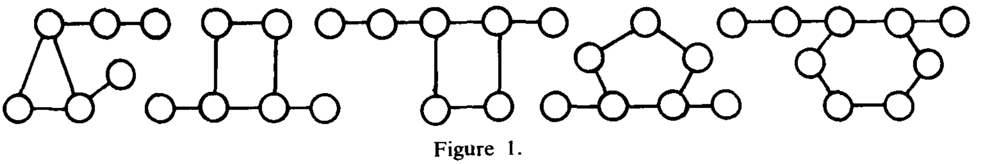


Figure 1.

If a graph is stable then it has a transposition automorphism, Holton and Grant (to appear). The converse is true for trees, Holton (1973b) and unicyclic graphs, McAvaney, Grant and Holton (1974). In Section 4 we show that it is true for all cacti.

Throughout the following sections, we use implicitly a characteristic of semi-stability demonstrated in Holton and Grant (1975): a graph G is semi-stable at v if and only if v is an isolated point or the set of points adjacent to V is a union of orbits of $\Gamma(G_v)$. We also use the following terminology. If $v_1, v_2, \cdots, v_k$ is a partial stabilising sequence in a graph G and $H = G_{v_1 v_2 \cdots v_k}$, we say G is *reducible* to H. The *distance* between a point v in G and a disjoint set A of points in G is the minimum $d(v, a)$ over all points a in A. A *penultimate point* v in G is a point in G such that G_v contains just two components one of which is a single point. If the number of points in graph G is less than the number of points in graph H, we say G is *smaller* than H. We denote by $\underline{P}_n$, a path P_n rooted at its endpoint.

2. Preliminaries

Before proving our main results we need to establish three lemmas. They require the following ideas. A *branch at a point* b of a cactus C is a maximal subcactus B of C with two or more points such that just one block of B contains b. A *branch at a block* D of C is a maximal subcactus of C with two or more points and which has just one point b in common with D. In both cases b is called the *root* of the branch.

LEMMA 1. *A rooted cactus C is semi-stable at a point c which is not a cutpoint or the root.*

PROOF. Let b_1 be the root of C and B_1 a smallest branch at b_1. Let B denote the block in B_1 that contains b_1. Let b_2 be a cutpoint in B closest to b_1 and let B_2' be the branch at B containing b_2. If b_2' is another cutpoint in B such that $d(b_1, b_2) = d(b_1, b_2')$ we assume B_2' is not larger than the branch at B containing b_2'. If B_2' is $\underline{P}_2$ and C is not semi-stable at its endpoint we redefine b_2 as a next closest cutpoint (if it exists) to b_1 and redefine B_2' accordingly. Let B_2 denote a smallest branch at b_2 that does not contain b_1. Thus, repeating this procedure, we generate a sequence of cutpoints $b_2, b_3, \cdots, b_n$ with associated branches $B_1, B_2, \cdots, B_n$, where B_n has just one cutpoint b_n.

Now every automorphism of C maintains the distance of each point from b_1. Moreover, a path between b_1 and a point a of C contains all the cutpoints w such that b_1 and a lie in different components of C_w. Hence, from the choice by size of B_i, if B_i is semi-stable at a point c then C is semi-stable at c. Therefore C is semi-stable at a point in B_n that is adjacent to b_n, unless B_n is $\underline{P}_2$ and the block D that contains b_n and b_{n-1} is a cycle and contains no other cutpoints. To examine the latter case let m and l denote the number of consecutive points in D between b_n and b_{n-1}. We may assume $m > l$, for if $m = l$ then B_{n-1}, and hence C, is semi-stable at the endpoint of B_n. Then B_{n-1}, hence C, is semi-stable at c defined as follows:

if $l = 0$, $m \neq 2$, the bare point adjacent to b_{n-1},
if $l = 0$, $m = 2$, the bare point adjacent to b_n,
if $l = 1$, the bare point adjacent to b_{n-1} and b_n,
if $l > 1$, the bare point adjacent to b_{n-1} and farthest from b_n.

