The Shoemaker's Knife.

By J. S. MACKAY, M A., F.R.S.E.

§ 1. Some of the properties of the figure which, on account of its shape, the Greeks named the Shoemaker's Knife (ἄρβηλος) are given in the *Lemmas* attributed to Archimedes; others occur in the fourth book of Pappus's *Mathematical Collection*. The *Lemmas* (which are not extant in Greek, but have been translated from the Arabic) are generally considered to be spurious; it is, however, regarded as possible, if not probable, that the theorems among them relating to the Arbelos may be due to Archimedes. Whether they are or not, the figure and the principal proposition respecting it which Pappus gives are said by him to be "ancient." It may be added that the Arbelos does not seem to have attracted much notice from geometers, few of them having treated of it, and fewer still having added to the properties known to the ancients. (See Steiner's *Gesammelte Werke*, Vol. I., pp. 47-76, and *The Lady's and Gentleman's Diary* for 1842 and 1845).

The object of the present paper is to collect together the principal and simplest properties of the figure, and to demonstrate them in a uniform manner.

§ 2. *Figure* 1. In the arbelos AGBJCM, that is, the curvilineal figure contained by the three semicircumferences AGB, BJC, CMA, the two semicircumferences BJC, CMA are together equal to the semicircumference AGB.

For $AB = AC + BC$;

and the circumferences of circles are proportional to their diameters; therefore the semicircumference AGB = semicircumference AMC + semicircumference BJC.

§ 3. *Figure* 1. The arbelos AGBJCM is equal to the circle whose diameter is CG, the common tangent at C to BJC, CMA.

For
$$AB^2 = AC^2 + BC^2 + 2\ AC \cdot BC,$$
$$= AC^2 + BC^2 + 2\ CG^2.$$

Now circles are proportional to the squares on their diameters, therefore the semicircle on AB = semicircle on AC + semicircle on BC + circle on CG;

therefore the arbelos AGBJCM = circle on CG.

Archimedes, *Lemma* 4.

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§ 4. *Figure* 1. The two circles inscribed in the arbelos and touching CG are equal.

Let HJK, LMN be the two circles. Draw the diameters HP, LQ parallel to AB.

Because N is the external centre of similitude of the circles AGB, LMN, and because AB, QL are parallel diameters;
therefore the points A, Q, N, and B, L, N are collinear.
Because M is the internal centre of similitude of the circles AMC, LMN, and because AC, LQ are parallel diameters;
therefore the points A, M, L, and C, M, Q are collinear.

Let AN meet CG produced at Y, and let AL meet the semicircumference AGB at R. Join BR.

Because YC is perpendicular to AB,
and BN is perpendicular to AY;
therefore L is the orthocentre of triangle YAB;
therefore AL produced will be perpendicular to BY.
But AR is perpendicular to BR;
therefore BR and RY form one straight line.
Now since angles AMC, ARB are right,
therefore CQ is parallel to BY;

therefore $AB : BC = AY : YQ,$
$= AC : QL;$

therefore $$LQ = \frac{AC \cdot BC}{AB}.$$

Similarly $$HP = \frac{BC \,.\, AC}{AB};$$

therefore $HP = LQ.$

Archimedes, *Lemma* 5.

Cor. *Figures* 2, 3. If the circles AMC_1, BJC_2 intersect, or have no point in common, the circles HJK, LMN are equal, provided CG be the radical axis of AMC_1, BJC_2.

For it may be proved as before that
$$AB : BC_1 = AY : YQ,$$
$$= AC : QL;$$

therefore $$LQ = \frac{AC \cdot BC_1}{AB}.$$

Similarly $$HP = \frac{BC \cdot AC_2}{AB}.$$

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Now $AC \cdot BC_1 = AC \cdot BC \mp AC \cdot CC_1,$
and $BC \cdot AC_2 = BC \cdot AC \mp BC \cdot CC_2.$
But $AC \cdot CC_1 = BC \cdot CC_2$, since C, being a point on the radical axis of AMC_1 and BJC_2, has equal potencies with respect to these two circles;
therefore $HP = LQ$.

This extension of the theorem of § 4, due to an Arabian mathematician, Alkauhi, is given in Borelli's *Apollonii Pergaei Conicorum Lib. V. VI. VII. et Archimedis Assumptorum Liber* (Florentiæ, 1661) pp. 393-5.

§ 5. *Figure* 1. The common tangent to the two circles AMC, LMN at M passes through B, and the common tangent to the two circles BJC, HJK at J passes through A.

