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CONSTRUCTIONS OF BRAUER-SEVERI VARIETIES AND NORM HYPERSURFACES

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1. Introduction. Let k be any field, A a central simple k-algebra of degree m (i.e., $\dim_k A = m^2$). Several methods of constructing the generic splitting fields for A are proposed and Saltman proves that these methods result in almost the same generic splitting field [**8**, Theorems 4.2 and 4.4]. In fact, the generic splitting field constructed by Roquette [**7**] is the function field of the Brauer-Severi variety $V_m(A)$ while the generic splitting field constructed by Heuser and Saltman [**4** and **8**] is the function field of the norm surface $W(A)$. In this paper, to avoid possible confusion about the dimension, we shall call it the norm hypersurface instead of the norm surface.

By the result of Saltman mentioned above $W(A)$ is birational to $V_m(A) \times P^{m^2-m-1}$. Biregularly, the Brauer-Severi varieties and the norm hypersurfaces are quite different, since the former are k-forms of the projective space [**9**, p. 152] and the latter are those of the determinantal variety Γ_m, which is a singular variety when $m \geqq 3$. (Please see Definition 3 are more details.)

In this paper we are concerned with the constructions of Brauer-Severi varieties and norm hypersurfaces. In Section 2 we shall give an explicit construction of Brauer-Severi varieties (Theorem 1). Previously a Brauer-Severi variety is usually defined either by Galois descent or as some subvariety of the Grassmann variety [**1** and **9**]. We hope that an explicit construction will help the understanding of Brauer-Severi varieties. After we finished our work we found that this construction was anticipated by M. Artin in his survey talk using certain Hochschild-Serre spectral sequence [**3**]. As a consequence, we show that the function fields of Brauer-Severi varieties are actually the generic splitting field constructed by Amitsur and Roquette [**7**], a fact mentioned in [**7**, §1] although we couldn't locate this result in the literature. In Section 3 we consider the norm hypersurface. We give a cohomological interpretation of Heuser and Saltman's construction of norm hypersurfaces (Theorem 4). In [**5**], N. Jacobson shows that the reduced norm of a central simple algebra determines uniquely the isomorphism or anti-isomorphism class of this algebra. We can show that the norm hypersurface plays a similar role as the reduced norm (Theorem 5).

Standing terminology: For terminologies about central simple algebras we adopt those in Draxl's book [**3**]. Hence $\exp(A)$ and $\operatorname{ind}(A)$ will be the exponent and the index of a central simple algebra A. $K[x]$ always means $K[x_1, \ldots, x_m]$, the polynomial ring of m variables. $PGL_m(K)$ is the quotient group of $GL_m(K)$ by the scalar matrices.

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2. The Brauer-Severi varieties. Let k be any field, K a finite Galois extension of k and $G = \mathrm{Gal}(K/k)$ the Galois group of K over k.

Given a central simple k-algebra A which is split by K, the similarity class of A determines a 2-cocycle $\gamma \in H^2(G, K^\times)$. If $\exp(A) = e$ and m is a multiple of $\mathrm{ind}(A)$, we shall construct a Brauer-Severi variety of dimension $m-1$ corresponding to A, which is denoted by $V_m(A)$. Since $V_m(A)$ depends only on the 2-cocycle γ, we denote it also by $V_m(\gamma)$.

To construct $V_m(\gamma)$ we shall determine its homogeneous coordinate ring as follows.

By [**7**, Theorem 1], γ is in

$$\mathrm{Image}\{H^1(G, PGL_m(K)) \longrightarrow H^2(G, K^\times)\}.$$

Choose the unique 1-cocycle

$$\beta = \{b_\sigma : \sigma \in G\} \in H^1(G, PGL_m(K))$$

which is mapped to γ. For each $b_\sigma \in PGL_m(K)$ choose a preimage $a_\sigma \in GL_m(K)$. Then

$$\gamma = \{c_{\sigma,\tau} \in K^\times : \sigma, \tau \in G\}$$

with

$$c_{\sigma,\tau} = a_\sigma \cdot \sigma(a_\tau) \cdot a_{\sigma\tau}^{-1}$$

where $a_\sigma \cdot \sigma(a_\tau) \cdot a_{\sigma\tau}^{-1}$ is a scalar matrix in $GL_m(K)$ and is identified with an element of $K^\times$.