The next lemma requires the following concepts. A cactus C rooted at b and containing at least one copy of a cycle U is called a *U-pole* if (i) all blocks in C are U or P_2, (ii) all points in the block-cutpoint tree (Harary (1969)) of C, $bc(C)$, have degree at most 3 and (iii) if u is a point in $bc(C)$ of degree 3 then u is a copy of U and of the three branches at u one contains b, one contains a copy of U and one is $\underline{P}_2$. C is called a *U-pillar* if, for all pairs of points in the block-cutpoint tree of C that are copies of U, one lies on the path between the other and the block containing b.

LEMMA 2. *A U-pillar is reducible to a U-pole.*

PROOF. Let C be the U-pillar and b_1 its root. By Lemma 1, we reduce to b_1, in increasing order of size, each branch at b_1 that does not contain a copy of U. Let D be the block that contains b_1. We define b_2 and B_2' as in the proof of Lemma 1. If B_2' does not contain U we reduce it to $\underline{P}_2$. If C is now semi-stable at the endpoint of B_2', we remove it. Otherwise we redefine b_2 as the next closest cutpoint to b_1 and redefine B_2' accordingly. Then we reduce B_2' to b_2 and repeat this procedure until all branches (except possibly for a single $\underline{P}_2$) that do not contain U are removed from D. Finally, let b_2 denote the root of the branch (if it exists) at D that contains U. Then b_2 is fixed in C and we repeat the above procedure on all branches at b_2 except the branch containing b_1. In this way we generate a sequence of cutpoints $b_2, b_3, \cdots, b_n$ where b_n lies in a copy of U.

If the block containing b_i and b_{i+1} is a cycle which is not U and has $\underline{P}_2$ as a branch, let b_i' denote the root of that $\underline{P}_2$. Then, for each i in turn for which b_i' is defined, we remove b_i' followed by the isolated point and reduce the resulting path branches at b_i and b_{i+1} to their roots. Similarly, if the cycle containing b_i and b_{i+1} is not U and has no branch $\underline{P}_2$, we remove a point adjacent to b_i and reduce the resulting path branch at b_{i+1} to its root. The resulting cactus is a U-pole.

LEMMA 3. *A U-pole is semi-stable at a non-cutpoint c in the copy of U farthest from the root.*

PROOF. Define c as in the proof of Lemma 1.

3. Semi-stable Cacti

Our aim in this section is to show that the graphs in Figure 1 are the only cacti with a cycle that are not semi-stable. We shall use the following notation. Let C denote a cactus with at least one cycle. For a cycle R in C, let $n(C, R)$

denote the number of copies of R in C. Let $m(C)$ be the minimum $n(C, R)$ over all cycles R in C. Finally, let U denote the smallest cycle in C for which $n(C, U) = m(C)$.

We first establish

THEOREM 1. *If $m(C) = 1$ then C is semi-stable unless it is one of the cacti in Figure 1. Moreover C is semi-stable at a point which is a penultimate point or non-cutpoint.*

PROOF. We assume $m(C) = 1$ and that C is not semi-stable. If there is only one branch B at U then its root b is fixed in C and hence, by Lemma 1, B is $\underline{P}_2$. But then C is semi-stable at b. Hence there are at least two branches at U.

We shall call a point in U of degree 2 a *bare point*. Let t denote the maximum number of consecutive bare points in U. Then $t > 0$; otherwise a smallest branch at U is reducible to its root, by Lemma 1. Let S denote the collection of branches at U whose roots are adjacent to a string of t consecutive bare points. Let B_1 denote a smallest branch in S and b_1 its root. Let B_2 denote a smallest branch in S whose root b_2 is adjacent to the same string of t consecutive bare points as is the root of a branch in S that is isomorphic to B_1. We may assume b_2 is adjacent to the same string of t consecutive bare points as b_1. Noting that the set of roots of the branches in S is a union of orbits of $\Gamma(C)$, Lemma 1 implies that B_1 is $\underline{P}_2$.

Let B_3 denote the branch at U whose root b_3 is the closest to b_1. Let $r = d(b_1, b_3) - 1$ (the number of bare points between b_1 and b_3). Note that $r < t$, for if $r = t$ then C is semi-stable at the endpoint e of B_1. It follows from the definition of t that the removal of e introduces the "reflection" automorphism g that maps b_2 into b_3. Hence B_2 and B_3 are isomorphic if not the same branch (see Figure 2).

