Canad. Math. Bull. Vol. **60** (3), 2017 pp. 604–612
http://dx.doi.org/10.4153/CMB-2016-078-3

Stackings and the W-cycles Conjecture

Larsen Louder and Henry Wilton

Abstract. We prove Wise's W-cycles conjecture. Consider a compact graph Γ' immersing into another graph Γ. For any immersed cycle $\Lambda: S^1 \to \Gamma$, we consider the map Λ' from the circular components $\mathbb{S}$ of the pullback to Γ'. Unless Λ' is reducible, the degree of the covering map $\mathbb{S} \to S^1$ is bounded above by minus the Euler characteristic of Γ'. As a corollary, any finitely generated subgroup of a one-relator group has a finitely generated Schur multiplier.

1 Introduction

As part of his work on the coherence of one-relator groups, Wise made a conjecture about the number of lifts of a cycle in a free group along an immersion, which we will call the *W-cycles conjecture*. If $f_1: \Gamma_1 \looparrowright \Gamma$ and $f_2: \Gamma_2 \looparrowright \Gamma$ are immersions of graphs, then the fibre product

$$\Gamma_1 \times_\Gamma \Gamma_2 = \{ (x,y) \in \Gamma_1 \times \Gamma_2 \mid f_1(x) = f_2(y) \}$$

immerses into Γ_1 and Γ_2, and is the pullback of f_1 and f_2. An immersed loop $\Lambda: S^1 \looparrowright \Gamma$ is *primitive* if it does not factor properly through any other immersion $S^1 \looparrowright \Gamma$.

With this definition, the W-cycles conjecture can be stated as follows.

Conjecture 1.1 (Wise [Wis05]) *Let $\rho: \Gamma' \to \Gamma$ be an immersion of finite connected core graphs and let $\Lambda: S^1 \to \Gamma$ be a primitive immersed loop. Let $\mathbb{S}$ be the union of the circular components of $\Gamma' \times_\Gamma S^1$. Then the number of components of $\mathbb{S}$ is at most the rank of Γ'.*

The purpose of this note is to prove Wise's conjecture; indeed, we prove a stronger statement. As usual, if π is a covering map, then $\deg \pi$ denotes its degree, the number of preimages of a point. An immersion of a union of circles $\Lambda: \mathbb{S} \to \Gamma$ is called *reducible* if there is an edge of Γ that is traversed at most once by Λ.

Theorem 1.2 *Let $\rho: \Gamma' \looparrowright \Gamma$ be an immersion of finite connected core graphs and let $\Lambda: S^1 \to \Gamma$ be a primitive immersed loop. Suppose that $\mathbb{S}$, the union of the circular components of $\Gamma' \times_\Gamma S^1$, is non-empty, so there is a natural covering map $\sigma: \mathbb{S} \looparrowright S^1$. Then either $\deg \sigma \le -\chi(\Gamma')$ or the pullback immersion $\Lambda': \mathbb{S} \to \Gamma'$ is reducible.*

Received by the editors October 9, 2014; revised January 15, 2016.
Published electronically April 13, 2017.
Author H. W. is supported by the EPSRC..
AMS subject classification: **20F65**.
Keywords: free groups, one-relator groups, right-orderability.

The statement of the conjecture is a corollary of this theorem. Indeed, the inequality in the theorem is strictly stronger than the inequality in the conjecture; alternatively, in the reducible case, we can remove an edge and proceed by induction.

Wise's notion of *nonpositive immersions* provides a connection with a famous question of Baumslag [Bau74]: is every one-relator group coherent? (Recall that a group is *coherent* if every finitely generated subgroup is finitely presented.) As in the case of graphs, an immersion of cell complexes is a locally injective cellular map.

Definition 1.3 (Wise) A cell complex X has *nonpositive immersions* (*NPI*) if, for every immersion of compact, connected complexes $Y \looparrowright X$, either $\chi(Y) \leq 0$ or Y has trivial fundamental group.

