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Part 7. Stochastic geometry

SIZE DISTRIBUTIONS IN RANDOM TRIANGLES

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SIZE DISTRIBUTIONS IN RANDOM TRIANGLES

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Abstract

The random triangles discussed in this paper are defined by having the directions of their sides independent and uniformly distributed on $(0, \pi)$. To fix the scale, one side chosen arbitrarily is assigned unit length; let a and b denote the lengths of the other sides. We find the density functions of a/b, $\max\{a, b\}$, $\min\{a, b\}$, and of the area of the triangle, the first three explicitly and the last as an elliptic integral. The first two density functions, with supports in $(0, \infty)$ and $(\frac{1}{2}, \infty)$, respectively, are unusual in having an infinite spike at 1 which is interior to their ranges (the triangle is then isosceles).

Keywords: Random directions

2010 Mathematics Subject Classification: Primary 60D05
Secondary 52A22; 53C65

1. Introduction

The problem discussed in this paper is classical, yet it has apparently escaped attention: we found no relevant discussion in Mathai (1999). By 'classical' we mean that its topic has a Croftonian flavour as in Kendall and Moran (1963, p. 5); this can be contrasted with 'modern' stochastic geometry where measure-theoretic techniques are more common (cf. Kendall's (1974) introduction to the collection of papers by Davidson and others). Our solution is also purely classical.

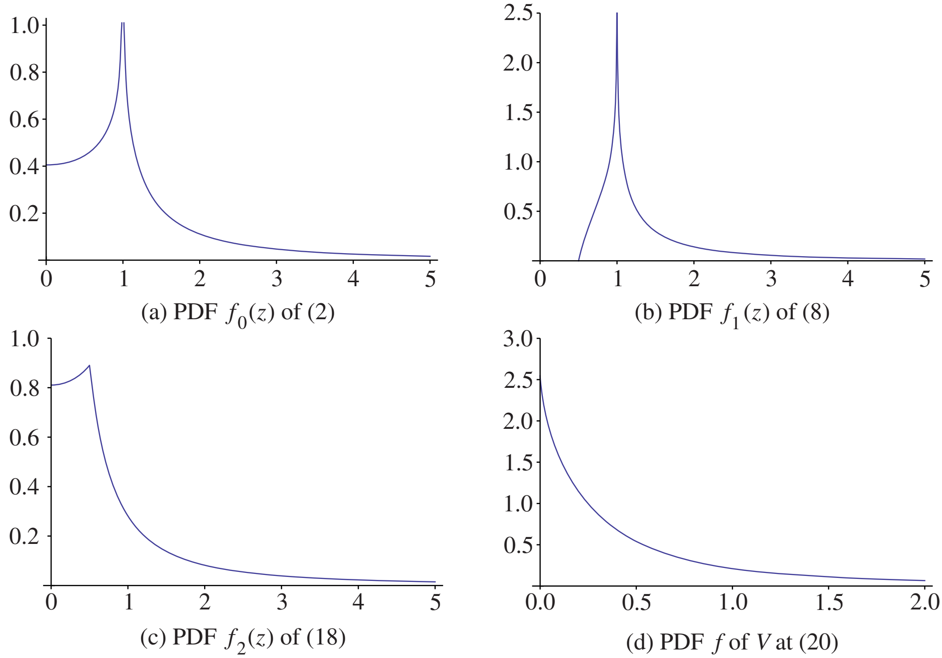
Our problem concerns *random triangles* defined by three coplanar lines that intersect and whose directions are independent and uniform on $(0, \pi)$. Almost surely, no two lines are parallel, so every pair of lines has a finite intersection point: these points are the vertices of our random triangle. Richard Cowan (email communication to DJD) remarked that, for 'random triangles', matters of shape have been of more concern than the relative lengths of the sides.

Given the angles α, β, and γ of a triangle, the sine rule enables us to determine the relative lengths a, b, and c of its sides:

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}. \tag{1}$$

Shape is described by just two angles α and β, say (because $\alpha + \beta + \gamma = \pi$), and these two angles transform directly into the relative length $Z_0 = a/b$ of two sides, so we might as well put $c = 1$. In Proposition 1 we find the probability density function (PDF) f_0 of Z_0 as just defined.