For the angles ACL, ANL are right;
therefore the points A, C, L, N are concyclic;
therefore $BA \cdot BC = BN \cdot BL$,
that is, B has equal potencies with respect to the two circles AMC, LMN;
therefore B is on the radical axis of the two circles.
Now M is also on the radical axis;
therefore BM is the radical axis, or common tangent at M.

Similarly for AJ.

Cor. 1. $BM = BG$, and $AJ = AG$.
For $BM^2 = BA \cdot BC = BG^2$; and $AJ^2 = AB \cdot AC = AG^2$.
Cor. 2. BM bisects CL at V, and AJ bisects CH at U.
For the radical axis of two circles bisects their common tangents.
Cor. 3. Hence is derived a method of finding O_2 and O_1, the centres of the circles LMN, HJK.
From B draw BM tangent to the circle AMC, and cutting CG in V; make $VL = CV$; and through L draw LQ parallel to AB. If F be the centre of the circle AMC, FM produced will meet LQ in O_2,

Similarly for O_1.

Cor. 4. *Figure* 4. If AG cuts AMC at T, and BG cuts BJC at W, TW is a common tangent to AMC, BJC.

Join CT, CW, and let CG, TW intersect at I.

Since angles ATC, AGB, CWB are right,
therefore CTGW is a rectangle;
therefore IT and IW are each equal to IC;
therefore IT and IW are tangents to AMC, BJC.

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§ 6. *Figure* 5. The first corollary of § 5 is a particular case of the following theorem :

Let AGB be a semicircle, and CG be perpendicular to AB. If a variable circle HJK be described to touch CG and the arc BG, and from A a tangent AJ be drawn to it, the length of AJ is constant.

Let D and O be the centres of AGB and HJK ; DO will pass through K. Join OA, OH, OJ, and draw OX perpendicular to AB.

Then $AO^2 = AD^2 + DO^2 + 2\ AD \cdot DX,$

$$= AD^2 + (AD - CX)^2 + 2\ AD\ (DC + CX),$$

$$= AD^2 + AD^2 - 2\ AD \cdot CX + CX^2 + 2\ AD \cdot DC + 2AD \cdot CX,$$

$$= 2\ AD^2 + 2\ AD \cdot DC + CX^2,$$

$$= 2\ AD \cdot AC + CX^2;$$

therefore $AO^2 - CX^2 = 2\ AD \cdot AC;$

therefore $AJ^2 = AB \cdot AC.$

The theorem is still true if the variable circle touch CG produced and the arc AG externally. It is also true when CG the perpendicular to AB touches the semicircle, or falls entirely outside it (*Figure* 6), the contact in these cases being necessarily external.

Leybourn's *Mathematical Repository* (New Series), Vol VI., Part I., pp. 209-211.

§ 7. *Figure* 1. The theorem of § 4 may also be proved thus :

Let D, E, F be the centres of the semicircles AGB, BJC, CMA, and let O_1, O_2 be the centres of the circles HJK, LMN.

From O_1, O_2 draw O_1X_1, O_2X_2 perpendicular to AB ; and join DO_1, EO_1, DO_2, FO_2.

Then DO_1 passes through K, EO_1 through J, DO_2 through N, and FO_2 through M.

Also CX_1 = the radius of HJK, CX_2 = the radius of LMN ;

$$AD = \tfrac{1}{2}AB = \tfrac{1}{2}AC + \tfrac{1}{2}BC = FE;$$

$$FD = AD - AF = FE - FC = CE;$$

$$AF = FD + DC = CE + DC = DE.$$

Now $FO_2^2 - DO_2^2 = FX_2^2 - DX_2^2.$

But $FO_2^2 - DO_2^2 = (FM + MO_2)^2 - (DN - NO_2)^2,$

$$= (FC + CX_2)^2 - (AD - CX_2)^2,$$

$$= FC^2 - AD^2 + 2\ AE \cdot CX_2;$$

and $FX_2^2 - DX_2^2 = (FC - CX_2)^2 - (DC - CX_2)^2,$

$$= FC^2 - DC^2 - 2\ FD \cdot CX_2;$$

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therefore $FC^2 - DC^2 - 2FD \cdot CX_2 = FC^2 - AD^2 + 2\ AE \cdot CX_2$;

therefore $AD^2 - DC^2 = 2\ (AE + FD) \cdot CX_2,$

$= 2\ AB \cdot CX_2.$

Again $DO_1^2 - EO_1^2 = DX_1^2 - EX_1^2.$

But $DO_1^2 - EO_1^2 = (DK - KO_1)^2 - (EJ + JO_1)^2,$

$= (AD - CX_1)^2 - (CE + CX_1)^2,$

$= AD^2 - CE^2 - 2\ FB \cdot CX_1$;

