SETS OF INTEGERS CONTAINING NO n TERMS IN GEOMETRIC PROGRESSION

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1. Introduction. R. A. Rankin [3] considered the problem of finding, for each integer $n \geqq 3$, a sequence of positive integers containing no n-term geometric progression. He constructed such sets B_n having asymptotic density

$$A_n = \frac{1}{\zeta(n-1)} \prod_{k=1}^{\infty} \frac{\zeta\{(2n-3)^k\}}{\zeta\{(n-1)(2n-3)^k\}}.$$

For example $A_3 \doteqdot 0{\cdot}71975$, $A_4 \doteqdot 0{\cdot}8626$, and $A_n \to 1$ as $n \to \infty$.

Let $H(n)$ denote the class of all sequences of positive integers that contain no n-term geometric progression. Rankin wondered whether A_n is the highest density possible for members of $H(n)$. In this paper we find members having higher density, in the cases $n \geqq 4$, and also find upper estimates for the possible density in all cases $n \geqq 3$.

If E is a set of non-negative integers containing 0, let $Q(E)$ denote the set of all integers N of the form

$$N = \prod_{i=1}^{\infty} p_i^{a_i},$$

where p_i is the ith prime and each a_i is chosen from E. We call $Q(E)$ the set of integers developed from the *exponent choice set E*. We shall simplify the notation by writing

$$Q(\{a, b, \ldots\}) = Q(a, b, \ldots).$$

If E contains no n-term arithmetic progression, then $Q(E)$ contains no n-term geometric progression. Rankin's B_n is the set $Q(C_n)$, where C_n is the set of all non-negative integers which, when expressed in the scale of $2n-3$, contain no digit greater than $n-2$.

For any real x and set Q of positive integers we let $Q(x)$ denote the number of elements of Q that do not exceed x. If Q has asymptotic density we shall denote it by $D(Q)$.

In Section 2, we either estimate or find the density of a member $Q(E_n)$ of $H(n)$ after proving the following lemma:

LEMMA 1. *If E is any exponent choice set, then* $D(Q(E))$ *exists.*

For each $n \geqq 4$ we find a set E_n such that $Q(E_n) \in H(n)$ and

$$D(Q(E_n)) > A_n = D(Q(C_n)).$$

In fact for each $n \geqq 4$ we observe that there are many sets having these properties of E_n. In Section 3 we find an upper estimate for the possible density of members of $H(n)$ for each $n \geqq 3$. A table comparing some few of the densities we obtain with the corresponding upper estimates is included at the end of the paper.

† These results are contained in the author's Ph.D. thesis written at the University of Alberta in 1967 under the direction of Leo Moser.

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2. Members of $H(n)$ with density exceeding $A_n (n \geqq 4)$. We prove the following theorem.

THEOREM 1. (i) *If n is prime, there exists $Q \in H(n)$ such that*

$$D(Q) = \frac{\zeta(n)}{\zeta(n-1)\zeta\{(n-1)n\}}. \tag{1}$$

(ii) *If n is composite, there exists $Q \in H(n)$ such that*

$$D(Q) > \frac{\zeta(n)}{\zeta(n-1)\zeta(hn)} - \left(\frac{1}{\zeta(hn-1)} - \frac{1}{\zeta(hn-h)}\right), \tag{2}$$

where h is the smallest prime divisor of n.

(iii) *There exists $Q \in H(4)$ such that*

$$D(Q) > 0{\cdot}8952. \tag{3}$$

The estimate (3) is somewhat larger than that provided by (2) with $n = 4$. We give the proof that the respective densities exceed A_n in Section 2.1. We first prove part (iii) of the theorem.

Proof of (iii). The set

$$E_4': 0, 1, 2, 4, 5, 7, 8, 9$$

contains no 4-term arithmetic progression. We shall find a lower estimate for $D(Q(E_4'))$. The set $Q(E_4')$ will not contain m if and only if there is a prime p such that

$$p^3 \mid m \quad \text{and} \quad p^4 \nmid m, \quad \text{or} \quad p^6 \mid m \quad \text{and} \quad p^7 \nmid m, \quad \text{or} \quad p^{10} \mid m.$$