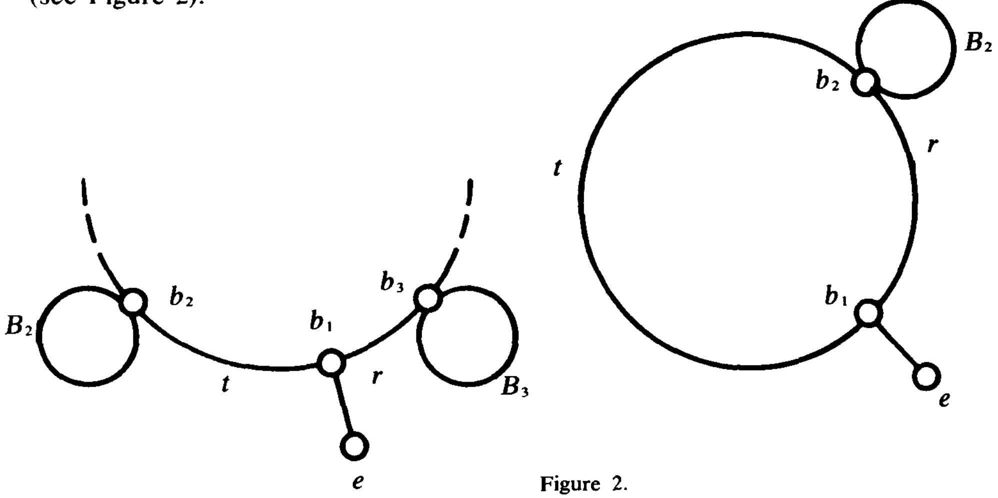


Figure 2.

It follows from Lemma 1 and the definition of B_2 that B_2 has at most 3 points, otherwise C is semi-stable at a point in B_2. Thus B_2 is $\underline{P}_2$, $\underline{P}_3$ or one of the branches in Figure 3.

Figure 3.

If $b_2 \neq b_3$ and B_2 is not $\underline{P}_2$, then the removal of one of its non-cutpoints introduces the "reflection" automorphism f that maps b_1 into b_2. Define inductively the points $b_4, b_5, \cdots$ on U as follows: $b_{2i} = f(b_{2i-1})$ and $b_{2i+1} = g(b_{2i})$ for $i = 1, 2, 3, \cdots$. Then the branch at U with root b_j is isomorphic to B_2 for $j \neq 1$, and (excluding b_1) all other points on U are bare. But then C is semi-stable at a non-cutpoint in B_3. Hence B_2 is $\underline{P}_2$ or $b_2 = b_3$. Thus, if $b_2 = b_3$, C falls into one of the three cases indicated in Figure 4 which we now examine in turn. (By the definition of U, B_2 is not isomorphic to the second branch in Figure 3.)

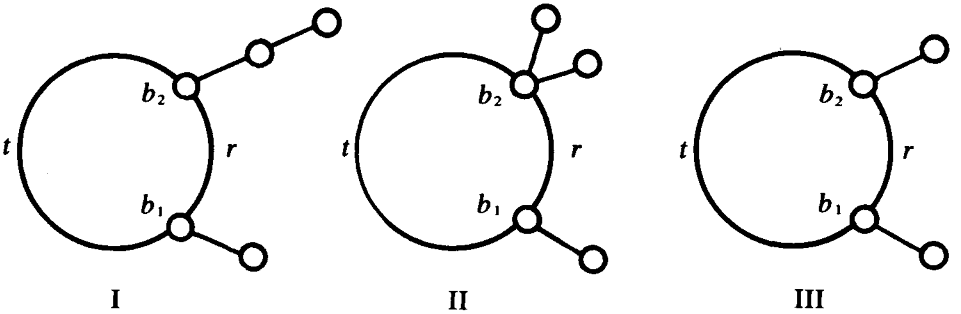


Figure 4.