Our interest in the problem arose from a different setting (see Daley *et al.* (2014)), and is akin to the following. Given the side length $c = 1$ say, find the distribution of the larger of a and b when the angles α and β can be regarded as being uniformly and independently distributed on $(0, \frac{1}{2}\pi)$ and $(0, \pi)$, respectively, where we require the triangle to be constructed on one side or the other of the given side c according to whether $\alpha + \beta$ is less than or greater than π (and in the latter event, it is not β but $\pi - \beta$ that is the angle internal to the triangle, but the sine rule as



(a) PDF $f_0(z)$ of (2)

(b) PDF $f_1(z)$ of (8)

(c) PDF $f_2(z)$ of (18)

(d) PDF f of V at (20)

FIGURE 1: PDFs.

at (1) remains). In Proposition 2 we find the PDF f_1 of $Z_1 := \max(a, b)$, and in Proposition 3 deduce the PDF f_2 of $Z_2 := \min(a, b)$.

The three PDFs are plotted in Figure 1(a), (b), and (c). Each PDF involves a logarithmic function of its argument z. All three distributions are heavy tailed: all moments of positive order less than 1 are finite, and the means are infinite. Z_0 and Z_1 have PDFs with infinite spikes at $z = 1$ when the triangle from which they come is isosceles. Z_0 and $1/Z_0$ have the same distribution, so their median equals 1. $\mathbb{P}\{Z_1 \le 1\} = \frac{1}{3} = \mathbb{P}\{Z_2 \ge 1\}$, Z_1 and Z_2 have medians approximately 1.145 and 0.608, and $\mathbb{P}\{Z_1 \le 2\} = \frac{2}{3} + \frac{1}{2}(\log 2)^2/\pi^2 = 0.7640$. Some further numerical illustrations of the tail behaviour are shown in Table 1 in Section 6.

In Section 5 we apply the conditional probability argument used to find the density functions of the Z_i to study the area V of the random triangle as defined. The integral for its density function leads to an elliptic integral from which we have computed the PDF of V; see Figure 1(d). In principle, there is a power series representation for the PDF; we give the simplest case.

2. The simplest random triangle

Proposition 1. *For a random triangle as defined, the PDF f_0 of $Z_0 := a/b$ is given by*

$$f_0(z) = \frac{2}{\pi^2 z} \log \frac{z+1}{|1-z|}, \qquad 0 < z < \infty. \tag{2}$$

Proof. Write $\psi = \alpha$ for the angle we choose to constrain by symmetry to $0 \le \psi \le \frac{1}{2}\pi$, and $\theta = \beta$ lying in $(0, \pi)$. From (1), it follows that $Z_0 = \sin\psi / \sin\theta$, and then by the symmetry about $\frac{1}{2}\pi$ of the sine function we can also constrain θ to $(0, \frac{1}{2}\pi)$ in seeking the distribution of Z_0 which is a function on the square $\mathcal{R}_0 := \{(\theta, \psi) \in (0, \frac{1}{2}\pi) \times (0, \frac{1}{2}\pi)\}$, where (θ, ψ) has the joint density function $\mathrm{d}\theta\, \mathrm{d}\psi / \frac{1}{4}\pi^2$. This makes it easy to evaluate $\mathbb{P}\{Z_0 \le z\}$ via a

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conditional probability argument:

$$\frac{1}{2}\pi\mathbb{P}\{Z_0 \le z\} = \int_0^{\pi/2} \mathbb{P}\{Z_0 \le z \mid \theta\}\,\mathrm{d}\theta. \tag{3}$$

Subject to $z\sin\theta \le 1$, or, equivalently, $\theta \le \operatorname{arsin}(1/\max(1,z))$, $\{Z_0 \le z \mid \theta\} = \{\sin\psi \le z\sin\theta \mid \theta\} = \{\psi \le \operatorname{arsin}(z\sin\theta) \mid \theta\}$. Consequently, the right-hand side of (3) is expressible as

$$\begin{cases} \displaystyle\int_0^{\pi/2} \operatorname{arsin}(z\sin\theta)\,\mathrm{d}\theta & \text{for } 0 < z \le 1, \\ \displaystyle\int_0^{\operatorname{arsin}(1/z)} \operatorname{arsin}(z\sin\theta)\,\mathrm{d}\theta + \left[\frac{1}{2}\pi - \operatorname{arsin}\left(\frac{1}{z}\right)\right]\frac{1}{2}\pi & \text{for } 1 < z < \infty. \end{cases} \tag{4}$$

Each case of (4) is differentiable in z and yields

$$\tfrac{1}{4}\pi^2 f_0(z) = \begin{cases} \displaystyle\int_0^{\pi/2} \frac{\sin\theta}{\sqrt{1 - z^2\sin^2\theta}}\,\mathrm{d}\theta & \text{for } 0 < z \le 1, \\ \displaystyle\int_0^{\operatorname{arsin}(1/z)} \frac{\sin\theta}{\sqrt{1 - z^2\sin^2\theta}}\,\mathrm{d}\theta & \text{for } 1 < z < \infty. \end{cases} \tag{5}$$