Presentation complexes of one-relator groups with torsion do not have non-positive immersions. Let C_k be the presentation complex of $\mathbb{Z}/k\mathbb{Z}$ associated with the presentation $\langle a \mid a^k \rangle$, and for $l \mid k$, let $C_{k,l}$ be the l–fold cover of C_k.

Definition 1.4 A cell complex X has *not too positive immersions* (*NTPI*) if, for every immersion of compact, connected complexes $Y \looparrowright X$, Y is homotopy equivalent to a wedge of subcomplexes of $C_{k,l}$s and a compact 2-complex $Y' \subset Y$ with $\chi(Y') \leq 0$.

For $k = 1$ this reduces to NPI, since $C_{1,l}$ is a disk. Our main theorem implies that presentation complexes associated with one-relator groups have NTPI; in particular, in the torsion-free case, they have NPI.

Corollary 1.5 *Let X be compact 2-complex with one 2-cell e^2 and suppose that the attaching map $\Lambda\colon S^1 \to X^{(1)}$ of e^2 is an immersion. Then X has NTPI.*

Proof Suppose that $\rho\colon Y \looparrowright X$ is an immersion of a compact 2-complex Y into X. Let $\Gamma = X^{(1)}$, $\Gamma' = Y^{(1)}$, and $\Lambda'\colon \mathbb{S} \to \Gamma'$ be the pullback immersion, in the notation of Theorem 1.2. Let $\mathbb{S}'$ be the union of the components $S_1, \dots, S_m$ of $\mathbb{S}$ that are realized by boundaries of 2-cells of Y. If $\chi(Y) > 0$, then $\deg(\sigma) > -\chi(\Gamma')$, and so, by Theorem 1.2, Λ' is reducible. That is, there is some edge e of Γ' traversed by at most one component S of $\mathbb{S}$.

If S is not contained in $\mathbb{S}'$, we can remove the edge e and proceed by induction on the size of the one-skeleton of Y.

We can therefore suppose that S is a component of $\mathbb{S}'$. Suppose that Λ is realized (up to conjugacy) by a k-th power w^k in $\pi_1\Gamma$, and that the covering map $S \to S^1$ has degree l. Then l divides k, and Y is homotopy equivalent to a wedge $D_{k,l} \vee Y'$, where $D_{k,l}$ is a subcomplex of $C_{k,l}$ and Y' is the subcomplex of Y with the edge e and all 2-cells attached to S removed. We now proceed by induction on the number of 2-cells of Y. ∎

Wise has conjectured that if a 2-complex X has nonpositive immersions, then its fundamental group is coherent. Although Baumslag's conjecture remains open, we do obtain a weaker statement: every finitely generated subgroup of a one-relator group has a finitely generated Schur multiplier.

Corollary 1.6 *Let G be a one-relator group. If $H < G$ is finitely generated, then*

$$\operatorname{rank}\big(H_2(H,\mathbb{Z})\big) \le b_1(H) - 1.$$

In his proof that three-manifold groups are coherent [Sco73], Scott introduces the notion of *indecomposable covers*: If G is a finitely generated freely indecomposable group, then $K \twoheadrightarrow G$ is an indecomposable cover if it does not factor (surjectively) through a free product. The next lemma is a straightforward consequence of the existence of indecomposable covers.

Lemma 1.7 *Let $G = G_1 * \cdots * G_n * \mathbb{F}_k$ be the Grushko decomposition of a finitely generated group G, with G_i freely indecomposable. There is a finitely presented group $H = H_1 * \cdots * H_n * \mathbb{F}_k$ and a surjective homomorphism $\varphi: H \twoheadrightarrow G$ such that $\varphi|_{H_i}: H_i \twoheadrightarrow G_i$ is an indecomposable cover.*

Let X be the presentation complex of a one-relator group G, and let $Y \looparrowright X$ be a covering map corresponding to a finitely generated subgroup H. By a trivial generalization of Stallings' folding technique [Sta83], there is a sequence of immersions of finite complexes obtained by first immersing a graph Y_1 in X and repeatedly adding relations and folding

$$Y_1 \looparrowright Y_2 \looparrowright \cdots \looparrowright Y_n \looparrowright \cdots \looparrowright Y$$

with the property that each immersion $Y_i \looparrowright Y_{i+1}$ induces a surjection on fundamental groups and such that $Y = \varinjlim Y_i$. If H is one-ended, by Lemma 1.7, we can assume that each Y_i has one-ended fundamental group and, by Corollary 1.5, that $\chi(Y_i) \le 0$.