Case I:

If $r = 0$ and $t = 1$, C is the first graph in Figure 1.
If $r = 0$ and $t = 2$, C is the third graph in Figure 1.
If $r = 0$ and $t = 3$, C is semi-stable at the bare point adjacent to b_2.
If $r = 0$ and $t = 4$, C is the fifth graph in Figure 1.
If $r = 0$ and $t > 4$, C is semi-stable at the bare point adjacent to b_2.
If $r = 1$ and $t = 2$, C is semi-stable at the bare point adjacent to b_1 but not b_2.

https://doi.org/10.1017/S1446788700016141 Published online by Cambridge University Press

If $r = 1$ and $t > 2$, C is semi-stable at b_1.
If $r = 2$, C is semi-stable at the bare point adjacent to b_1 that is closer to b_2.
If $r > 2$, C is semi-stable at b_1.

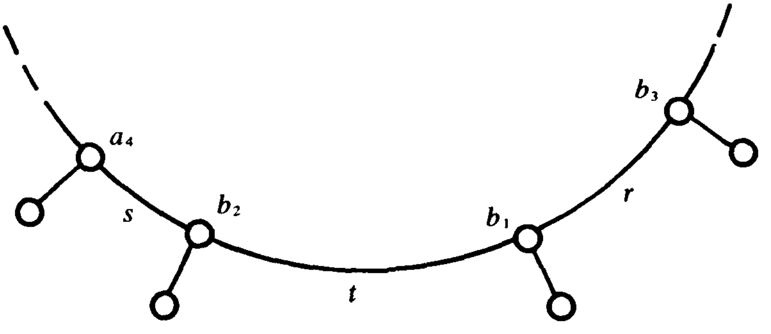
Case II:

If $r = 0$ and $t = 1$, C is semi-stable at the bare point.
If $r = 0$ and $t > 1$, C is semi-stable at b_1.
If $r = 1$, C is semi-stable at the bare point adjacent to b_1 and b_2.
If $r > 1$, C is semi-stable at b_1.

Case III:

If $r = 0$ and $t = 1$, C is semi-stable at the bare point.
If $r = 0$ and $t = 2$, C is the second graph in Figure 1.
If $r = 0$ and $t = 3$, C is the fourth graph in Figure 1.
If $r = 0$ and $t > 3$, C is semi-stable at the bare point adjacent to b_1.
If $r = 1$, C is semi-stable at the bare point adjacent to b_1 and b_2.
If $r > 1$, C is semi-stable at b_1.

If $b_2 \neq b_3$ and B_2 is $\underline{P}_2$ then the removal of its endpoint introduces the "reflection" automorphism h that maps b_1 into a_4, the point in U that is closest to b_2 and which is not a bare point. Thus we define inductively the points $a_4, a_5, a_6, \cdots$ in U as follows: $g(b_2) = b_3$, $h(b_1) = a_4$, $g(a_4) = a_5$, $h(b_3) = a_6$, etc. Then the branches at U with roots b_1, b_2, b_3, a_j ($j \geqq 4$) are $\underline{P}_2$ and all other points in U are bare (see Figure 5). If $s = d(b_2, a_4) - 1$ then $s < t$ and we may assume that $s \leqq r$. Thus:



IV

Figure 5.

https://doi.org/10.1017/S1446788700016141 Published online by Cambridge University Press

Case IV:

If $r = 0$ and $t = 1$, C is semi-stable at the bare point.
If $r = 0$ and $t > 1$, C is semi-stable at b_3.
If $r = 1$, C is semi-stable at the bare point adjacent to b_1 and b_3.
If $r > 1$, C is semi-stable at b_1.

This completes the proof of Theorem 1.

Our second main result is

THEOREM 2. *If* $m(C) > 1$ *then* C *is reducible to a cactus* C' *for which* $m(C') = m(C) - 1$.

PROOF. The rationale of this proof is to remove a sufficient number of branches of C in order to allow us to remove a point of some copy of U. The resulting cactus suffices for C'. A variety of cases present themselves according to the distribution in C of its copies of U.

Let $bc(C)$ denote the block-cutpoint tree of C. Let N be the set of points in $bc(C)$ which are either copies of U or points of degree 3 or more and at which there exists 3 or more distinct branches each containing a copy of U. Let $T(C, U)$ denote the tree whose points are the points in N and in which two points a, b are adjacent if and only if the path in $bc(C)$ joining a to b does *not* contain a point in N. Finally, let E be the centre of $T(C, U)$.