For the first case of (5), the substitutions $v = z\sin\theta$ followed by $x = \sqrt{(z^2 - v^2)/(1 - v^2)}$, for which $v = 0$ and z for $\theta = 0$ and $\frac{1}{2}\pi$, and then $x = 0$ and z for $v = z$ and 0, facilitate explicit evaluation of $f_0(z)$ for $0 \le z < 1$ as follows:

$$\begin{aligned} \frac{1}{4}\pi^2 f_0(z) &= \int_0^z \frac{(v/z)\,\mathrm{d}v}{\sqrt{(1 - v^2)(z^2 - v^2)}} \qquad (6) \\ &= \int_0^z \frac{1}{z(1 - z^2)}\left[\sqrt{\frac{1 - v^2}{z^2 - v^2}} - \sqrt{\frac{z^2 - v^2}{1 - v^2}}\right] v\,\mathrm{d}v \\ &= \int_{x=0}^z \frac{1}{z(1 - z^2)}\left[\frac{1}{x} - x\right]\frac{1 - z^2}{(1 - x^2)^2}\,x\,\mathrm{d}x \\ &= \frac{1}{2z}\int_0^z \left(\frac{1}{1 - x} + \frac{1}{1 + x}\right)\mathrm{d}x \\ &= \frac{1}{2z}\log\frac{1 + z}{1 - z}. \qquad (7) \end{aligned}$$

For $1 < z < \infty$,

$$\begin{aligned} \frac{1}{4}\pi^2 f_0(z) &= \int_0^1 \frac{(v/z)\,\mathrm{d}v}{\sqrt{(1 - v^2)(z^2 - v^2)}} \\ &= \int_0^1 \frac{1}{z(z^2 - 1)}\left[\sqrt{\frac{z^2 - v^2}{1 - v^2}} - \sqrt{\frac{1 - v^2}{z^2 - v^2}}\right] v\,\mathrm{d}v \\ &= \int_{x=0}^{1/z} \frac{1}{z(z^2 - 1)}\left(\frac{1}{x} - x\right)\frac{z^2 - 1}{(1 - x^2)^2}\,x\,\mathrm{d}x \quad \left(\text{here } x = \sqrt{\frac{1 - v^2}{z^2 - v^2}}\right) \\ &= \frac{1}{2z}\int_0^{1/z} \left(\frac{1}{1 - x} + \frac{1}{1 + x}\right)\mathrm{d}x \\ &= \frac{1}{2z}\log\frac{z + 1}{z - 1}. \qquad (8) \end{aligned}$$

Combining (6) and (8) yields (2), as asserted.

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Two simple checks are available. First, $1/Z_0$ has the same density as Z_0. Second, we check that $\int_0^\infty f_0(z)\,\mathrm{d}z = 1$ by computing separately

$$\frac{1}{2z}\int_0^1 \log\frac{1+z}{1-z}\,\mathrm{d}z = \int_0^1 \sum_{n=1}^\infty \frac{z^{2n-2}}{2n-1}\,\mathrm{d}z = \sum_{n=1}^\infty \frac{1}{(2n-1)^2} = \frac{\pi^2}{6}\Big(1-\frac{1}{4}\Big) = \frac{\pi^2}{8},$$

$$\frac{1}{2z}\int_1^\infty \log\frac{1+1/z}{1-1/z}\,\mathrm{d}z = \int_1^\infty \sum_{n=1}^\infty \frac{z^{-2n}}{2n-1}\,\mathrm{d}z = \sum_{n=1}^\infty \frac{1}{(2n-1)^2} = \frac{\pi^2}{8};$$

the sum of the two contributions equals $\frac{1}{4}\pi^2$, as required.

