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Dr THIRD, Vice-President, in the Chair.

Note on the Theorems of Menelaus and Ceva.

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Menelaus's theorem is

If a transversal DEF cut the sides of triangle ABC at the points D E F, then

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = -1,$$

and conversely.

It will be observed that in naming the sides of ABC, the letters A B C are successively omitted, so that the sides in order are

BC CA AB

The points in which the transversal intersects these sides are in order

D E F

Hence in forming the ratios whose product is -1 the first letters of the sides are written in the numerator, the second letters of the sides are written in the denominator;

thus
$$\frac{B\quad}{\quad C} \cdot \frac{C\quad}{\quad A} \cdot \frac{A\quad}{\quad B}$$

The letters D E F

in order are then inserted in both numerator and denominator;

thus
$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB}$$

On this theorem Carnot makes the suggestive remark * that of

* *Essai sur la théorie des transversales*, p. 68 (1806).

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the four lines BC CA AB DEF, any three may be regarded as forming the sides of a triangle, and the fourth may be taken as the transversal; but that the four results obtained are not independent of each other, for any one of them may be deduced from the other three.

In verifying Carnot's remark those beginning the study of Transversals usually find some difficulty in obtaining the ratios which are to be multiplied together. The object of this note is to show how they may be written down easily and without the trouble of consulting a diagram.

Observe that to

the points A B C A B ... correspond

the points D E F D E ...

Hence if ABC be the triangle DEF is the transversal

,,	AEF	,,	,,	,,	DBC	,,	,,	,,
,,	BFD	,,	,,	,,	ECA	,,	,,	,,
,,	CDE	,,	,,	,,	FAB	,,	,,	,,

The results obtained are therefore

$$\frac{B\quad}{\quad C}\cdot\frac{C\quad}{\quad A}\cdot\frac{A\quad}{\quad B}=-1$$

$$\frac{E\quad}{\quad F}\cdot\frac{F\quad}{\quad A}\cdot\frac{A\quad}{\quad E}=-1$$

$$\frac{F\quad}{\quad D}\cdot\frac{D\quad}{\quad B}\cdot\frac{B\quad}{\quad F}=-1$$

$$\frac{D\quad}{\quad E}\cdot\frac{E\quad}{\quad C}\cdot\frac{C\quad}{\quad D}=-1$$

In the blank spaces there are inserted successively and in order the letters denoting the transversal

D E F, D B C, E C A, F A B;

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thus
$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = -1$$
$$\frac{ED}{DF} \cdot \frac{FB}{BA} \cdot \frac{AC}{CE} = -1$$
$$\frac{FE}{ED} \cdot \frac{DC}{CB} \cdot \frac{BA}{AF} = -1$$
$$\frac{DF}{FE} \cdot \frac{EA}{AC} \cdot \frac{CB}{BD} = -1$$

If now any three of these expressions, without rearrangement, be multiplied together, the result is the reciprocal of the fourth. No difficulty arises here with regard to the signs of the segments which cancel each other; in each case they are the same for any given figure.

It may be worth while to draw attention to the following four constructions, which will give four varieties of proof. The proofs need not be supplied, as the theorem is so well known.

(1) Through any of the vertices A draw a parallel to the opposite side BC, and let it meet the transversal at G.

(2) Through any of the vertices A draw a parallel to the transversal, and let it meet the opposite side BC at G.

(3) On any straight line project the vertices by means of parallels to the transversal.

(4) On the transversal project the vertices by means of parallels to any straight line.

Ceva's theorem is

If three concurrent straight lines AD BE CF be drawn from the vertices of triangle ABC to meet the opposite sides in D E F, then
$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = +1,$$
and conversely.

Let O be the point of concurrency; then of the four points A, B, C, O any three may be regarded as forming the vertices of a triangle, and the fourth may be taken as the point of concurrency of the angular transversals. The four results obtained however are not independent of each other, for any one of them may be deduced from the other three.