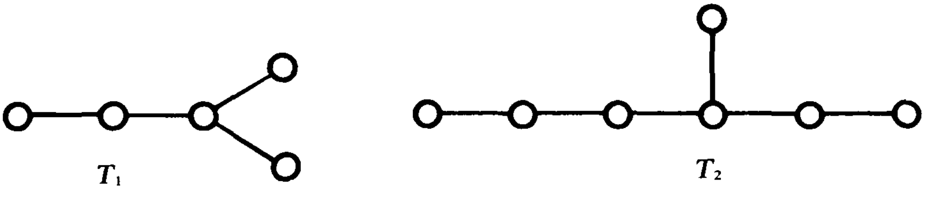


Figure 6.

We now assume for the moment that $T(C, U)$ is not a path $P_n (n \geqq 2)$ or one of the trees T_1 or T_2 in Figure 6. Heffernan (1972) has shown that all trees T except $P_n (n > 2)$, T_1 and T_2 are semi-stable at an endpoint u. His argument is constructive and ensures that u lies in a smallest branch at each point v on the path in T from u to the closer centre point in T. We tighten this condition for $T(C, U)$ by choosing from the smallest branches at v a branch that corresponds to a smallest branch at v in C. We note that u is a copy of U.

Let w be the point in $T(C, U)$ that is adjacent to u. If w is not in E, let a denote the point of C in w that is closest to E in C. If w is in E and $T(C, U)$ is

https://doi.org/10.1017/S1446788700016141 Published online by Cambridge University Press

bicentral, let a denote the point of C in w that is closest to $E\backslash\{w\}$ in C. In both cases let b denote the point of C in w that is closest to u in C. If w is a cycle in C, let b' denote the point of C in w for which $d(a,b) = d(a,b')$. Let $B(B')$ denote the maximal subcactus of C that contains $b(b')$ but not the rest of the branch at b that contains E. If B' contains one copy of U, then by the definition of u, we may assume B' is not smaller than B. Thus we may regard b as fixed in C and use Lemma 2 to reduce B to a U-pole at b. If B' contains no copy of U, we use Lemma 1 to reduce B' to its root. By Lemma 3, B is semi-stable at a non-cutpoint c in u. Hence, from the definition of u, C is semi-stable at c.

If w is in E and $T(C,U)$ is unicentral, let b denote the point of C in w that is closest to u. By Lemma 2, we may assume the branch B at w that contains u is a U-pole. (Also, if w is a *point* in C, we use Lemma 1 to reduce to b the branches at w, in increasing order of size, that do not contain a copy of U.) Let c be a point in u at which B is semi-stable. Then two cases present themselves.

Case I: w is not fixed in C_c. Then there are just three branches in $T(C,U)$ at w. Let v and v' be the other two points in $T(C,U)$ adjacent to w. Let $A(A')$ be the branch in C at w that contains v (v') and let $a(a')$ be its root. Let $e(e')$ denote the point in $v(v')$ that is closest to $a(a')$. Let w map into w' in C_c. We assume that w' is in A'. Thus $d(a',e') > d(a,e)$ and a' is therefore fixed in C. Then by the methods of Lemma 2, we reduce A' so that all points between a' and e' have degree 2 and no branches remain at e' that do not contain a copy of U. Note that, while reducing A' in this way, $d(a',e')$ is non-decreasing. Thus we may assume the w', and hence w, is a *point* in C. But then $w(=b)$ can not map into w' as B_c is a branch that does not contain a copy of U and is not smaller than $\underline{P}_2$ and no such branch exists at a point between a' and e'. Hence C is now semi-stable at c.

Case II: w is fixed in C_c. If w is a *point* in C we reduce, in increasing order of size, all the branches at w that do not contain U. Then C is semi-stable at c or Case I applies.

If w is a *cycle* in C, we use the methods of Theorem 1. Define a bare point as a point in w of degree 2 or a root of a branch at w that contains U. Then, as in the proof of Theorem 1, several cases present themselves. Note that there are at least three branches at w containing U.