3. The length of the larger of two sides

Proposition 2. *For a random triangle with* $c = 1$*, the PDF* f_1 *of* $Z_1 := \max(a, b)$ *is given by*

$$f_1(z) = \frac{4}{\pi^2 z}\log\frac{z}{|1-z|}, \qquad \tfrac{1}{2} < z < \infty. \tag{9}$$

Proof. In our original setting for Z_1 (see Daley *et al.* (2014)), a point P_0 is at the origin, with a line through P_0 coincident with the x-axis, and P_1 is the point of a unit-rate Poisson process closest to the origin. Without loss of generality, take P_1 in the first quadrant and let P_0P_1 make an angle ψ with the x-axis. Let a line through P_1, for the third side of the triangle (P_0, P_1, P_{01}), make an angle θ to the x-axis, meeting it at the point P_{01}. Then the larger of the two sides of the triangle meeting in P_{01} is of length RZ_1, where

$$Z_1 \sin\theta = \begin{cases} \sin\psi & \text{when } \psi \ge \frac{1}{2}\theta, \\ \sin(\theta-\psi) & \text{when } \psi < \frac{1}{2}\theta, \end{cases} \tag{10}$$

and $R = P_0P_1$ is of no concern here. We shall ultimately deduce that this function Z_1 of the pair of independent random variables (θ, ψ) has density as at (9).

Start by examining Z_1 on the rectangle $\mathcal{R} = \{(\theta,\psi)\colon 0 \le \theta \le \pi,\ 0 \le \psi \le \frac{1}{2}\pi\}$; partition $\mathcal{R}$ by its diagonal $\psi = \frac{1}{2}\theta$, along which $Z_1 = \frac{1}{2}\sec\frac{1}{2}\theta$. We have $Z_1 = 1$ on the base $(\{0 < \theta < \pi\},\ \psi = 0)$, on the equidiagonal $0 < \theta = \psi < \frac{1}{2}\pi$, and on two lines meeting in $(\frac{2}{3}\pi, \frac{1}{3}\pi)$, one from $(\frac{1}{2}\pi, 0)$ and the other from $(\frac{1}{2}\pi, \frac{1}{2}\pi)$, i.e. excepting a spur on the right-hand side of the base, $Z_1 = 1$ on a quadrilateral, inside which $Z_1 < 1$ and outside which $Z_1 > 1$. We have $Z_1 = \operatorname{cosec}\theta$ on the top of $\mathcal{R}$ and on the shifted diagonal where $0 \le \psi = \theta - \frac{1}{2}\pi \le \frac{1}{2}\pi$, and $Z_1 = \infty$ on the two sides $\theta = 0$ and π, $0 < \psi < \frac{1}{2}\pi$. The quadrilateral inside which $Z_1 < 1$ is of area $\frac{1}{6}\pi^2$, so, since $\mathcal{R}$ is of area $\frac{1}{2}\pi^2$, $\mathbb{P}\{Z \le 1\} = \frac{1}{3}$.

To find the conditional distributions $\mathbb{P}\{Z_1 \le z \mid \theta\}$ analogous to the integrand at (3), we examine the behaviour of Z_1 on each line segment $L_\theta := \{(\theta,\psi)\colon 0 < \psi < \frac{1}{2}\pi\}$. As ψ increases on L_θ, Z_1 first decreases from 1 to its minimum $\frac{1}{2}\sec\frac{1}{2}\theta$ at $\psi = \frac{1}{2}\theta$, then increases through 1 at $\psi = \theta$ to its maximum on L_θ of $\operatorname{cosec}\theta$ at $\psi = \frac{1}{2}\pi$. As ψ increases but for $\theta > \frac{1}{2}\pi$, Z_1 increases from 1 to a local maximum $\operatorname{cosec}\theta$ where $\psi = \theta - \frac{1}{2}\pi$, decreases to a local minimum at $\psi = \frac{1}{2}\theta$, and then increases to regain its maximum value $\operatorname{cosec}\theta$ at $\psi = \frac{1}{2}\pi$.

It follows from the last paragraph that, for given $z \in (\frac{1}{2}, \infty)$ and given θ, the equation $Z_1 = z$ may have 0, 1, 2, or 3 different roots for $\psi \in (0, \frac{1}{2}\pi)$ (i.e. the same number of intercepts in ψ on L_θ). When $\theta \le \frac{1}{2}\pi$ and $z < \frac{1}{2}\sec\frac{1}{2}\pi$ (which, it will be noted, is less than 1 for all $\theta < \frac{2}{3}\pi$), no roots exist. If $\frac{1}{2}\sec\frac{1}{2}\theta < z < 1$, two roots exist, at $\psi_1 = \theta - \arcsin(z\sin\theta)$ and $\psi_2 = \arcsin(z\sin\theta)$, but if $1 < z < \operatorname{cosec}\theta$, one root exists, at $\psi_3 = \arcsin(z\sin\theta)$. No root