Proof of Corollary 1.6 Let Y and Y_i be the spaces constructed in the previous paragraph. By [Lyn50], both $H_2(G,\mathbb{Z})$ and $H_2(H,\mathbb{Z})$ are torsion-free, so it suffices to show that $b_2(Y) \le b_1(H) - 1$. Combining Corollary 1.5 with Lemma 1.7, we can assume that H is one-ended and that $\chi(Y_i) \le 0$. No Y_i is simply connected and so, since X has NTPI and H is one-ended, $\chi(Y_i) \le 0$ for all i. Since homology commutes with direct limits, it follows that $\operatorname{rank}(H_2(Y,\mathbb{Z})) \le b_1(H) - 1$, as claimed. ■

Our proof of Theorem 1.2 was inspired by the proof of the following theorem of Duncan and Howie. In particular, the punch line in Lemma 2.6 is essentially their proof of [DH91, Lemma 3.1].

The *genus* of an element w in a free group F is the minimal number g so that $w = \Pi_{i=1}^{g}[x_i, y_i]$ has a solution in F, or equivalently, the minimal genus of a once-holed surface mapping into a graph representing F with boundary w.

Theorem ([DH91, Corollary 5.2]) *Let w be an indivisible element in a free group F. Then the genus of w^m is at least $m/2$.*

While this work was in preparation, we learned that Helfer and Wise have also proved the W-cycles conjecture [HW16] and its generalization to staggered presentations (see Remark 3.5).

https://doi.org/10.4153/CMB-2016-078-3 Published online by Cambridge University Press

2 Stackings

2.1 Computing the Characteristic of a Free Group

By a *circle*, we mean a graph homeomorphic to S^1.

Definition 2.1 Let Γ be a finite graph, let $\mathbb{S}$ be a disjoint union of finitely many circles, and let $\Lambda\colon \mathbb{S} \looparrowright \Gamma$ be a map of graphs. Consider the trivial $\mathbb{R}$-bundle $\pi\colon \Gamma \times \mathbb{R} \to \Gamma$. A *stacking* is an embedding $\widehat{\Lambda}\colon \mathbb{S} \hookrightarrow \Gamma \times \mathbb{R}$ such that $\pi\widehat{\Lambda} = \Lambda$.

Although this definition is very simple, it leads to a natural way of estimating the Euler characteristic of a graph.

Let π and ι be the projections of $\Gamma \times \mathbb{R}$ to Γ and $\mathbb{R}$, respectively. Let

$$\mathcal{A}_{\widehat{\Lambda}} = \Big\{ x \in \mathbb{S} \mid \forall y \neq x \big(\Lambda(x) = \Lambda(y) \Rightarrow \iota(\widehat{\Lambda}(x)) > \iota(\widehat{\Lambda}(y)) \big) \Big\},$$
$$\mathcal{B}_{\widehat{\Lambda}} = \Big\{ x \in \mathbb{S} \mid \forall y \neq x \big(\Lambda(x) = \Lambda(y) \Rightarrow \iota(\widehat{\Lambda}(x)) < \iota(\widehat{\Lambda}(y)) \big) \Big\}$$

Intuitively, $\mathcal{A}_{\widehat{\Lambda}}$ is the set of points of $\widehat{\Lambda}(\mathbb{S})$ that one sees if one looks at $\widehat{\Lambda}(\mathbb{S})$ from above, and likewise $\mathcal{B}_{\widehat{\Lambda}}$ is the set of points of $\widehat{\Lambda}(\mathbb{S})$ that one sees from below.