Given a prime p, the number of such m not exceeding x is

$$K(x, p) = \left[\frac{x}{p^3}\right] - \left[\frac{x}{p^4}\right] + \left[\frac{x}{p^6}\right] - \left[\frac{x}{p^7}\right] + \left[\frac{x}{p^{10}}\right],$$

and the density of the set of such numbers m is

$$K(p) = \lim_{x \to \infty} \frac{K(x, p)}{x} = \frac{1}{p^3} - \frac{1}{p^4} + \frac{1}{p^6} - \frac{1}{p^7} + \frac{1}{p^{10}}.$$

By the principle of inclusion and exclusion,

$$D(Q(E_4')) = 1 - \sum_p K(p) + \sum_{p<q} K(p)K(q) - \sum_{p<q<r} K(p)K(q)K(r) + \dots, \tag{4}$$

where the sums are respectively taken over all the tuples $(p), (p, q), (p, q, r), \dots$ of primes satisfying the indicated inequalities. Since

$$\sum_p K(p) < \sum_p \frac{1}{p^3} < 1,$$

$$\sum_{\substack{p<q<r<\dots \\ (j \text{ primes})}} (K(p)K(q)K(r)\dots) < \sum_p K(p) \sum_{\substack{q<r<\dots \\ (j-1 \text{ primes})}} (K(q)K(r)\dots) < \sum_{\substack{q<r<\dots \\ (j-1 \text{ primes})}} (K(q)K(r)\dots), \tag{5a}$$

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and

$$\sum_{\substack{p<q<r<\dots \\ (j \text{ primes})}} (K(p)K(q)K(r)\dots) < (\sum_p K(p))^j = o(1) \quad \text{as} \quad j \to \infty. \tag{5b}$$

Hence the series (4) converges. We have estimated the first four terms and found that

$$D(Q(E_4')) > 1 - 0{\cdot}107569 + 0{\cdot}002875 - 0{\cdot}000023 > 0{\cdot}8952.$$

Before proceeding with the remaining parts of the theorem, we prove Lemma 1, using the method developed above.

Proof of Lemma 1. Given an exponent choice set E and a prime p, we can define a quantity $K_E(p)$ corresponding to $K(p)$ above. If E contains 1, then the series S in K_E corresponding to (4) converges. For, the first term of $K_E(p)$ will be $1/p^a$ with $a \geqq 2$, so that $\sum_p K_E(p) < \sum_p 1/p^2 < 1$; hence we can obtain the inequalities (5) with K_E in place of K. Therefore $D(Q(E))$ exists and is the sum of the series S. If $1 \notin E$, then $\sum_p K_E(p)$ diverges, for the first term of $K_E(p)$ is $1/p$. However, in this case $D(Q(E)) = 0$, because $Q(E) \subset Q(0, 2, 3, 4, \dots)$, the set of squarefull numbers, and this set has density 0. We refer the reader to the solution by P. T. Bateman [2] of a problem proposed by D. J. Newman which shows that, if $Q = Q(0, 2, 3, 4, \dots)$, then $Q(x) = O(x^{\frac{1}{2}})$. Hence Lemma 1.

Proof of (i). If n is prime, then the set

$$E_n: \quad \begin{array}{cccc} 0, & 1, & 2, \dots, & n-2, \\ n, & n+1, & n+2, \dots, & 2n-2, \\ 2n, & 2n+1, & 2n+2, \dots, & 3n-2, \\ \cdot & \cdot & \cdot & \cdot \\ (n-2)n, & (n-2)n+1, & (n-2)n+2, \dots, & (n-1)n-2 \end{array}$$

contains no n-term arithmetic progression. For if E_n contained such progressions, one of them would have its first term among $0, 1, \dots, n-2$, and all of them would have common difference less than n. However, if $0 \leqq a \leqq n-2$ and $1 \leqq d \leqq n-1$, some term of the progression

$$a, \quad a+d, \quad a+2d, \dots, a+(n-1)d$$

is congruent to -1 modulo n and hence is not in E_n. This is because $(d, n) = 1$, whence $0, d, 2d, \dots, (n-1)d$ form a complete residue system modulo n.