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exists for $z > \operatorname{cosec}\theta$. Thus, on L_θ for given $\theta \le \frac{1}{2}\pi$,

$$\frac{1}{2}\pi\,\mathbb{P}\{Z_1 \le z \mid \theta\} = \begin{cases} 0 & \text{for } z \le \frac{1}{2}\sec\frac{1}{2}\theta, \\ \psi_2 - \psi_1 = 2\big(\operatorname{arsin}(z\sin\theta) - \frac{1}{2}\theta\big) & \text{for } \frac{1}{2}\sec\frac{1}{2}\theta < z \le 1, \\ \psi_3 = \operatorname{arsin}(z\sin\theta) & \text{for } 1 < z \le \operatorname{cosec}\theta, \\ \frac{1}{2}\pi & \text{for } \operatorname{cosec}\theta < z. \end{cases} \tag{11}$$

On L_θ for $\theta > \frac{1}{2}\pi$, there are no roots for $z < \frac{1}{2}\sec\frac{1}{2}\theta < 1$, and two roots when $\frac{1}{2}\sec\frac{1}{2}\theta < z < 1$, at $\psi_1 = \operatorname{arsin}(z\sin\theta) - (\pi - \theta)$ and $\psi_2 = \theta - \operatorname{arsin}(z\sin\theta)$. If $1 < z < \frac{1}{2}\sec\frac{1}{2}\theta$ (and this is possible only for $\theta > \frac{2}{3}\pi$), there are three roots, at ψ_1 and ψ_2 as already detailed, and $\psi_3 = \operatorname{arsin}(z\sin\theta)$. When $\frac{1}{2}\sec\frac{1}{2}\theta < z < \operatorname{cosec}\theta$, there is one root, at ψ_3 as just given; there are no roots when $z > \operatorname{cosec}\theta$. So, for $\theta > \frac{1}{2}\pi$,

$$\begin{aligned} &\frac{1}{2}\pi\,\mathbb{P}\{Z_1 \le z \mid \theta\} \\ &\quad = \begin{cases} 0 & \text{for } z < \frac{1}{2}\sec\frac{1}{2}\theta, \\ \psi_2 - \psi_1 = 2\big(\operatorname{arsin}(z\sin\theta) - \frac{1}{2}\theta\big) & \text{for } \frac{1}{2}\sec\frac{1}{2}\theta < z < 1, \\ \psi_1 + (\psi_3 - \psi_2) = 3\operatorname{arsin}(z\sin\theta) - \pi & \text{for } 1 < z < \frac{1}{2}\sec\frac{1}{2}\theta, \\ \psi_3 = \operatorname{arsin}(z\sin\theta) & \text{for } 1 < \frac{1}{2}\sec\frac{1}{2}\theta < z < \operatorname{cosec}\theta, \\ \frac{1}{2}\pi & \text{for } \operatorname{cosec}\theta < z. \end{cases} \end{aligned} \tag{12}$$

Analogously to the first case of (4), the first two cases of (11) and (12) imply that, for $z \le 1$,

$$\frac{1}{2}\pi^2\,\mathbb{P}\{Z \le z\} = \int_0^{2\pi/3} 2\left(\operatorname{arsin}(z\sin\theta) - \frac{1}{2}\theta\right)_+ \mathrm{d}\theta.$$

Then, by differentiation and with $g(z,\theta) = \sin\theta/\sqrt{1 - z^2\sin^2\theta}$ (cf. (5)), for $z \le 1$,

$$\frac{1}{2}\pi^2 f_1(z) = \begin{cases} 2\displaystyle\int_0^{2\arccos(1/2z)} g(z,\theta)\,\mathrm{d}\theta & \text{for } \frac{1}{2} < z < \frac{1}{2}\sqrt{2}, \\ 2\left[\displaystyle\int_0^{\pi/2} + \int_{2\operatorname{arsin}(1/2z)}^{\pi/2}\right] g(z,\theta)\,\mathrm{d}\theta & \text{for } \frac{1}{2}\sqrt{2} \le z \le 1, \end{cases} \tag{13}$$

the latter expression exploiting the symmetry about $\frac{1}{2}\pi$ of $\sin\theta$. Proceeding much as around (6) with $v = z\sin\theta$, for the first case of (13) (with $\frac{1}{2} \le z \le \frac{1}{2}\sqrt{2}$),

