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A COMMUTATIVITY CONDITION FOR SEMI PRIME RINGS-II

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It is shown that if R is a semi prime ring in which $(xy)^2-xy$ is central for every $x,y \in R$, then R is commutative.

1. Introduction

Throughout the paper R will represent a nonzero associative ring with centre $Z(R)$. It is well known that a Boolean ring satisfies $x^2 = x$, for all $x \in R$ and this implies commutativity. Now the question arises as to what we can say about the rings R in which $(xy)^2 = xy$, for each pair of elements $x,y \in R$. In this direction we prove the following theorem:

THEOREM. *Let R be a semi prime ring in which $(xy)^2 - xy \in Z(R)$, for all $x,y \in R$, then R is commutative.*

2. Preliminary Results

We begin with the following lemmas:

LEMMA 2.1. *Let R be a prime ring and $x \neq 0$ be an element in $Z(R)$. If for any $y \in R$, $xy \in Z(R)$, then $y \in Z(R)$.*

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Proof. x and xy in $Z(R)$ give $xR(yz-zy) = 0$, for all $z \in R$. But since $x \neq 0$ and R is prime, this forces $yz - zy = 0$. Hence $y \in Z(R)$.

LEMMA 2.2. *Let R be a semi prime ring in which $xy^2x = yx^2y$, for all, $x,y \in R$. Then R is commutative.*

Proof. A particular case of the first author's theorem [3].

LEMMA 2.3. *Let R be a semi prime ring satisfying $(xy)^2 - xy \in Z(R)$, for all $x,y \in R$. Then R has no nonzero nilpotent elements.*

Proof. Let $x \in R$ such that $x^2 = 0$. By our hypothesis we have $\{(xy)^2 - xy\} y = y \{(xy)^2 - xy\}$. On replacing y by $(x-yx)$ and using the fact that $x^2 = 0$, we get $(xy)^2x = 0$ or $(xy)^3 = 0$, for all $y \in R$. If $xR \neq 0$, then xR is a nonzero nil right ideal in R satisfying the identity $z^3 = 0$, for all $z \in xR$. Now by lemma 1.1 of [2] R has a nonzero nilpotent ideal which is a contradiction since R is semi prime. Thus $xR = 0$ and hence $xRx = 0$. This implies that $x = 0$.

Now lemma 1.1.1 of [2] together with the above result readily yield the following

LEMMA 2.4. *Let R be a prime ring satisfying $(xy)^2 - xy \in Z(R)$, for all $x,y \in R$. Then R has no zero divisors.*

3. Proof of the Theorem

Since R is semi prime, it is isomorphic to a subdirect sum of prime rings R_α each of which as a homomorphic image of R satisfies the hypothesis of the theorem. Hence we can assume that R is a prime ring satisfying $(xy)^2 - xy \in Z(R)$, for all $x,y \in R$. First we assert that $Z(R) \neq (0)$. Assume on the contrary that $Z(R) = (0)$. In that case,

$$(xy)^2 = xy, \quad \text{for all } x,y \in R. \qquad \ldots (1)$$

Replacing x by $(x + y)$ in (1) and simplifying we get,

$$(xy^2 + y^2x)y = 0 . \qquad \ldots. (2)$$

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With $x = xr$, (2) gives

$$(xry^2 + y^2xr) \quad \ldots\ldots (3)$$

But from (2), $ry^2y = -y^2ry$ and so (3) yields that $(xy^2 - y^2x)ry = 0$ or $(xy^2 - y^2x)Ry = 0$. Since R is prime, either $y = 0$ or $(xy^2-y^2x) = 0$. But $y = 0$ also gives $(xy^2 - y^2x) = 0$. This implies that $y^2 \in Z(R) = (0)$ or $y^2 = 0$ for every $y \in R$ which gives that $(x + y)^2 y = 0$ or $yRy = 0$. Again R prime forces $y = 0$ that is $R = (0)$, a contradiction. Hence $Z(R) \neq (0)$.

Now let c be a nonzero element in $Z(R)$. Replacing x by $(x + c)$ in $(xy)^2 - xy\ Z(R)$, we get $c(xy^2 + yxy) \in Z(R)$. Thus by lemma 2.1, $(xy^2 + yxy) \in Z(R)$, for all $x, y \in R$ and we get,

$$\{(xy^2 + yxy\}\ y = y\{xy^2 + yxy\}\ ,$$

$$\text{that is} \quad (xy^2 - y^2x)y = 0. \quad \ldots\ldots (4)$$

Therefore by lemma 2.4, we have either $y = 0$ or $(xy^2 - y^2x) = 0$. But $y = 0$ also gives $(xy^2 - y^2x) = 0$ and so in every case

$$xy^2 = y^2x\ . \quad \ldots\ldots (5)$$

Now putting $y = x + y$ in (4) and using $x^2y^2 = y^2x^2$, $x^2yx = yx^3$, easy consequences of (5), we get $xy^2x = y\ x^2y$, for every $x, y \in R$. Hence by lemma 2.2, R is commutative.

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