Consider the polynomial ring $K[x] = K[x_1, \ldots, x_m]$. Elements in G or $GL_m(K)$ can be regarded as k-algebra automorphisms of $K[x]$. In fact, when $\sigma \in G$, $c \in K$, $\sigma(c)$ is defined as before and define $\sigma(x_i) = x_i$ for $1 \leqq i \leqq m$. When $\tilde{a} = (a_{ij})_{1 \leqq i,j \leqq m}$ is in $GL_m(K)$, define a K-automorphism of $K[x]$ by

$$\tilde{a}(x_j) = \sum_{1 \leqq i \leqq m} a_{ij} x_i \quad \text{for } 1 \leqq j \leqq m.$$

For each $\sigma \in G$, define u_σ by $u_\sigma = a_\sigma \cdot \sigma$. Then each u_σ is a k-algebra automorphism of $K[x]$ satisfying the following conditions

$$(1) \qquad u_\sigma \cdot u_\tau = c_{\sigma,\tau} \cdot u_{\sigma\tau} \quad \text{for all } \sigma, \tau \in G,$$

$$(2) \qquad u_\sigma \cdot c = \sigma(c) \cdot u_\sigma \quad \text{for all } c \in K.$$

Since $\exp(\gamma) = e$, choose a 1-cochain $\chi = \{c_\sigma : \sigma \in G\}$ in $C^1(G, K^\times)$ so that

$$c_{\sigma,\tau}^e = c_\sigma \cdot \sigma(c_\tau) \cdot c_{\sigma\tau}^{-1} \quad \text{for all } \sigma, \tau \in G.$$

The monomials of degree e in $K[x]$ generate a graded sub-algebra over K. Call it $K[\bar{x}]$.

Now we shall define a k-algebra automorphism v_σ on $K[\bar{x}]$ for each $\sigma \in G$. Namely, if $f \in K[\bar{x}]$ is a homogeneous polynomial of degree er in $x_1, \ldots, x_m$, define

$$v_\sigma(f) = c_\sigma^{-r} \cdot u_\sigma(f).$$

It is easy to check that

$$(1) \qquad v_\sigma \cdot v_\tau = v_{\sigma\tau} \quad \text{for all } \sigma, \tau \in G,$$

$$(2) \qquad v_\sigma(c \cdot f) = \sigma(c) \cdot v_\sigma(f) \quad \text{for all } c \in K \text{ and for all } f \in K[\bar{x}].$$

The group of automorphisms $\{v_\sigma : \sigma \in G\}$ is isomorphic to G. By abusing the notations we still call it G.

Definition 1. Let R be the ring of invariants of $K[\bar{x}]$ under the actions of G, i.e.,

$$R = K[\bar{x}]^G = \{f \in K[\bar{x}] : v_\sigma(f) = f \text{ for all } \sigma \in G\}.$$

R is a graded k-algebra. We shall show that R is the homogeneous coordinate ring of the Brauer-Severi variety $V_m(\gamma)$.

Note that it may be possible that similar arguments could be applied to the construction of forms of some other projective varieties.

LEMMA 1. (1) *R is an affine normal domain over k.*
(2) $K[\bar{x}] = R \bigotimes_k K$.

Proof. (1) Since $K[\bar{x}]$ is normal, its ring of invariants is also normal.
(2) By a theorem of Speiser [**3**, Theorem 1, page 36]

$$K[\bar{x}] = K[\bar{x}]^G \bigotimes_k K.$$

In [**7**, §5] Roquette defines the following generic splitting field of γ,

$$F_m(\gamma) = \{f \cdot g^{-1} : f \text{ and } g \text{ are homogeneous polynomials in } K[x], \\ \deg f = \deg g \text{ and } u_\sigma(f \cdot g^{-1}) = f \cdot g^{-1} \text{ for all } \sigma \in G\}.$$

LEMMA 2. $F_m(\gamma) = \{f \cdot g^{-1}$: *f and g are homogeneous polynomials in R with* $\deg f = \deg g\}$.