$$\begin{aligned} \frac{1}{2}\pi^2 f_1(z) &= 2\int_0^{\sqrt{1-1/4z^2}} \frac{(v/z)\,\mathrm{d}v}{\sqrt{(1-v^2)(z^2-v^2)}} \\ &= 2\int_0^{\sqrt{1-1/4z^2}} \frac{1}{z(1-z^2)}\left[\sqrt{\frac{1-v^2}{z^2-v^2}} - \sqrt{\frac{z^2-v^2}{1-v^2}}\right] v\,\mathrm{d}v \\ &= 2\int_{x=1-2z^2}^{z} \frac{1}{z(1-z^2)}\left(\frac{1}{x} - x\right)\frac{1-z^2}{(1-x^2)^2}\,x\,\mathrm{d}x \quad \left(x = \sqrt{\frac{z^2-v^2}{1-v^2}}\right), \\ &= \frac{1}{z}\int_{1-2z^2}^{z}\left(\frac{1}{1-x} + \frac{1}{1+x}\right)\mathrm{d}x \\ &= \frac{1}{z}\log\left[\frac{1+z}{1-z}\,\frac{2z^2}{2(1-z^2)}\right] \\ &= \frac{2}{z}\log\frac{z}{1-z}. \end{aligned} \tag{14}$$

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For the second case of (13), so $\frac{1}{2}\sqrt{2} < z < 1$,

$$\begin{aligned}
\frac{1}{2}\pi^2 f_1(z) &= 2\bigg[\int_0^z + \int_{\sqrt{1-1/4z^2}}^z\bigg]\frac{(v/z)\,\mathrm{d}v}{\sqrt{(1-v^2)(z^2-v^2)}} \\
&= 2\bigg[\int_{x=0}^z + \int_0^{2z^2-1}\bigg]\frac{1}{z(1-z^2)}\bigg(\frac{1}{x}-x\bigg)\frac{1-z^2}{(1-x^2)^2}\,x\,\mathrm{d}x \quad \bigg(x=\sqrt{\frac{z^2-v^2}{1-v^2}}\bigg) \\
&= \frac{1}{z}\bigg[\int_0^z + \int_0^{2z^2-1}\bigg]\bigg(\frac{1}{1-x}+\frac{1}{1+x}\bigg)\,\mathrm{d}x \\
&= \frac{1}{z}\bigg[\log\frac{1+z}{1-z} + \log\frac{2z^2}{2(1-z^2)}\bigg] \\
&= \frac{2}{z}\log\frac{z}{1-z}.
\end{aligned} \tag{15}$$

Equations (14) and (15) establish (9) for the case $z < 1$.

The remaining cases of (11) and (12) show first, from (11), that

$$\frac{1}{2}\pi^2\,\mathbb{P}\bigg\{Z_1 \le z,\ \theta \le \frac{1}{2}\pi\bigg\} = \int_0^{\operatorname{arsin}(1/z)} \operatorname{arsin}(z\sin\theta)\,\mathrm{d}\theta + \bigg[\frac{1}{2}\pi - \operatorname{arsin}\bigg(\frac{1}{z}\bigg)\bigg]\frac{1}{2}\pi.$$

When $\theta < \frac{1}{2}\pi$ and $z > 1$, differentiation now gives as a contribution to $\frac{1}{2}\pi^2 f_1(z)$ the quantity

$$\begin{aligned}
\int_0^{\operatorname{arsin}(1/z)} \frac{\sin\theta\,\mathrm{d}\theta}{\sqrt{1-z^2\sin^2\theta}} &= \int_0^1 \frac{(v/z)\,\mathrm{d}v}{\sqrt{(1-v^2)(z^2-v^2)}} \\
&= \int_{x=0}^{(1/z)} \frac{1}{z(z^2-1)}\bigg(\frac{1}{x}-x\bigg)\frac{z^2-1}{(1-x^2)^2}\,x\,\mathrm{d}x \quad \bigg(x=\sqrt{\frac{1-v^2}{z^2-v^2}}\bigg) \\
&= \frac{1}{2z}\int_0^{1/z}\bigg(\frac{1}{1-x}+\frac{1}{1+x}\bigg)\,\mathrm{d}x \\
&= \frac{1}{2z}\log\frac{z+1}{z-1}.
\end{aligned} \tag{16}$$

From (12) we have for the rest of $\mathcal{R}$ where $z > 1$ and now $\theta > \frac{1}{2}\pi$,

$$\begin{aligned}
&\tfrac{1}{2}\pi^2\,\mathbb{P}\{Z_1 \le z,\ \theta > \tfrac{1}{2}\pi\} \\
&\quad = \bigg[\pi - \operatorname{arsin}\bigg(\frac{1}{z}\bigg) - \frac{1}{2}\pi\bigg]\frac{1}{2}\pi + \int_{\pi-\operatorname{arsin}(1/z)}^{2\arccos(1/2z)} 3\bigg(\operatorname{arsin}(z\sin\theta) - \frac{1}{3}\pi\bigg)\,\mathrm{d}\theta \\
&\qquad + \int_{2\arccos(1/2z)}^{\pi} (\operatorname{arsin}(z\sin\theta) - (\pi-\theta))\,\mathrm{d}\theta.
\end{aligned} \tag{17}$$