As in the case of Menelaus's theorem, these results may be written down easily and without the trouble of consulting a diagram.

Observe that whichever of the points A B C O is taken as the point of concurrency, the points D E F are to be in order the feet of the angular transversals. Thus

DEF DEF DEF DEF

OOO AAA BBB CCC

ABC OCB COA BAO Hence

If O be chosen as the point of concurrency, DO, EO, FO pass in order through the vertices of the triangle, that is, ABC is the triangle.

If A be chosen as the point of concurrency, DA, EA, FA pass in order through the vertices of the triangle, that is, OCB is the triangle.

If B be chosen as the point of concurrency, DB, EB, FB pass in order through the vertices of the triangle, that is, COA is the triangle.

If C be chosen as the point of concurrency, DC, EC, FC pass in order through the vertices of the triangle, that is, BAO is the triangle.*

The results obtained are therefore

$$\frac{B\ \ }{\ \ C} \cdot \frac{C\ \ }{\ \ A} \cdot \frac{A\ \ }{\ \ B} = +1$$

$$\frac{C\ \ }{\ \ B} \cdot \frac{B\ \ }{\ \ O} \cdot \frac{O\ \ }{\ \ C} = +1$$

$$\frac{O\ \ }{\ \ A} \cdot \frac{A\ \ }{\ \ C} \cdot \frac{C\ \ }{\ \ O} = +1$$

$$\frac{A\ \ }{\ \ O} \cdot \frac{O\ \ }{\ \ B} \cdot \frac{B\ \ }{\ \ A} = +1$$

* With regard to the naming of the triangles ABC OCB COA BAO see *Proceedings of the Edinburgh Mathematical Society*, Vol. I., p. 62.

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In the blank spaces there are inserted successively and in order the letters denoting the feet of the angular transversals D E F;

thus
$$\frac{\mathrm{BD}}{\mathrm{DC}} \cdot \frac{\mathrm{CE}}{\mathrm{EA}} \cdot \frac{\mathrm{AF}}{\mathrm{FB}} = +1$$
$$\frac{\mathrm{CD}}{\mathrm{DB}} \cdot \frac{\mathrm{BE}}{\mathrm{EO}} \cdot \frac{\mathrm{OF}}{\mathrm{FC}} = +1$$
$$\frac{\mathrm{OD}}{\mathrm{DA}} \cdot \frac{\mathrm{AE}}{\mathrm{EC}} \cdot \frac{\mathrm{CF}}{\mathrm{FO}} = +1$$
$$\frac{\mathrm{AD}}{\mathrm{DO}} \cdot \frac{\mathrm{OE}}{\mathrm{EB}} \cdot \frac{\mathrm{BF}}{\mathrm{FA}} = +1$$

If now any three of these expressions, without rearrangement, be multiplied together, the result is the reciprocal of the fourth. Here also it may be said that no difficulty arises with regard to the signs of the segments which cancel each other; in each case they are different for any given figure. It may be worth while to write down the results which are obtained by multiplying together every three of the given expressions. They are

$$\frac{\mathrm{CD}}{\mathrm{DB}} \cdot \frac{\mathrm{AE}}{\mathrm{EC}} \cdot \frac{\mathrm{BF}}{\mathrm{FA}} = +1$$
$$\frac{\mathrm{CF}}{\mathrm{FO}} \cdot \frac{\mathrm{OE}}{\mathrm{EB}} \cdot \frac{\mathrm{BD}}{\mathrm{DO}} = +1$$
$$\frac{\mathrm{AD}}{\mathrm{DO}} \cdot \frac{\mathrm{CE}}{\mathrm{EA}} \cdot \frac{\mathrm{OF}}{\mathrm{FC}} = +1$$
$$\frac{\mathrm{AF}}{\mathrm{FB}} \cdot \frac{\mathrm{BE}}{\mathrm{EO}} \cdot \frac{\mathrm{OD}}{\mathrm{DA}} = +1$$

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