Proof. It suffices to prove the following assertion. Let f and g be relatively prime homogeneous polynomials in $K[x]$ with $\deg f = \deg g$ and

$$u_\sigma(f \cdot g^{-1}) = f \cdot g^{-1} \quad \text{for all } \sigma \in G.$$

Then there is some element $c \in K^\times$ so that both cf and cg are in R.

We shall prove the above assertion in three steps.

Step 1. $u_\sigma(f) = \lambda_\sigma \cdot f$, $u_\sigma(g) = \lambda_\sigma \cdot g$ where $\lambda_\sigma \in K^\times$.

In fact, $f \cdot u_\sigma(g) = g \cdot u_\sigma(f)$. Since f and g are relatively prime, hence $u_\sigma(f) = \lambda_\sigma \cdot f$ for some $\lambda_\sigma \in K^\times$.

Step 2. $e \mid \deg f$.

Apply the relation $u_\sigma \cdot u_\tau = c_{\sigma,\tau} u_{\sigma\tau}$ to f and use the result of Step 1. Then

$$c^r_{\sigma,\tau} = \lambda_\sigma \cdot \sigma(\lambda_\tau) \cdot \lambda^{-1}_{\sigma\tau} \quad \text{where } r = \deg f.$$

Thus $\{c^r_{\sigma,\tau} : \sigma, \tau \in G\}$ is cohomologously trivial in $H^2(G, K^\times)$. It follows that $e \mid r$.

Step 3. There is an element $c \in K^\times$ so that $cf \in R$.

Write $r = e \cdot r'$. In the proof of Step 2, we have

$$c^r_{\sigma,\tau} = \lambda_\sigma \cdot \sigma(\lambda_\tau) \cdot \lambda^{-1}_{\sigma\tau}.$$

On the other hand

$$c^r_{\sigma,\tau} = (c^e_{\sigma,\tau})^{r'} = c^{r'}_\sigma \cdot \sigma(c^{r'}_\tau) \cdot c^{-r'}_{\sigma\tau}$$

where $\{c_\sigma : \sigma \in G\}$ is the 1-cochain used in the definition of v_σ. Hence $\{\lambda_\sigma \cdot c^{-r'}_\sigma : \sigma \in G\}$ is a 1-cocycle in $H^1(G, K^\times)$. By Hilbert Theorem 90 we can find $c \in K^\times$ so that

$$\lambda_\sigma \cdot c^{-r'}_\sigma = c \cdot \sigma(c^{-1}).$$

Then $cf \in R$.

It is not difficult to show that, up to isomorphism, the graded algebra R depends only on the cohomology class of the 2-cocycle λ, i.e., it is independent of the choice of the 1-cochain $\{c_\sigma : \sigma \in G\}$, etc. Hence we arrive at our definition of $V_m(\gamma)$.

Definition 2. For a 2-cocycle $\gamma \in H^2(G, K^\times)$, define $V_m(\gamma) = \operatorname{Proj}(R)$. $V_m(\gamma)$ is a projective variety defined over k.

THEOREM 1. (1) *$V_m(\gamma)$ is a Brauer-Severi variety of dimension $m-1$ defined over k, i.e., $V_m(\gamma)$ is a form of P^{m-1}_k, the projective $(m-1)$-space over k. In fact,*

$$V_m(\gamma) \bigotimes_k K \simeq P^{m-1}_K.$$

(2) *$V_m(\gamma)$ can be embedded in P^{N-1}_k as a projectively normal subvariety where*

$$N = \binom{e+m-1}{e} \quad \textit{and} \quad e = \exp(\gamma).$$

(3) *The function field of* $V_m(\gamma)$ *is* $F_m(\gamma)$, *a generic splitting field of* γ.