and $DX_1^2 - EX_1^2 = (DC + CX_1)^2 - (CE - CX_1)^2,$

$= DC^2 - CE^2 + 2\ AF \cdot CX_1$;

therefore $AD^2 - CE^2 - 2\ FB \cdot CX_1 = DC^2 - CE^2 + 2\ AF \cdot CX_1$;

therefore $AD^2 - DC^2 = 2\ (AF + FB) \cdot CX_1,$

$= 2\ AB \cdot CX_1.$

Hence $2\ AB \cdot CX_1 = 2\ AB \cdot CX_2$, and $CX_1 = CX_2$.

The Gentleman's Diary for 1833 p. 40.

§ 8. *Figure* 1. E and X_2 are inverse points with respect to circle AMC; F and X_1 are inverse points with respect to circle BJC.

For $FE = \frac{AB}{2}$;

$$FX_2 = FC - CX_2,$$

$$= \frac{AC}{2} - \frac{AC \cdot BC}{2AB},$$

$$= \frac{AC \cdot AB - AC \cdot BC}{2\ AB},$$

$$= \frac{AC^2}{2AB}.$$

Hence $FE \cdot FX_2 = \frac{AB}{2} \cdot \frac{AC^2}{2\ AB},$

$$= \frac{AC^2}{4} = FC^2.$$

Similarly for F and X_1.

Cor. 1. $EO_1 + EX_1 = BC$, and $FO_2 + FX_2 = AC$.

For $EO_1 + EX_1 = EX_2 + EX_1 = 2\ EC = BC$;

$FO_2 + FX_2 = FX_1 + FX_2 = 2\ FC = AC$.

Cor. 2. $DO_1 + DX_1 = AC$, and $DO_2 - DX_2 = BC$.

For $DO_1 = DK - O_1K$, and $DX_1 = DC + CX_1$;

therefore $DO_1 + DX_1 = DK + DC = DA + DC = AC$.

And $DO_2 = DN - O_2N$, and $DX_2 = DC - CX_2$;

therefore $DO_2 - DX_2 = DN - DC = DB - DC = BC$.

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§ 9. *Figure* 1. Relations between FO_2 and EO_1.

(*a*) Their values.

$$FO_2 = FM + O_2M,$$
$$= \frac{AC}{2} + \frac{AC \cdot BC}{2AB} = \frac{AC \cdot AB + AC \cdot BC}{2AB} = \frac{AC\,(AB + BC)}{2AB}.$$

Similarly $EO_1 = \dfrac{BC\,(AB + AC)}{2\,AB}.$

(*b*) Their sum.

$$FO_2 + EO_1 = \tfrac{1}{2}\,(AC + LQ) + \tfrac{1}{2}\,(BC + HP),$$
$$= \tfrac{1}{2}\,AB + HP.$$

(*c*) Their difference.

$$FO_2 - EO_1 = \tfrac{1}{2}\,(AC + LQ) - \tfrac{1}{2}\,(BC + HP),$$
$$= \tfrac{1}{2}\,(AC - BC) = CD.$$

(*d*) Their rectangle.

$$FO_2 \cdot EO_1 = FX_1 \cdot EX_2,$$
$$= (FC + CX_1)\,(EC + CX_2),$$
$$= FC \cdot EC + FC \cdot CX_2 + EC \cdot CX_1 \quad + CX_1 \cdot CX_2,$$
$$= \frac{AC \cdot BC}{4} + \qquad FE \cdot CX_1 \qquad + CX_1^2,$$
$$= \frac{AC \cdot BC}{4} + \frac{AC + BC}{2} \cdot \frac{AC \cdot BC}{2\,AB} + \frac{HP^2}{4},$$
$$= \frac{AC \cdot BC}{2} + \frac{HP^2}{4} = \tfrac{1}{4}\,(2CG^2 + HP^2).$$

(*e*) Their ratio.

$$FO_2 : EO_1 = \frac{AC\,(AB + BC)}{2\,AB} : \frac{BC\,(AB + AC)}{2\,AB},$$
$$= AC\,(AB + BC) \;:\; BC\,(AB + AC),$$
$$= 2\,AF \cdot 2\,FB \;:\; 2\,EB \cdot 2\,AE,$$
$$= AF \cdot FB \;:\; AE \cdot EB.$$

§ 10. *Figure* 1. Relations between CL and CH.

(*a*) Their values.