Case II. 1: There is no branch at w not containing U. Then C is semi-stable at c.

Case II. 2: There is only one branch A at w not containing U and it is $\underline{P}_2$. If C is not semi-stable at c then B_c is $\underline{P}_2$ and B is Q, the second branch in Figure 3. If B is the only branch at w isomorphic to Q and C is not semi-stable at the endpoint of A, then a "reflection" argument similar to that in the proof of Theorem 1 guarantees at least three isomorphic branches at w containing U.

Let B' be one of these branches with its root closest to the root of A. Then, noting that the root of A is fixed in C, we use Lemma 1 to reduce B' to Q or until a point in a copy of U is removed. If there are at least two branches at w isomorphic to Q and C is not semi-stable at a point with degree 2 in any of them, then another "reflection" argument shows that their roots together with the root of A are distributed uniformly in w a constant distance apart. Hence we may remove the endpoint of A and then Case II.1 applies.

Case II.3: There are only two branches A and A' at w not containing U; A is $\underline{P}_2$ and A' is $\underline{P}_2$, $\underline{P}_3$ or one of the branches in Figure 3. If C is not semi-stable at c then B_c maps into A or A'. If the former, we may remove the endpoint of A and then Case II.2 applies. If the latter, then because B is a U-pole, A' is either $\underline{P}_2$ or $\underline{P}_3$. Hence we may remove respectively the endpoint or point of degree 2 in A' and then Case II.2 applies.

Case II.4: There are three or more branches at w not containing U. These are distributed about w as in Case IV of Theorem 1. If C is not semi-stable at c then B is C_3 (rooted at one point) and b is one of the t consecutive bare points at distance $r+1$ from b_2 or distance $s+1$ from b_1. But then C is semi-stable at the endpoint of B_1 or B_2 respectively, giving a contradiction.

To complete the proof of Theorem 2 we now investigate the cases where $T(C, U)$ is $P_n (n \geqq 2)$, T_1 or T_2 (Figure 6).

If $T(C, U)$ is P_2, let u and u' be the copies of U. Let $b'(b)$ be the point of C in $u(u')$ closest to $u'(u)$. Let $B(B')$ be the maximal subcactus of C that contains $b(b')$ but no other point in the branch $A(A')$ at $b(b')$ that contains $u'(u)$. We assume B is not larger than B' and use Lemma 2 to reduce B to a U-pole. Then, by the methods of Lemma 2, A is reducible to u' with at most two other points, that is, at most one more point than A'. Let c be a point in u at which B is semi-stable. If C is not semi-stable at c then B_c maps into a branch at u'. Then B_c is $\underline{P}_2$, $\underline{P}_3$ or one of the branches in Figure 3. The cacti that satisfy these constraints consist of just two copies of either C_3 or C_4 together with at most three other points. It can be shown exhaustively that all such cacti are semi-stable at some point in u or u'.

If $T(C, U)$ is P_n with $n > 2$ let u and u' be the endpoints of $T(C, U)$. Let $v(v')$ be the point in $T(C, U)$ adjacent to $u(u')$. Let $b(b')$ be the point of C in $v(v')$ closest to $u'(u)$. Let $B(B')$ be the maximal subcactus of C containing $b(b')$ but no other point in the branch at $b(b')$ that contains $u'(u)$. We assume B is not greater than B' and use Lemma 2 to reduce B to a U-pole. Let $a(a')$ be the point of C in $u'(u)$ closest to $u(u')$ and let $A(A')$ be the maximal subcactus of C that contains $a(a')$ but no other point of the branch at $a(a')$ that contains $u(u')$. Then, by the methods of Lemma 2, A is reducible to u' with at most two other points, that is, at most one more point than A'. Let c be a point in u at which B is semi-stable. If C is not semi-stable at c then B_c maps

https://doi.org/10.1017/S1446788700016141 Published online by Cambridge University Press

into A. Hence B_c consists of v with a branch that is $\underline{P}_2$, $\underline{P}_3$ or one of the branches in Figure 3. Hence U is C_3 or C_4, and it can be shown exhaustively that all cacti satisfying these constraints are semi-stable at some point in u, u' or v.