Henceforth, assume that $\Lambda\colon \mathbb{S} \to \Gamma$ is an immersion. The stacking $\widehat{\Lambda}$ is called *good* if $\mathcal{A}_{\widehat{\Lambda}}$ and $\mathcal{B}_{\widehat{\Lambda}}$ each meet every connected component of $\mathbb{S}$. For brevity, we will call a subset $s \subseteq \mathbb{S}$ an *open arc* if it is connected, simply connected, open, and a union of vertices and interiors of edges.

Lemma 2.2 *If Λ is an immersion, then each connected component of $\mathcal{A}_{\widehat{\Lambda}}$ or $\mathcal{B}_{\widehat{\Lambda}}$ is either a connected component of $\mathbb{S}$ or an open arc in $\mathbb{S}$.*

Proof It suffices to prove the lemma for $\mathcal{A}_{\widehat{\Lambda}}$. Let $s \subseteq \mathbb{S}$ be a connected component of $\mathcal{A}_{\widehat{\Lambda}}$. It follows from the definition that s is open. Note also that if one point p in the interior of an edge e is contained in $\mathcal{A}_{\widehat{\Lambda}}$, then the whole interior of e is contained in $\mathcal{A}_{\widehat{\Lambda}}$. This completes the proof. ∎

The next lemma characterizes reducible maps in terms of a stacking; in particular, reducibility is reduced to non-disjointness of $\mathcal{A}_{\widehat{\Lambda}}$ and $\mathcal{B}_{\widehat{\Lambda}}$.

Lemma 2.3 *If $\widehat{\Lambda}$ is a stacking of an immersion $\Lambda\colon \mathbb{S} \to \Gamma$, then $\mathcal{A}_{\widehat{\Lambda}} \cap \mathcal{B}_{\widehat{\Lambda}}$ contains the interior of an edge if and only if Λ is reducible. If $\widehat{\Lambda}$ is a good stacking and $\mathcal{A}_{\widehat{\Lambda}}$ or $\mathcal{B}_{\widehat{\Lambda}}$ contains a circle, then $\widehat{\Lambda}$ is reducible.*

Proof The first assertion is immediate from the definitions. It suffices to prove the second assertion for $\mathcal{A}_{\widehat{\Lambda}}$. Let S be a component of $\mathbb{S}$ contained in $\mathcal{A}_{\widehat{\Lambda}}$. Since $\mathbb{S}$ is good, there is an edge e of S contained in $\mathcal{B}_{\widehat{\Lambda}}$. Therefore, e is contained in both $\mathcal{A}_{\widehat{\Lambda}}$ and $\mathcal{B}_{\widehat{\Lambda}}$. It follows that e is traversed exactly once $\widehat{\Lambda}$, so $\widehat{\Lambda}$ is reducible. ∎

The final lemma of this section is completely elementary, but is the key observation in the proof. It asserts that the number of open arcs in $\mathcal{A}_{\Lambda}$ or $\mathcal{B}_{\Lambda}$ computes the Euler characteristic of the image of Λ.

Lemma 2.4 *Let $\widehat{\Lambda}\colon \mathbb{S} \to \Gamma \times \mathbb{R}$ be a stacking of a surjective immersion $\Lambda\colon \mathbb{S} \to \Gamma$. The number of open arcs in $\mathcal{A}_{\widehat{\Lambda}}$ or $\mathcal{B}_{\widehat{\Lambda}}$ is equal to $-\chi(\Gamma)$.*

Proof As usual, it suffices to prove the lemma for $\mathcal{A}_{\widehat{\Lambda}}$. Let x be a vertex of Γ of valence $v(x)$. Because Λ is surjective, exactly $v - 2$ edges incident at x are covered by open arcs of $\mathcal{A}_{\widehat{\Lambda}}$ that end at x. Therefore, the number of open arcs is