The right-angled triangles BCV, BMF are similar;

therefore $BC : 2\,CV = BM : 2\,MF;$

therefore $BC : CL = BG : AC;$

therefore $CL = \dfrac{AC \cdot BC}{BG} = \dfrac{CG^2}{BG}.$

Similarly $CH = \dfrac{AC \cdot BC}{AG} = \dfrac{CG^2}{AG}.$

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(*b*) Their sum and difference.

$$CL \pm CH = CG^2\left(\frac{1}{BG} \pm \frac{1}{AG}\right) = CG^2\left(\frac{AG \pm BG}{AG \cdot BG}\right),$$

$$= CG^2\left(\frac{AG \pm BG}{AB \cdot CG}\right) = \frac{CG}{AB}(AG \pm BG).$$

(*c*) The sum of their squares.

From the theorem, If a straight line be a common tangent to two circles which touch each other externally, that part of the tangent between the points of contact is a mean proportional between the diameters of the circles,

there results $CL^2 = AC \cdot LQ$ and $CH^2 = BC \cdot HP$;

therefore $CL^2 + CH^2 = (AC + BC)\ HP,$

$= AB \cdot HP = AC \cdot BC = CG^2.$

Cor. $AB : BC = CL^2 : LQ^2$, and $AB : AC = CH^2 : HP^2$.

For $AB : BC = AC : LQ,$

$= AC \cdot LQ : LQ^2,$

$= CL^2 : LQ^2,$

Similarly $AB : AC = CH^2 : HP^2$

Pappus, Book IV. Prop. 17.

(*d*) Their rectangle.

Since $CL^2 = AC \cdot LQ$, and $CH^2 = BC \cdot HP$;

therefore $CL^2 \cdot CH^2 = AC \cdot BC \cdot HP^2 = CG^2 \cdot HP^2$;

therefore $CL \cdot CH = CG \cdot HP$.

(*e*) Their ratio.

Since $CL \cdot BG = CG^2 = CH \cdot AG$;

therefore $CL : CH = AG : BG$.

§ 11. *Figure* 1. The arbelos is equal to the least circle which can be circumscribed to touch the circles HJK, LMN.

The diameter of the least circle which can be circumscribed to touch HJK, LMN will pass through O_1 and O_2, and will be equal to $O_1O_2 + HP$.

Now $O_1O_2^2 = PL^2,$

$= HL^2 + HP^2,$

$= (CL - CH)^2 + HP^2,$

$= \frac{CG^2}{AB^2}(AG - BG)^2 + \frac{CG^4}{AB^2},$

$= \frac{CG^2}{AB^2}(AG^2 + BG^2 - 2\ AG \cdot BG + CG^2),$

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$$= \frac{CG^2}{AB^2}(AB^2 - 2\,AB\cdot CG + CG^2),$$

$$= \frac{CG^2}{AB^2}(AB - CG)^2;$$

therefore $$O_1O_2 = \frac{CG}{AB}(AB - CG),$$

$$= CG - \frac{CG^2}{AB},$$

$$= CG - HP;$$

therefore $O_1O_2 + HP = CG.$

Leybourn's *Mathematical Repository* (New Series), Vol. VI., Part 1., pp. 155, 214.

§ 12. *Figure* 7. If from O_1, O_2, the centres of two circles which touch the semicircles AKB, AMC, and each other, there be drawn O_1X_1, O_2X_2 perpendicular to AB, and if r_1, r_2 denote the radii of circles O_1, O_2, then

$$O_1X_1 + 2\,r_1 : 2\,r_1 = O_2X_2 : 2\,r_2.$$

Of the six centres of similitude of any three circles, every two internal centres are collinear with one external centre, and the three external centres are collinear. Hence, since J is the internal centre of similitude of the circles O_1 and AMC, and M is the internal centre of similitude of the circles O_2 and AMC, JM produced passes through E, the external centre of similitude of the circles O_1 and O_2. Since K is the external centre of similitude of the circles O_1 and AKB, and N the external centre of similitude of the circles O_2 and AKB, therefore KN produced passes also through E. Now with reference to E the external centre of similitude of circles O_1 and O_2, the points J and M are anti-homologous points, as also are K and N, and two anti-homologous points coincide at H;

therefore $EJ\cdot EM = EH^2 = EK\cdot EN.$

Hence E has equal potencies with respect to the circles AMC, AKB;

therefore E is a point on the radical axis of AMC, AKB;

therefore EA is the radical axis of AMC, AKB.

Therefore $EA^2 = EJ\cdot EM = EH^2$;

therefore $EA = EH.$

Now since $EA = EH$, and O_2Q is parallel to EA and $= O_2H$, therefore A, Q, H are collinear.