Proof.

$$(1) \qquad V_m(\gamma)\bigotimes_k K \simeq \mathrm{Proj}\left(R\bigotimes_k K\right) \simeq \mathrm{Proj}(K[\bar{x}]) \simeq \mathrm{Proj}(K[x]) \simeq P_K^{m-1}.$$

(2) Note that $R = R_0 \oplus R_e \oplus R_{2e} \oplus \cdots \oplus R_{re} \oplus \cdots$ where R_i is the set of homogeneous polynomials of degree i in R and $R_0 = k$. Since $\dim_k R_e =$ the dimension of monomials of degree $e = N$ and each element in R_e defines a hyperplane section in the given projective embedding of Proj(R). Thus $V_m(\gamma)$ is a subvariety of P_k^{N-1}. Now apply Lemma 1 (1).

(3) Use Lemma 2.

THEOREM 2. *Let*

$$\Omega_m(K/k) = \mathrm{Image}\{H^1(G, PGL_m(K)) \longrightarrow H^2(G, K^\times)\}.$$

Then there are one-to-one correspondences among the following sets; the set of all central simple k-algebras of degree m which are split by K, the set of all Brauer-Severi varieties of dimension $m-1$ *over k which are split by K,*

$$\{F_m(\gamma) : \gamma \in_n (K/k)\}.$$

Moreover the correspondences are given as follows. If A is a central simple k-algebra of degree m and with a splitting field K, let γ *be the associated 2-cocycle in* $H^2(G, K^\times)$. *Then A corresponds to* $V_m(\gamma)$ $(= V_m(A))$ *and* $V_m(\gamma)$ *corresponds to its function field.*

Proof. Almost everything in this theorem follows from Galois descent. The essence of this theorem is to make sure that everything corresponds to the right one as we expect, that is, a central simple algebra A should go to $V_m(A)$ constructed in Theorem 1 and $V_m(A)$ should goes to its function field. As to the proof, note that the correspondences are functorial and it has been proved that A and $F_m(A)$ correspond bijectively in [**7**].

3. The norm hypersurfaces.

Definition 3. Let k be any field, $(x_{ij})_{1\leqq i,j\leqq m}$ the generic $m\times m$ matrix over k, i.e., $k(x_{11}, x_{12}, \ldots, x_{mm})$ are the rational function field of m^2 variables over k. The determinantal variety Γ_m of dimension m^2-2 is the projective subvariety of $P_k^{m^2-1}$ defined by the equation $\det(x_{ij}) = 0$.

Note that Γ_2 is just the quadratic surface $xu - yz = 0$ in P^3, and therefore is biregular to $P^1 \times P^1$.

Definition 4. Let k be any field and A a central simple k-algebra of degree m. Then the reduced norm of A is a homogeneous polynomial $f_A(x_1, \ldots, x_{m^2})$ of degree m, which is called the *norm polynomial for* A [**8**, Section 1]. The *norm hypersurface associated to* A, denoted by $W(A)$, is the projective subvariety of $P_K^{m^2-1}$ defined by the equation

$$f_A(x_1, \ldots, x_{m^2}) = 0.$$

Note that our definition of a norm hypersurface is in accordance with that of [**4**]. However the norm hypersurface in [**8**] is the affine subvariety in the affine m^2-space defined by the same equation

$$f_A(x_1, \ldots, x_{m^2}) = 0.$$

When K is a splitting field of A, then $f_A(x_1, \ldots, x_{m^2})$ is equivalent to $\det(x_{ij})$ under some linear change of variables over K. Hence $W(A)$ is a k-form of Γ_m and $W(A)$ is K-isomorphic to Γ_m [**9**, page 152].

Let K be a finite Galois extension of the field k with Galois group G, $E(K/k)$ the set of all k-forms which are K-isomorphic to Γ_m, and $\Omega_m(K/k)$ the set of all central simple k-algebras of degree m which are split by k.

By [**7**, Theorem 1],

$$\Omega_m(K/k) \simeq \operatorname{Image}\{H^1(G, PGL_m(K)) \longrightarrow H^2(G, K^\times)\}.$$

$E(K/k)$ contains $\{W(A) : A \in \Omega_m(K/k)\}$. But it may be possible that the norm hypersurfaces cannot exhaust $E(K/k)$.