$$\frac{1}{2} \sum_{x \in V(\Gamma)} (v(x) - 2),$$

which is easily seen to be $-\chi(\Gamma)$. ∎

2.2 Computing the Characteristic of a Subgroup

As in the previous section, Γ is a finite graph, $\Lambda\colon \mathbb{S} \looparrowright \Gamma$ is an immersion, and $\widehat{\Lambda}\colon \mathbb{S} \hookrightarrow \Gamma \times \mathbb{R}$ is a stacking. Consider now an immersion of finite graphs $\rho\colon \Gamma' \to \Gamma$, and let $\mathbb{S}'$ be the circular components of the fibre product $\mathbb{S} \times_\Gamma \Gamma'$, which is equipped with a map $\sigma\colon \mathbb{S}' \to \mathbb{S}$ and an immersion $\Lambda'\colon \mathbb{S}' \to \Gamma'$. Note that if $\mathbb{S}'$ is non-empty, then σ is a covering map. In order to prove Theorem 1.2, we would like to estimate the characteristic of Γ' in terms of $\widehat{\Lambda}$.

The stacking $\widehat{\Lambda}$ of Λ naturally pulls back to a stacking $\widehat{\Lambda}'$ of Λ'. More precisely, there is a natural isomorphism

$$(\Gamma \times \mathbb{R}) \times_\Gamma \Gamma' \cong \Gamma' \times \mathbb{R}$$

and the universal property of the fibre bundle defines a map $\widehat{\Lambda}'\colon \mathbb{S}' \to \Gamma \times \mathbb{R}$, so we have the following commutative diagram.

$$\begin{array}{ccccc} & \Gamma' \times \mathbb{R} & \xrightarrow{\widehat{\rho}} & \Gamma \times \mathbb{R} & \\ {\scriptstyle\widehat{\Lambda}'}\nearrow & \downarrow{\scriptstyle\pi'} & {\scriptstyle\widehat{\Lambda}}\nearrow & & \downarrow{\scriptstyle\pi} \\ \mathbb{S}' & \xrightarrow{\sigma} & \mathbb{S} & & \\ {\scriptstyle\Lambda'}\searrow & & {\scriptstyle\Lambda}\searrow & & \\ & \Gamma' & \xrightarrow{\rho} & \Gamma & \end{array}$$

Lemma 2.5 *If $\widehat{\Lambda}$ is a stacking, then $\widehat{\Lambda}'$ is also a stacking. Furthermore, if $\widehat{\Lambda}$ is good, then $\widehat{\Lambda}'$ is also good.*

Proof The proof of the first assertion is a diagram chase, which we leave as an exercise to the reader. The second assertion follows immediately from the observation that $\sigma^{-1}(\mathcal{A}_{\widehat{\Lambda}}) \subseteq \mathcal{A}_{\widehat{\Lambda}'}$ and $\sigma^{-1}(\mathcal{B}_{\widehat{\Lambda}}) \subseteq \mathcal{B}_{\widehat{\Lambda}'}$. ∎

The final lemma in this section estimates the Euler characteristic of Γ' using a stacking of the pullback immersion Λ'. Since all finitely generated subgroups of free groups can be realized by immersions of finite graphs, this can be thought of as an estimate for the rank of a subgroup of a free group; this point of view motivates the title of this subsection.

https://doi.org/10.4153/CMB-2016-078-3 Published online by Cambridge University Press

Lemma 2.6 *If $\widehat{\Lambda}$ is a good stacking, then either $\Lambda' : \mathbb{S}' \to \Gamma'$ is reducible or*

$$-\chi(\Lambda'(\mathbb{S}')) \geq \deg \sigma.$$

Proof Suppose Λ' is not reducible; in particular, Λ' is surjective.

Let e be an edge in $\mathcal{A}_{\widehat{\Lambda}}$ and consider its $\deg \sigma$ preimages $\{e'_j\}$. Since Λ' is not reducible, no component of $\mathcal{A}_{\widehat{\Lambda}'}$ is a circle, by Lemma 2.3, and so every e'_j is contained in an open arc of $\mathcal{A}_{\widehat{\Lambda}'}$.