Now, with $s = \sigma + it$ (σ, t real),

$$\sum_{N \in Q(E_n)} \frac{1}{N^s} = \prod_p \left(\sum_{a \in E_n} \frac{1}{p^{as}} \right) = \prod_p \frac{1 - 1/p^{(n-1)s}}{1 - 1/p^s} \frac{1 - 1/p^{(n-1)ns}}{1 - 1/p^{ns}} = \frac{\zeta(s)\zeta(ns)}{\zeta\{(n-1)s\}\zeta\{(n-1)ns\}}. \tag{6}$$

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We now employ the following lemma (see Ayoub [**1**]):

LEMMA 2. *If*

$$f(s)=\sum_{N=1}^{\infty}\frac{a(N)}{N^s}\quad \text{and}\quad \lim_{x\to\infty}\frac{\sum_{N\leqq x}a(N)}{x}=A,$$

then

$$\lim_{s\to 1}(s-1)f(s)=A. \tag{7}$$

Defining

$$a(N)=\begin{cases}1 \text{ if } N\in Q(E_n),\\ 0 \text{ otherwise,}\end{cases}$$

we have

$$\sum_{N\in Q(E_n)}\frac{1}{N^s}=\sum_{N=1}^{\infty}\frac{a(N)}{N^s},$$

and with $Q=Q(E_n)$,

$$Q(x)=\sum_{N\leqq x}a(N).$$

Lemma 1 assures us that $D(Q(E_n))=\lim_{x\to\infty}Q(x)/x$ exists, and by Lemma 2 we can find this limit from (7). It is the residue of (6) at the simple pole $s=1$. Thus

$$D(Q(E_n))=\frac{\zeta(n)}{\zeta(n-1)\zeta\{(n-1)n\}},$$

and hence part (i) of Theorem 1.

Note that we could adjoin integers to the above set E_n and still have a set free of n-term progressions, thus obtaining an even denser member of $H(n)$.

Proof of (ii). Suppose that n is composite, and that h is the smallest prime divisor of n. Then the set

$$E_n:\quad\begin{array}{llll} 0, & 1, & 2,\ldots, & n-2,\\ n, & n+1, & n+2,\ldots, & 2n-2,\\ \cdots & \cdots & \cdots & \cdots\\ (h-2)n, & (h-2)n+1, & (h-2)n+2,\ldots, & (h-1)n-2,\\ (h-1)n, & (h-1)n+1, & (h-1)n+2,\ldots, & hn-h-1\end{array}$$

contains no n-term arithmetic progression. For consider any progression with first term a and common difference d such that

$$0\leqq a<a+d<a+2d<\ldots<a+(n-1)d\leqq h(n-1)-1.$$

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Evidently $d<h$. Hence $(d,n)=1$. Therefore $a, a+d, \ldots, a+(n-1)d$ form a complete residue system modulo n, whence this progression contains one of $n-1, 2n-1, \ldots, (h-1)n-1$, and is not contained in E_n.

We shall obtain the lower estimate (2) for $D(Q(E_n))$. The numbers $hn-h, hn-h+1, \ldots, hn-2$ cannot be included in E_n since each of these is the nth term of an arithmetic progression having difference h and first $n-1$ terms in E_n. Using Lemma 2 we find that

$$D(Q(E_n \cup \{hn-h, hn-h+1, \ldots, hn-2\})) = \frac{\zeta(n)}{\zeta(n-1)\zeta(hn)}. \tag{8}$$

and (2) will follow when we make allowance for the exclusion of $hn-1, hn-h+1, \ldots, hn-2$.

Given any exponent choice set E and set of positive integers F disjoint from E, we shall denote by $Q(E \& F)$ the set of integers developed from the exponent choice set $E \cup F$ with always at least one element from each of E and F included among the exponents chosen (we always include the 0 from E). Then

$$Q(E) \cup Q(E \& F) = Q(E \cup F). \tag{9}$$

By Lemma 1, $D(Q(E))$ and $D(Q(E \cup F))$ exist. Therefore, since the sets on the left side of (9) are disjoint, $D(Q(E \& F)$ exists and

$$D(Q(E)) + D(Q(E \& F)) = D(Q(E \cup F)). \tag{10}$$

Now, writing $G = \{hn-h, hn-h+1, \ldots, hn-2\}$, we have by (10)

$$D(Q(E_n)) = D(Q(E_n \cup G)) - D(Q(E_n \& G)). \tag{11}$$

Furthermore,

$$Q(E_n \& G) \subset Q(\{0,1,2,\ldots,hn-h-1\} \& G) = Q(0,1,2,\ldots,hn-2) - Q(0,1,2,\ldots,hn-h-1),$$

where we have applied (9). Hence, by (10),

$$D(Q(E_n \& G)) \leqq D(Q(0,1,2,\ldots,hn-2)) - D(Q(0,1,2,\ldots,hn-h-1))$$
$$= \frac{1}{\zeta(hn-1)} - \frac{1}{\zeta(hn-h)}.$$

Hence, from (8) and (11), the result follows, and the proof of Theorem 1 is now complete.