By Galois descent [**9**, Proposition 4, page 153] there is a bijection between $E(K/k)$ and $H^1(G, \operatorname{Aut}(\Gamma_m \bigotimes_k K))$ where $\operatorname{Aut}_K(\Gamma_m \bigotimes_k K)$ is the group of K-automorphisms of $\Gamma_m \bigotimes_k K$.

By a theorem of Dieudonne [**2**], the automorphism group of $\Gamma_m \bigotimes_k K$ is the semi-direct product of $PGL_m(K) \times PGL_m(K)$ and $\Sigma = \{\mathrm{id}, \eta\}$ where

$$(\xi_1, \xi_2) \in PGL_m(K) \times PGL_m(K)$$

acts on Γ_m by sending $(a_{ij}) \in \Gamma_m$ to $\xi_1(a_{ij}) \cdot \xi_2^{-1}$ and η is the automorphism on Γ_m by sending (a_{ij}) to $(a_{ij})^t = (a_{ji})$. Hence we obtain the following.

PROPOSITION 3. *Let K be a finite Galois extension of k with Galois group G. Then the following sequence is exact as pointed sets:*

$$H^1(G, PGL_m(K) \times PGL_m(K)) \longrightarrow H^1\left(G, \operatorname{Aut}_K\left(\Gamma_m \bigotimes_k K\right)\right) \longrightarrow H^1(G, \Sigma) \longrightarrow 1.$$

Now we shall define a map from $H^1(G, PGL_m(K))$ by sending a 1-cocycle $\beta = \{b_\sigma \in PGL_m(K) : \sigma \in G\}$ to $\beta' = \{(b_\sigma, b_\sigma) : \sigma \in G\}$ which is in $H^1(G, PGL_m(K) \times PGL_m(K))$. Call $\tilde{\Psi}$ the composite map of

$$H^1(G, PGL_m(K)) \longrightarrow H^1(G, PGL_m(K) \times PGL_m(K))$$

and

$$H^1(G, PGL_m(K) \times PGL_m(K)) \longrightarrow H^1\left(G, \mathrm{Aut}_K\left(\Gamma_m \bigotimes_k K\right)\right).$$

THEOREM 4. *Consider the folowing diagram*

$$\begin{array}{ccc} H^1(G, PGL_m(K)) & \xrightarrow{\tilde{\psi}} & H^1\left(G, \mathrm{Aut}_K\left(\Gamma_m \bigotimes_k K\right)\right) \\ \downarrow & & \downarrow \\ \Omega_m(K/k) & \xrightarrow{\psi} & E(K/k) \end{array}$$

where the vertical bijections are the canonical ones given by Galois descent, and ψ is defined so that the diagram commutes. If A is a central simple k-algebra in $\Omega_m(K/k)$, then the image of A under the map ψ is the norm hypersurface $W(A)$.

Proof. Let $\beta = \{b_\sigma \in PGL_m(K) : \sigma \in G\}$ be a 1-cocycle in $H^1(G, PGL_m(K))$ and A its associated central simple algebra of degree m. Then there is a K-linear isomorphism

$$\phi : M_m(K) \longrightarrow A \bigotimes_k K.$$

For each $\sigma \in G$, id $\otimes \sigma$ denotes the automorphism of $A \otimes K$ keeping A pointwise fixed and σ is the automorphism of $M_n(K)$ by acting on the entries of matrices in $M_m(K)$. Let

$${}^\sigma\phi = (\mathrm{id} \otimes \sigma) \circ \phi \circ \sigma^{-1} \quad \text{for each } \sigma \in G.$$

Then

$${}^\sigma\phi = \phi \circ b_\sigma \quad \text{for each } \sigma \in G.$$

Let

$$\tilde{\Gamma} = \{t \in M_m(K) : \det(t) = 0\} \quad \text{and} \quad \tilde{W} = \phi(\tilde{\Gamma})$$

and denote the restriction map of ϕ by $\tilde{\phi}$, i.e., $\tilde{\phi} : \tilde{\Gamma} \longrightarrow \tilde{W}$ is a K-linear isomorphism of vector spaces.