If $-\chi(\Gamma') < \deg \sigma$ then, by Lemma 2.4 and the pigeonhole principle, two distinct preimages e'_i and e'_j are contained in the same open arc A. But then, for any f an edge of $\mathbb{S}$ contained in $\mathcal{B}_{\widehat{\Lambda}}$ (which again exists because $\widehat{\Lambda}$ is good), A also contains an edge f' that maps to f. Therefore, $\mathcal{A}_{\widehat{\Lambda}'} \cap \mathcal{B}_{\widehat{\Lambda}'}$ contains f', and so Λ' is reducible by Lemma 2.3. See Figure 1. ■

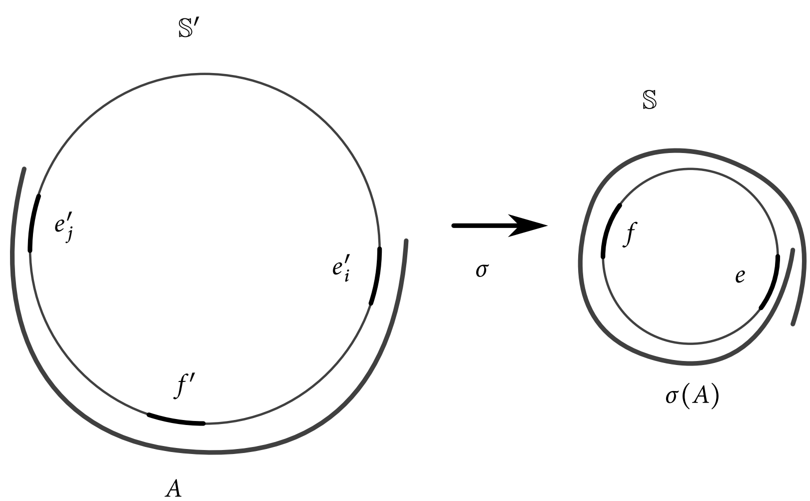

Figure 1: If $-\chi(\Gamma')$ is smaller than the sum of the degrees, then Λ' is reducible.

3 A Tower Argument

In order to apply Lemma 2.6 to prove Theorem 1.2, we need to prove that stackings exist. The proof here employs a cyclic tower argument of the kind used by Brodskiĭ and Howie to prove that one-relator groups are right-orderable and locally indicable [Bro80, How82].

Definition 3.1 Let X be a complex. A *(cyclic) tower* is the composition of a finite sequence of maps

$$X_0 \looparrowright X_1 \looparrowright \cdots \looparrowright X_n = X$$

such that each map $X_i \looparrowright X_{i+1}$ is either an inclusion of a subcomplex or a covering map (resp. a normal covering map with infinite cyclic deck group).

https://doi.org/10.4153/CMB-2016-078-3 Published online by Cambridge University Press

One can argue by induction with towers because of the following lemma of Howie (building on ideas of Papakyriakopoulos and Stallings) [How81].

Lemma 3.2 *Let $Y \to X$ be cellular map of compact complexes. Then there exists a maximal (cyclic) tower map $X' \looparrowright X$ such that $Y \to X$ lifts to a map $Y \to X'$.*

As in the previous, sections let Γ be a graph. To apply a cyclic tower argument, one needs to know that the phenomena of interest are preserved by cyclic coverings. In our case, that control is provided by the following lemma.

Lemma 3.3 *Consider an infinite cyclic cover of a graph Γ. Then there is an embedding $\widetilde{\Gamma} \times \mathbb{R} \hookrightarrow \Gamma \times \mathbb{R}$ such that the diagram*

$$\begin{array}{ccc} \widetilde{\Gamma} \times \mathbb{R} & \xrightarrow{\widetilde{\pi}} & \widetilde{\Gamma} \\ \downarrow & & \downarrow \\ \Gamma \times \mathbb{R} & \xrightarrow{\pi} & \Gamma \end{array}$$

commutes where, as usual, π and $\widetilde{\pi}$ denote coordinate projections onto Γ and $\widetilde{\Gamma}$, respectively. (Note that the embedding $\widetilde{\Gamma} \times \mathbb{R} \hookrightarrow \Gamma \times \mathbb{R}$ is usually not *natural with respect to the coordinate projections onto $\mathbb{R}$.)*

Proof Elements g of the group $\pi_1\Gamma$ act by deck transformations $x \mapsto gx$ on the covering space $\widetilde{\Gamma}$. The infinite cyclic covering $\widetilde{\Gamma} \to \Gamma$ also defines a homomorphism $\pi_1\Gamma \to \mathbb{Z}$, which in turn allows elements g of $\pi_1\Gamma$ to act by translation on $\mathbb{R}$.