If $T(C, U)$ is T_1 (Figure 6) let w be the point in $T(C, U)$ of degree 3 and v the point of degree 2. Let a denote the point of C in w closest to v. Let A denote the branch of C at w that contains v. Since a is fixed in C, we may use Lemma 2 to reduce A to a U-pole. Let B and B' be the branches at w, with roots b and b' respectively, that contain one copy of U. Assuming B is not larger than B' we reduce B to a U-pole. Let e be the point of C in w for which $d(a, b) = d(a, e)$. In increasing order of size, the branches at e that do not contain U are reduced to e using Lemma 1. Let c be a point in u, the copy of U in B, at which B is semi-stable. Suppose C is not semi-stable at c. If w is a *point* in C, then $w = a = b$ and maps, in C_c, into a point w' between v and the end copy of U in A. But B_c is not smaller than $\underline{P}_2$ and no branch exists at w' that does not contain U. Hence w is a cycle, and w maps into v in C_c. Then there is a branch $\underline{P}_2$ at v, B_c is $\underline{P}_2$ and therefore U is C_3. Hence, in reducing A to a U-pole, we can reduce $\underline{P}_2$ at v to its root. This contradiction ensures C is semi-stable at c.

If $T(C, U)$ is T_2 (Figure 6) let w be the point in $T(C, U)$ of degree 3. Let $A(B)$ be the branch at w, with root $a(b)$, that contains one (two) copy(ies) of U. Using Lemma 2 we may reduce A and B to U-poles. Let c be a point in the end copy of U in B at which B is semi-stable. If C is not semi-stable at c then B_c maps into A, there is a branch $\underline{P}_2$ at the end copy of U in A and U in C_3. Then C is semi-stable at the point of degree 2 in the penultimate copy of U in B.

This concludes the proof of Theorem 2.

As a corollary to Theorem 2 we have

THEOREM 3. *All cacti C with at least one cycle are semi-stable except those in Figure 1. Also, C is semi-stable at a point which is a penultimate point or non-cutpoint.*

PROOF. If $m(C) = 1$ then the result follows immediately from Theorem 1. In the case where $m(C) > 1$ the proof of Theorem 2 tacitly secures the required point.

4. Stable Cacti

Our aim now is to show that a cactus with a transposition automorphism is stable. We first characterise these cacti. To do this, we define a certain class of subcacti. A cactus C is said to contain a *transfig* at a point b if there is (i) a branch at b isomorphic to C_3 (rooted at one point), or (ii) a branch in which the block D containing b is C_4 and in which the two points in D adjacent to b have degree 2, or (iii) two or more branches at b isomorphic to $\underline{P}_2$, or (iv) any combination of (i), (ii) and (iii).

We note that, in this section, the cacti include trees.

THEOREM 4. *A cactus C with at least three points has a transposition automorphism if and only if it contains a transfig.*

PROOF. If C contains a transfig and a and a' denote its points of degree 2 (in types (i) and (ii)) or its endpoints (in type (iii)), then clearly the transposition (aa') is an automorphism of C.

Conversely, suppose $g = (aa')$ is a transposition automorphism of C. Then, because C is connected and it has at least three points, there is another point u of C adjacent to a or a'. Suppose $u \sim a$. Then $g(u) \sim g(a)$, that is $u \sim a'$. If $a \sim a'$, then u is unique, otherwise the line (a, a') lies on more than one cycle in C. If $a \not\sim a'$, then there is at most one other point u' of C for which $u' \sim a$ and $u' \sim a'$, otherwise the line (u, a) lies on more than one cycle in C. Either way u is a root of a transfig.

Using this characterisation we can now prove

THEOREM 5. *A cactus containing just one transfig is stable.*

PROOF. Let C denote such a cactus and b the root of the transfig. Let A denote the maximal subcactus of C that contains b but not the rest of the transfig. If A_b is empty and C is C_3 or C_4 then clearly C is stable. If A_b is empty and C is neither C_3 nor C_4 then b is fixed and, by Lemma 1, C is reducible to one of the cacti in Figure 7. These cacti are stable; we delete the points in the indicated order.