One can again adjoin integers to E_n, in the case n is composite, and obtain a still denser member of $H(n)$. For example if n is even, and l is the smallest prime divisor of $n-1$, then the set

$$E_n' = \{0,1,2,\ldots,l(n-1)\} - \{n-1, 2(n-1), 3(n-1), \ldots, (l-1)(n-1)\}$$

contains no n-term arithmetic progression. We found earlier that $D(Q(E_4')) > 0{\cdot}8952$. By comparison the estimate (2) in the case $n=4$, found using $E_4 = \{0,1,2,4,5\}$, is $0{\cdot}88796$ to five places, and estimating from above, we find using (10) that

$$D(Q(E_4)) < 1/\zeta(6) - [1/\zeta(4) - 1/\zeta(3)] < 0{\cdot}89093.$$

2.1. Comparison of the densities. We shall show that

$$\frac{\zeta(n)}{\zeta(n-1)\zeta(2n)}-\left[\frac{1}{\zeta(2n-1)}-\frac{1}{\zeta(2n-2)}\right]>A_n \tag{12}$$

for $n \geqq 4$, and that

$$\frac{\zeta(n)}{\zeta(n-1)\zeta(hn)}-\left[\frac{1}{\zeta(hn-1)}-\frac{1}{\zeta(hn-h)}\right]>\frac{\zeta(n)}{\zeta(n-1)\zeta\{(h-1)n\}} \tag{13}$$

for $3 \leqq h \leqq n-2$. From (12) it will follow that the density in Theorem 1(ii), in the case n is even, exceeds A_n. If n is an odd composite number and h is the smallest prime divisor of n, then $3 \leqq h \leqq \sqrt{n}$ and (13) will hold. Furthermore in this case the right side of (13), and hence that of (2), exceeds $\zeta(n)/(\zeta(n-1)\zeta(2n))$, as does the quantity $\zeta(n)/[\zeta(n-1)\zeta\{(n-1)n\}]$ of (1) in the case n is prime. Hence by (12) the densities in Theorem 1(i), (ii) will have been shown to exceed A_n in any case.

We use the following easily proved lemma:

Lemma 3. *For integers $a>1$ and $b>0$,*

$$\zeta(a+b)<1+\frac{\zeta(a)-1}{2^b}.$$

To prove (12) we first show that

$$\prod_{k=1}^{\infty}\frac{\zeta\{(2n-3)^k\}}{\zeta\{(n-1)(2n-3)^k\}}<\frac{\zeta(2n-4)}{\zeta(2n^2)} \quad \text{for} \quad n \geqq 3 \tag{14}$$

and then that

$$\frac{2n-4}{\zeta(2n^2)}<\frac{\zeta(n)}{\zeta(2n)}-\zeta(n-1)\left[\frac{1}{\zeta(2n-1)}-\frac{1}{\zeta(2n-2)}\right] \tag{15}$$

for $n \geqq 5$. This will imply (12) for $n \geqq 5$. For $n=4$ we find, using tables, that the left side of (12) exceeds $0{\cdot}88796$ while $A_4<0{\cdot}8627$.

We observe that

$$\zeta(2n^2)<\zeta\{(n-1)(2n-3)\}<\prod_{k=1}^{\infty}\zeta\{(n-1)(2n-3)^k\},$$

so that if we prove

$$\prod_{k=1}^{\infty}\zeta\{(2n-3)^k\}<\zeta(2n-4) \tag{16}$$

for $n \geqq 3$ we shall have (14). Writing $m=2n-3$, we shall prove

$$\zeta(m-1)>1+2(\zeta(m)-1)>\prod_{k=1}^{\infty}\zeta(m^k) \tag{17}$$

for $m \geqq 3$, and this will yield (16). The first inequality of (17) is immediate from Lemma 3.