Now $\tilde{W}$ consists of elements of reduced norm zero in $A \bigotimes_k K$. Hence $\tilde{\Gamma}$ and $\tilde{W}$ are just the affine cones of the projective varieties Γ_m and $W(A)$. Therefore the map $\tilde{\phi}$ induces a splitting of the k-form $W(A)$ of Γ_m. In other words, the induced map

$$\phi_* : \Gamma_m \bigotimes_k K \longrightarrow W(A) \bigotimes_k K$$

is a regular isomorphism defined over K.

Again the relation ${}^{\sigma}\phi_* = \phi_* \circ b_\sigma$ holds for each $\sigma \in G$. Moreover, since b_σ acts as an automorphism of $M_m(K)$ by sending each $m \times m$ matrix t to $b_\sigma \cdot t \cdot b_\sigma^{-1}$ by Skolem-Noether's Theorem, hence the induced action of b_σ on $\Gamma_m(K)$ is just the same one when we embed $PGL_m(K)$ into $\mathrm{Aut}_K(\Gamma_m \bigotimes_k K)$. This proves that $W(A)$ corresponds to the 1-cocycle

$$\beta' = \{(b_\sigma, b_\sigma) : \sigma \in G\}.$$

Hence the result.

THEOREM 5. *Let A and B be central simple k-algebras of degree m and $W(A)$, $W(B)$ the norm hypersurfaces of A and B respectively. If $W(A)$ is isomorphic to $W(B)$ as subvarieties in $P_k^{m^2-1}$, then A and B are either isomorphic or anti-isomorphic.*

Proof. By assumption, $W(A)$ and $W(B)$ are isomorphic by some regular map of $P_k^{m^2-1}$. Such a regular map is of degree one since it is so on $W(B)$. (Recall the $W(B) \longrightarrow W(A)$ is one-to-one.) Hence there is a k-linear map ϕ so that

$$f_A(\phi(x_1, \ldots, x_{m^2})) = af_B(x_1, \ldots, x_{m^2})$$

for some $a \in k\backslash\{0\}$. After choosing basis for A and B, we can regard ϕ as a k-linear map of the vector spaces A into B so the

$$f_A(\phi(x_1, \ldots, x_{m^2})) = af_B(x_1, \ldots, x_{m^2}).$$

When both A and B are matrix rings, since the proof of Lemma 7 and Lemma 8 of [**6**] is still valid in our situation, it follows that ϕ is rank-preserving and hence $\phi = \beta \cdot \psi$ by [**6**, Theorem 1] where β is a non-singular matrix and ψ is an isomorphism or anti-isomorphism of the matrix ring. When A and B are any central simple algebras, the technique of descent of [**10**, Theorem] can be applied to show that $\phi = \beta \cdot \psi$ again for some isomorphism or anti-isomorphism ψ of the central simple algebras A and B.

REFERENCES

1. M. Artin, *Brauer-Severi varieties*, in *Brauer groups in ring theory and algebraic geometry*, Springer LNM *917* (Springer-Verlag, Berlin, 1982).

2. J. Dieudonné, *Sur une generalisation du group orthogonal a quatre variables*, Archiv. der Math. *1* (1948), 282–287.

3. P. K. Draxl, *Skew fields* (Cambridge University Press, Cambridge, 1983).

4. A. Heuser, *Uber den Funktionenkorper der Normflache einer zentral einfachen Algebra*, J. Reine Angew. Math. *301* (1978), 105–113.

5. N. Jacobson, *Structure groups and Lie algebras of Jordan algebras of symmetric elements of associative algebras with involution*, Advance in Math. *20* (1976), 106–150.

6. M. Marcus and B. N. Moyls, *Linear transformations on algebras of matrices*, Can. J. Math. *11* (1959), 61–66.

7. P. Roquette, *On the Galois cohomology of the projective linear group and its applications to the construction of generic splitting field of algebras*, Math. Ann. *150* (1962), 411–439.
8. D. Saltman, *Norm polynomials and algebras*, J. Algebra *62* (1980), 333–345.
9. J. P. Serre, *Local fields*, English translation, Springer GTM *67* (Springer-Verlag, New York, 1979).
10. W. C. Waterhouse, *Linear maps preserving reduced norms*, Linear Algebra and Its Appl. *43* (1982), 197–200.

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