Consider the diagonal action of $\pi_1\Gamma$ on $\widetilde{\Gamma} \times \mathbb{R}$. The quotient is homeomorphic to $\Gamma \times \mathbb{R}$. Let $X = \widetilde{\Gamma} \times (-1/2, 1/2) \subset \widetilde{\Gamma} \times \mathbb{R}$. Distinct translates of X are disjoint, and so the map $X \hookrightarrow \widetilde{\Gamma} \times \mathbb{R}$ descends to an embedding $X \hookrightarrow \Gamma \times \mathbb{R}$. Any choice of homeomorphism $(-1/2, 1/2) \cong \mathbb{R}$ identifies X with $\widetilde{\Gamma} \times \mathbb{R}$. It is straightforward to check that the claimed diagram commutes. ∎

We are now ready to prove that stackings exist. A very simple example of a stacking is illustrated in Figure 2.

Lemma 3.4 *Any primitive immersion $\Lambda\colon S^1 \to \Gamma$ has a stacking $\widehat{\Lambda}\colon S^1 \to \Gamma \times \mathbb{R}$.*

Proof Let $\Gamma_0 \looparrowright \Gamma_1 \looparrowright \cdots \looparrowright \Gamma_m = \Gamma$ be a maximal cyclic tower lifting of Λ, and let $\Lambda_n\colon S^1 \to \Gamma_n$ be the lift of Λ to Γ_n. Note that Γ_0 is a circle and Λ_0 is a finite-to-one covering map. Since Λ is primitive, it follows that Λ_0 is a homeomorphism and hence trivially stackable.

Proceeding by induction on n, let $\widehat{\Lambda}_{n-1}\colon S^1 \hookrightarrow \Gamma_{n-1} \times \mathbb{R}$ be a stacking of Λ_{n-1}. If $\Gamma_{n-1} \to \Gamma_n$ is an inclusion of subgraphs then it extends naturally to an inclusion $i\colon \Gamma_{n-1} \times \mathbb{R} \hookrightarrow \Gamma_n \times \mathbb{R}$, and so $\widehat{\Lambda} = i \circ \widehat{\Lambda}_{n-1}$ is a stacking.

Suppose therefore that $\Gamma_{n-1} \to \Gamma_n$ is an infinite cyclic covering map, and let $i\colon \Gamma_{n-1} \times \mathbb{R} \to \Gamma_n \times \mathbb{R}$ be the embedding provided by Lemma 3.3. Then $\widehat{\Lambda}_n = i \circ \widehat{\Lambda}_{n-1}$ is an embedding $S^1 \hookrightarrow \Gamma_n \times \mathbb{R}$, and a simple diagram chase confirms that $\widehat{\Lambda}_n$ is a lift of Λ_n. This completes the proof. ∎

Remark 3.5 Note that any stacking of a map of a single circle is automatically good. Lemma 3.4 (also implicit in [HW16]) holds for graphs and immersions associated with staggered presentations.

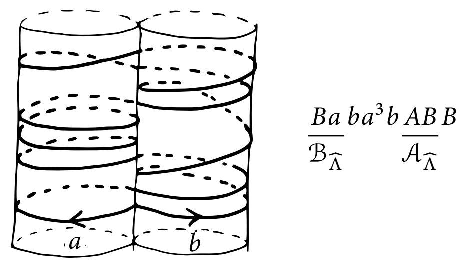

Figure 2: A stacking of the word $Baba^3bABB$.