Because $\theta > \frac{1}{2}\pi$ in the two integrals here and $\sin\theta$ is symmetric about $\frac{1}{2}\pi$, their sum equals

$$\int_0^{2\operatorname{arsin}(1/2z)} (\operatorname{arsin}(z\sin\theta) - \theta)\,\mathrm{d}\theta + \int_{2\operatorname{arsin}(1/2z)}^{\operatorname{arsin}(1/z)} 3\bigg(\operatorname{arsin}(z\sin\theta) - \frac{1}{3}\pi\bigg)\,\mathrm{d}\theta.$$

Differentiation of (17) (with the modified integrals) gives as the contribution to $\frac{1}{2}\pi^2 f_1(z)$ for

$z > 1$ from the part of $\mathscr{R}$ where $\theta > \frac{1}{2}\pi$ the quantity

$$\begin{aligned}
&\left[\int_0^{2\,\mathrm{arsin}(1/2z)} + 3\int_{2\,\mathrm{arsin}(1/2z)}^{\mathrm{arsin}(1/z)}\right] g(z,\theta)\,\mathrm{d}\theta \\
&\quad = \left[\int_0^{\sqrt{1-1/4z^2}} + 3\int_{\sqrt{1-1/4z^2}}^{1}\right] \frac{(v/z)\,\mathrm{d}v}{\sqrt{(1-v^2)(z^2-v^2)}} \\
&\quad = \left[\int_0^{\sqrt{1-1/4z^2}} + 3\int_{\sqrt{1-1/4z^2}}^{1}\right] \frac{1}{z(z^2-1)}\left[\sqrt{\frac{z^2-v^2}{1-v^2}} - \sqrt{\frac{1-v^2}{z^2-v^2}}\right] v\,\mathrm{d}v \\
&\quad = \left[\int_{x=1/(2z^2-1)}^{1/z} + 3\int_0^{1/(2z^2-1)}\right] \frac{1}{z(z^2-1)}\left(\frac{1}{x}-x\right)\frac{z^2-1}{(1-x^2)^2}\,x\,\mathrm{d}x \qquad \left(x = \sqrt{\frac{1-v^2}{z^2-v^2}}\right) \\
&\quad = \frac{1}{2z}\left[\int_{1/(2z^2-1)}^{1/z} + 3\int_0^{1/(2z^2-1)}\right]\left(\frac{1}{1-x}+\frac{1}{1+x}\right)\mathrm{d}x \\
&\quad = \frac{1}{2z}\log\left[\frac{1+1/z}{1-1/z}\,\frac{2(z^2-1)}{2z^2}\right] + \frac{3}{2z}\log\frac{2z^2}{2(z^2-1)} \\
&\quad = \frac{1}{2z}\log\left[\frac{(z+1)^2}{z^2}\,\frac{z^6}{(z^2-1)^3}\right].
\end{aligned} \tag{18}$$

Combining (18) and (16) gives, for $z > 1$,

$$\frac{1}{2}\pi^2 f_1(z) = \frac{1}{2z}\log\left[\frac{z+1}{z-1}\,\frac{z^4}{(z+1)(z-1)^3}\right] = \frac{2}{z}\log\frac{z}{z-1}.$$

This establishes (9) for $z > 1$.

As a check on (9), evaluate $\int_{1/2}^{\infty} f_1(z)\,\mathrm{d}z$, or, equivalently, find

$$\frac{\pi^2}{4}\int_{1/2}^{\infty} f_1(z)\,\mathrm{d}z = \int_{1/2}^{1}\frac{1}{z}\log\frac{z}{1-z}\,\mathrm{d}z + \int_1^{\infty}\frac{1}{z}\log\frac{z}{z-1}\,\mathrm{d}z := J_1 + J_2 \quad \text{say.}$$