Again, from the lemma,

$$\prod_{k=1}^{\infty} \zeta(m^k) < \prod_{k=1}^{\infty}\left(1+\frac{\zeta(m)-1}{2^{m^k-m}}\right),$$

and writing $x = 2^m(\zeta(m)-1)$ one can show by comparing logarithms that

$$\prod_{k=1}^{\infty}\left(1+\frac{x}{2^{m^k}}\right) < 1+\frac{x}{2^{m-1}}$$

for $m \geqq 3$, whence follows the second inequality of (17).

The inequality (15) is equivalent to

$$\frac{\zeta(2n-4)}{\zeta(2n^2)} + \zeta(n-1)\frac{\zeta(2n-2)-\zeta(2n-1)}{\zeta(2n-1)\zeta(2n-2)} < \frac{\zeta(n)}{\zeta(2n)}, \tag{18}$$

and the second term on the left side here is less than $\zeta(n-1)(\zeta(2n-2)-1)/\zeta^2(2n)$. Replacing that second term by this quantity and multiplying through by $\zeta^2(2n)\zeta(2n^2)$, we find that the left side of the resulting inequality is less than

$$\zeta(2n)\zeta^2(2n-4) + \zeta(2n)\zeta(n-1)(\zeta(2n-2)-1).$$

For $n \geqq 6$ one can show by Lemma 3 that this quantity is less than $\zeta(n)\zeta(2n)\zeta(2n^2)$, giving (18) for $n \geqq 6$, while for $n = 5$, (18) can be proved using tables. Hence (12).

The proof of (13) involves manipulations similar to those in the proof of (15).

3. Upper estimates.

Let

$$M_n = \sup\left\{\lim_{x\to\infty} \frac{Q(x)}{x} \,\middle|\, Q \in H_1(n)\right\},$$

where $H_1(n)$ is that subset of $H(n)$ whose members have asymptotic density. We prove the following theorem.

THEOREM 2. *For every* $n \geqq 3$,

$$M_n \leqq 1 - \frac{1}{2^n - 1}.$$

The proof of Theorem 2 depends on Theorem 3 below, which is concerned with geometric progressions of integral ratio r. Let I denote the set of positive integers. For any integer $r > 1$ let $R = R(n, r)$ denote the set of geometric progressions in I of n terms and ratio r, and let $H(n, r)$ denote the class of all sequences in I that contain no progression of R. Further, let $H_1(n, r)$ be the class of all sequences $Q \in H(n, r)$ for which $\lim_{x\to\infty} Q(x)/x$ exists. We define

$$M^*_{n,r} = \sup\left\{\limsup_{x\to\infty} \frac{Q(x)}{x} \,\middle|\, Q \in H(n, r)\right\},$$

$$M_{n,r} = \sup\left\{\lim_{x\to\infty} \frac{Q(x)}{x} \,\middle|\, Q \in H_1(n, r)\right\}.$$

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THEOREM 3. (i) *No integer appears in more than n progressions of R.*

(ii) *There is exactly one subset of I with the property: Each element of the set appears in n progressions of R and each progression of R contains exactly one element of the set.*

(iii) *Let S be the set in* (ii). *If* $T \subset I$ *and* $I - T \in H(n,r)$, *then* $T(x) \geqq S(x)$ *for every* x.

(iv) $I - S \in H_1(n,r)$ *and* $\lim_{x\to\infty} S(x)/x = (r-1)/(r^n-1)$.

(v) $M_{n,r} = M_{n,r}^* = 1-(r-1)/(r^n-1)$.

If analogously to $M_{n,r}^*$ we defined

$$M_n^* = \sup\left\{\limsup_{x\to\infty} \frac{Q(x)}{x} \,\middle|\, Q \in H(n)\right\},$$

we might expect that similarly $M_n = M_n^*$. Perhaps this would be so if one considered only geometric progressions with integral ratio, but it seems doubtful in the general case.