Let $L = \langle x_1, \dots, x_n \mid w \rangle$ be a one-relator group, where w is a cyclically reduced nonperiodic word $w = x_{i_1} \cdots x_{i_m}$ in the x_i. Duncan and Howie use right-orderability of L to assign heights to the (distinct, by [How82, Corollary 3.4]) elements $a_0 = 1$, $a_j = x_{i_1} \cdots x_{i_j}$, $j < m$, in L in the same way we use the embedding $\widehat{\Lambda}$ to find open arcs that remain above ($\mathcal{A}$) or below ($\mathcal{B}$) every point of S^1 with the same image in Γ. Lemma 3.4 is equivalent to the existence of a right-invariant pre-order on L that distinguishes between the elements a_j. Lemma 3.4 is also closely related to the main theorem of [Far76].

Our main theorem is now a quick consequence of Lemmas 2.6 and 3.4.

Proof of Theorem 1.2 Let Γ, Γ', etc., be as in Theorem 1.2, and let $\widehat{\Lambda}$ be the stacking provided by Lemma 3.4. Since S^1 is connected, the stacking $\widehat{\Lambda}$ is automatically good. By hypothesis Λ' is not reducible, and therefore by Lemma 2.6, $-\chi(\Gamma') \geq -\chi(\Lambda'(\mathbb{S}')) \geq \deg \sigma$, as claimed. ■

Acknowledgments The authors thank Warren Dicks for pointing out an error in an ancillary argument of the first version of this paper.

References

[Bau74] G. Baumslag, *Some problems on one-relator groups*. In: Proceedings of the Second International Conference on the Theory of Groups (Australian Nat. Univ., Canberra, 1973), Lecture Notes in Math., 372, Springer, Berlin, 1974, pp. 75–81.

[Bro80] S. D. Brodskiĭ, *Equations over groups and groups with one defining relation*. Uspekhi Mat. Nauk **35**(1980), no. 4(214), 183.

[DH91] A. J. Duncan and J. Howie, *The genus problem for one-relator products of locally indicable groups*. Math. Z. **208**(1991), no. 2, 225–237. http://dx.doi.org/10.1007/BF02571522

[Far76] F. T. Farrell, *Right-orderable deck transformation groups*. Rocky Mountain J. Math. **6**(1976), no. 3, 441–447. http://dx.doi.org/10.1216/RMJ-1976-6-3-441

[HW16] J. Helfer and D. T. Wise, *Counting cycles in labeled graphs: the nonpositive immersions property for one-relator groups*. Int. Math. Res. Not. IMRN **2016**, no. 9, 2813–2827. http://dx.doi.org/10.1093/imrn/rnv208

[How81] J. Howie, *On pairs of 2-complexes and systems of equations over groups*. J. Reine Angew. Math. **324**(1981), 165–174. http://dx.doi.org/10.1515/crll.1981.324.165

[How82] ———, *On locally indicable groups*. Math. Z. **180**(1982), no. 4, 445–461. http://dx.doi.org/10.1007/BF01214717

[Lyn50] R. C. Lyndon, *Cohomology theory of groups with a single defining relation*. Ann. of Math. (2) **52**(1950), 650–665. http://dx.doi.org/10.2307/1969440

[Sco73] G. P. Scott, *Finitely generated 3-manifold groups are finitely presented*. J. London Math. Soc. (2) **6**(1973), 437–440. http://dx.doi.org/10.1112/jlms/s2-6.3.437

[Sta83] J. R. Stallings, *Topology of finite graphs*. Invent. Math. **71**(1983), no. 3, 551–565. http://dx.doi.org/10.1007/BF02095993

[Wis05] D. T. Wise, *The coherence of one-relator groups with torsion and the Hanna Neumann conjecture*. Bull. London Math. Soc. **37**(2005), no. 5, 697–705. http://dx.doi.org/10.1112/S0024609305004376

University College London, London, UK
e-mail: l.louder@ucl.ac.uk

University of Cambridge, Cambridge, UK
e-mail: h.wilton@maths.cam.ac.uk