And since O_2Q and O_1G are parallel and opposite in direction, therefore Q, H, G are collinear.

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Let AO_2 meet O_1G at F.

Then $FG : O_2Q = AG : AQ,$
$= AX_1 : AX_2,$
$= O_1E : O_2E,$
$= r_1 : r_2.$

But $O_2Q = r_2$; therefore $FG = r_1$, and $FO_1 = 2\ r_1$.

Lastly $FX_1 : O_2X_2 = AX_1 : AX_2,$
$= r_1 : r_2;$

therefore $O_1X_1 + 2\ r_1 : 2\ r_1 = O_2X_2 : 2\ r_2.$

Cor. The figure given for the theorem is susceptible of various modifications; for example, AKB may touch AMC externally at A, and then the circle O_1 may touch both semicircles externally or both internally. Whether AKB touch AMC internally or externally, if its centre moves off to infinity in either direction along AB, instead of the two semicircles AKB, AMC, there will be the straight line AE and the semicircle AMC, and the theorem will still be true.

Pappus, Book IV., Prop. 15.

§ 13. *Figures* 8, 9. Let the semicircles AGB, AMC touch each other internally at A, and let a series of circles O_1, O_2, O_3, &c., whose radii are denoted by r_1, r_2, r_3, &c., touch AGB, AMC and each other consecutively; if from the centres O_1, O_2, O_3, &c., perpendiculars O_1X_1, O_2X_2, O_3X_3, &c., be drawn to AB, then

(*a*) When the centre of the first circle O_1 lies in BC

$$\frac{O_1X_1}{r_1},\ \frac{O_2X_2}{r_2},\ \frac{O_3X_3}{r_3},\ \frac{O_4X_4}{r_4},\ \ldots\ldots\ \frac{O_nX_n}{r_n}$$
$$= 0\ ,\ 2\ ,\ 4\ ,\ 6\ ,\ \ldots\ldots\ 2\,(n-1).$$

(*b*) When the first circle touches BC

$$\frac{O_1X_1}{r_1},\ \frac{O_2X_2}{r_2},\ \frac{O_3X_3}{r_3},\ \frac{O_4X_4}{r_4},\ \ldots\ldots\ \frac{O_nX_n}{r_n}$$
$$= 1\ ,\ 3\ ,\ 5\ ,\ 7\ ,\ \ldots\ldots\ 2\,n-1.$$

In other words, the quotients obtained in the manner above described from the Shoemaker's Knife are the even numbers, those obtained from the Shoemaker's Pastehorn (as the figure AGBCM may be called) are the odd numbers.

For $$\frac{O_1X_1 + 2\ r_1}{r_1} = \frac{O_2X_2}{r_2},$$

therefore $$\frac{O_1X_1}{r_1} + 2 = \frac{O_2X_2}{r_2}.$$

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Now when the centre O_1 lies in BC,

$$\frac{O_1X_1}{r_1}=0 \text{ ; therefore } \frac{O_2X_2}{r_2}=2.$$

Again $\frac{O_2X_2}{r_2}+2=\frac{O_3X_3}{r_3}$; therefore $\frac{O_3X_3}{r_3}=4$; and so on.

When the circle O_1 touches BC,

$$\frac{O_1X_1}{r_1}=1 \text{ ; therefore } \frac{O_2X_2}{r_2}=3, \frac{O_3X_3}{r_3}=5 \text{ ; and so on.}$$

Cor. It will be seen that in figures 8 and 9 there are three circles AGB, AMC, and O_1 in mutual contact, and that of the series of circles O_2, O_3, O_4..., the first touches O_1, the second O_2, and so on, while all touch AGB, AMC. If, out of the three circles of mutual contact, instead of choosing AGB, AMC to be touched by all the series O_2, O_3, O_4..., we choose AGB and O_1, we shall have O_2, as before, touching AMC, and a second series O_3, O_4..., consecutively inscribed in the curvilineal space bounded by the circumferences O_1, O_2, and AGB. If we choose AMC and O_1 to be touched by all the series O_2, O_3, O_4..., we shall have O_2, as before, touching AGB, and a third series, O_3, O_4..., consecutively inscribed in the curvilineal space bounded by the circumferences O_1, O_2, and AMC. With respect to these two series of circles the property enunciated in § 13 holds good. It also holds good with respect to the three series of circles that may be inscribed when the semicircles AGB, AMC are replaced by straight lines perpendicular to BC at the points B and C; these straight lines being the limits towards which the two semicircles tend when their centres move off to infinity in the direction BA.

Pappus, Book IV., Props. 16, 18.

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