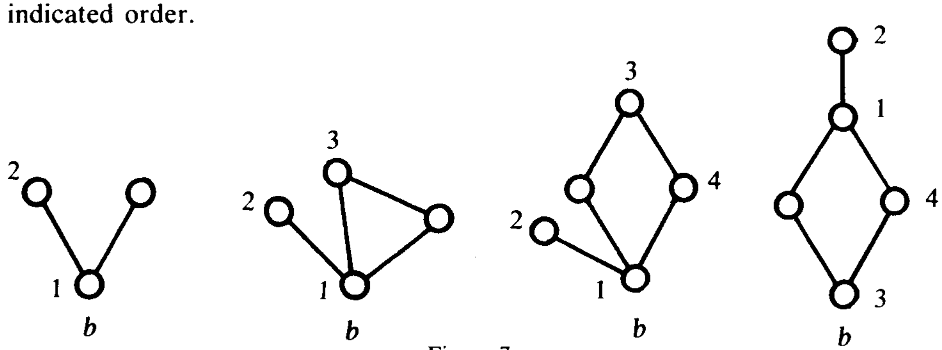
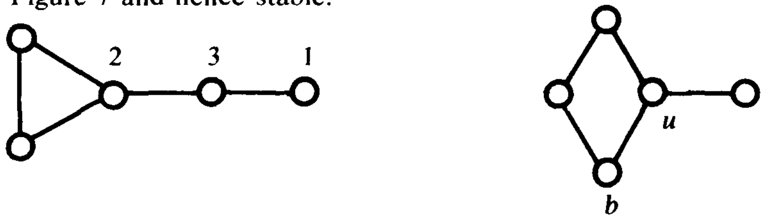


Figure 7.

Suppose now that A_b is not empty. Since A contains no transfig it is reducible, by Lemma 1, to $\underline{P}_3$ or the rooted cactus A' (Figure 8). In the latter case we then remove the cutpoint u followed by the isolated point, thus leaving $\underline{P}_3$. We assume in the case where the transfig is of type (ii), that A is not larger than the other branch at C_4.

https://doi.org/10.1017/S1446788700016141 Published online by Cambridge University Press

If the resulting cactus is B in Figure 8, we continue the stabilising sequence as indicated. Otherwise we remove the point of degree 2 in A followed by the isolated point. Then, as before, C is reducible to one of the cacti in Figure 7 and hence stable.



B A′

Figure 8.

Combining Theorem 5 and Theorem 3 we have

THEOREM 6. *A cactus C is stable if and only if it has a transposition automorphism.*

PROOF. We may assume from Theorem 4 and Theorem 5 that C has at least two transfigs. Then, by Theorem 3 and the analogous result (Heffernan (1972)) for trees, C is semi-stable at a non-cutpoint or penultimate point v. If C_v contains just one transfig (after removing any isolated point), the result follows from Theorem 5. Otherwise we continue reducing C until only one transfig remains.

References

F. Harary (1969), *Graph Theory* (Addison-Wesley, 1969).

P. Heffernan (1972), *Trees* (Masters Dissertation, University of Canterbury, New Zealand).

D. A. Holton (1973), 'Two applications of semi-stable graphs', *Discrete Math.* **4**, 151–158.

D. A. Holton (1973a), 'A report on stable graphs', *J. Austral. Math. Soc.* **15**, 163–171.

D. A. Holton (1973b), 'Stable trees', *J. Austral. Math. Soc.* **15**, 476–481.

D. A. Holton and D. D. Grant (1975), 'Regular graphs and stability', *J. Austral. Math. Soc.* **20**, 377–384.

D. A. Holton and D. D. Grant, (to appear), 'Products of graphs and stability', *J. Combinatorial Theory.*

K. L. McAvaney, D. D. Grant and D. A. Holton (1974), 'Stable and semi-stable unicyclic graphs', *Discrete Math.* **9**, 277–288.

N. Robertson and J. A. Zimmer (1972), 'Automorphisms of subgraphs obtained by deleting a pendant vertex', *J. Combinatorial Theory Ser. B.* **12**, 169–173.

Mathematics Department
Gordon Institute of Technology
Geelong, Vic. 3220
Australia.

https://doi.org/10.1017/S1446788700016141 Published online by Cambridge University Press