Proof of Theorem 3. Let us separate R into families F_k of progressions:

$$F_k: \quad \begin{array}{llll} k, & kr, & kr^2, & \ldots, kr^{n-1}; \\ kr, & kr^2, & kr^3, & \ldots, kr^n; \\ \cdot & \cdot & \cdot & \cdot \\ kr^{n-1}, & kr^n, & kr^{n+1}, & \ldots, kr^{2n-2}; \\ kr^n, & kr^{n+1}, & kr^{n+2}, & \ldots, kr^{2n-1}; \\ \cdot & \cdot & \cdot & \cdot \end{array}$$

where $k \in I$, $r \nmid k$. Clearly $\bigcup_{\substack{k=1 \\ r\nmid k}}^{\infty} F_k = R$, and no integer appears in more than n progressions of one family. Furthermore, if V_k denotes the set of all integers appearing in the progressions of F_k, then the sets V_k are pairwise disjoint. For if $kr^u = lr^v$ and $k \neq l$, then $u \neq v$ and either $r \mid k$ or $r \mid l$. (i) follows.

The integers $kr^{n-1}, kr^{2n-1}, kr^{3n-1}, \ldots$ each appear in exactly n of the progressions of F_k, and each progression of F_k contains exactly one of them; it is clear that this is the only set of integers with this property. Since the V_k are pairwise disjoint, the set

$$S = \bigcup_{\substack{k=1 \\ r\nmid k}}^{\infty} \{kr^{n-1}, kr^{2n-1}, kr^{3n-1}, \ldots\}$$

has the property required in (ii). It is clear that $I - S \in H(n,r)$.

Proceeding to (iii), we observe that if each F_k is separated into blocks of n progressions each, starting with the first member of the family, then in order that $I - T \in H(n,r)$, T must contain at least one integer from each block of each family. Since S contains exactly one integer from each block, $T(x) \geqq S(x)$ for every x.

The number of integers

$$r^{in-1}, 2r^{in-1}, \ldots, (r-1)r^{in-1}, (r+1)r^{in-1}, \ldots \tag{19}$$

not exceeding x is

$$a_i = \left[\frac{x-r^{in-1}}{r^{in}}+1\right]+\left[\frac{x-2r^{in-1}}{r^{in}}+1\right]+\ldots+\left[\frac{x-(r-1)r^{in-1}}{r^{in}}+1\right]$$

$$=\left[\frac{x}{r^{in}}+\frac{r-1}{r}\right]+\left[\frac{x}{r^{in}}+\frac{r-2}{r}\right]+\ldots+\left[\frac{x}{r^{in}}+\frac{1}{r}\right]$$

provided that $1 \leqq i \leqq m = \left[\dfrac{\log_r x+1}{n}\right]$, while if $i > m$, the integers (19) all exceed x. Hence

$$S(x) = \sum_{i=1}^{m} a_i = \sum_{i=1}^{m} \frac{(r-1)x}{r^{in}} + O(\log x),$$

so that S has density

$$\lim_{x\to\infty} \frac{S(x)}{x} = \sum_{i=1}^{\infty} \frac{r-1}{r^{in}} = \frac{r-1}{r^n-1},$$

and we have proved (iv).

From (iv), $M_{n,r} \geqq 1-(r-1)/(r^n-1)$. On the other hand, by (iii), if $U \in H(n,r)$, then $U(x) \leqq [I-S](x)$ for every x, so that $M^*_{n,r} \leqq 1-(r-1)/(r^n-1)$. Since $M_{n,r} \leqq M^*_{n,r}$ by definition, (v) follows.

Proof of Theorem 2. The theorem follows immediately from the observation that $M_n \leqq M_{n,r}$ for any r. We choose $r = 2$ since $M_{n,r}$ is smallest for that value of r.

In the case $n = 3$ we have obtained the better estimate

$$M_3 < 0.8339. \tag{20}$$

This compares with the estimate $6/7 = 0{\cdot}8571\ldots$ of Theorem 2. We find (20) by considering what integers must be removed from I in order to eliminate, in addition to all 3-term progressions of ratio 2, certain progressions of ratio 3. The details are too lengthy to be included here.

The most dense members of $H(n)$ discussed in Sections 1 and 2 provide lower estimates for M_n. We compare these with our upper esimates for M_n for some few values of n:

n	lower estimate	upper estimate
3	$A_3 = 0{\cdot}7197\ldots$	$0{\cdot}8339$
4	$0{\cdot}8952$	$14/15 = 0{\cdot}933\dot{3}$
5	$0{\cdot}9580$	$30/31 = 0{\cdot}9677\ldots$
8	$0{\cdot}9957$	$254/255 = 0{\cdot}9960\ldots$.

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