The second integral $J_2 = -\int_1^{\infty} z^{-1}\log(1-z^{-1})\,\mathrm{d}z$ and an argument similar to that at the end of Section 2 gives $J_2 = \frac{1}{6}\pi^2$. For the other term, substitute $z = \frac{1}{2}(1+w)$, so

$$J_1 = \int_0^1 \frac{1}{1+w}\log\frac{1+w}{1-w}\,\mathrm{d}w.$$

Now $\int_1^2 v^{-1}\log v\,\mathrm{d}v = \frac{1}{2}(\log 2)^2$, and, for the rest, absolutely convergent series arise on grouping terms in pairs:

$$\begin{aligned}
&\lim_{T\uparrow 1}\int_0^T (1-w+w^2-\cdots)\big(w+\tfrac{1}{2}w^2+\tfrac{1}{3}w^3+\tfrac{1}{4}w^4+\cdots\big)\,\mathrm{d}w \\
&\qquad = \lim_{T\uparrow 1}\sum_{n=1}^{\infty}\left[T^{2n}\left(\frac{1}{2n}-\frac{T}{2n+1}\right)\left(\sum_{j=1}^{2n-1}\frac{(-1)^{j-1}}{j}\right) + \frac{T^{2n+1}}{(2n+1)(2n)}\right]
\end{aligned}$$

$$
\begin{aligned}
&= 2\sum_{n=1}^{\infty}\frac{1}{2n(2n+2)} - \sum_{n=1}^{\infty}\frac{1}{2n(2n+1)}\sum_{j=1}^{n-1}\frac{1}{2j(2j+1)} \\
&= 2(1-\log 2) - \frac{1}{2}\bigg(\sum_{n=1}^{\infty}\frac{1}{2n(2n+1)}\bigg)^2 + \frac{1}{2}\sum_{n=1}^{\infty}\bigg(\frac{1}{2n(2n+1)}\bigg)^2 \\
&= 2(1-\log 2) - \tfrac{1}{2}(1-\log 2)^2 + \tfrac{1}{2}\big(\tfrac{1}{6}\pi^2 - 1 - 2[1-\log 2]\big) \\
&= \tfrac{1}{12}\pi^2 - \tfrac{1}{2}(\log 2)^2.
\end{aligned}
$$

Thus, $J_1 + J_2 = \frac{1}{4}\pi^2$, as is consistent with f_1 being a PDF.

As yet another check, MAPLE® evaluates

$$\int_{1/2}^{\infty} f_1(z)\,\mathrm{d}z = 1.000\,000\,000.$$

4. The length of the smaller of two sides

Proposition 3. *For a random triangle with $c = 1$, the PDF f_2 of $Z_2 = b := \min(a, b)$ is given by*

$$f_2(z) = \begin{cases} \dfrac{4}{\pi^2 z}\log\dfrac{1+z}{1-z} & \textit{for } 0 < z \le \frac{1}{2}, \\[2ex] \dfrac{4}{\pi^2 z}\log\dfrac{1+z}{z} & \textit{for } \frac{1}{2} < z < \infty. \end{cases} \tag{19}$$

Proof. We could adopt a similar setup to that of the last section, concentrating now on the smaller of the two sides that meet in P_{01}, of length RZ_2 say, for which analogously to (10) we have

$$Z_2 \sin\theta = \begin{cases} \sin|\theta - \psi| & \text{when } \psi \ge \frac{1}{2}\theta, \\ \sin\psi & \text{when } \psi < \frac{1}{2}\theta. \end{cases} \tag{20}$$

A simpler proof is based on the formulae in Propositions 1 and 2 and the following fact. In a random triangle with $c = 1$ fixed, the length $\widetilde{Z}$, say, of another specified side, opposite the (random) angle ψ, say, equals $\sin\psi/\sin\theta$ (in our standard description via angles (θ, ψ) as in Proposition 1). Then $\{\widetilde{Z} \le z\} = \{\sin\psi \le z\sin\theta\}$, and, thus (cf. below (3)), $\widetilde{Z}$ and Z_0 have the *same distribution*.

Let Z_2 at (20) have density function f_2. Then because $\widetilde{Z} = Z_1$ or Z_2 with equal probability, the density function $\tilde{f}$ of $\widetilde{Z}$ is related to the densities f_1 and f_2 via $f_0 = \tilde{f} = \frac{1}{2}(f_1 + f_2)$, and, thus,

$$f_2(z) = 2\,f_0(z) - f_1(z) = \frac{4}{\pi^2 z}\log\frac{z+1}{|1-z|} - \begin{cases} 0 & \text{for } 0 < z < \frac{1}{2}, \\ \dfrac{4}{\pi^2 z}\log\dfrac{z}{|1-z|} & \text{for } \frac{1}{2} \le z < \infty. \end{cases}$$

The relation asserted at (19) follows directly.

5. Area

In this section we discuss the area, V say, of a random triangle as defined. As for our earlier propositions, we find the density $f(y) = (\mathrm{d}/\mathrm{d}y)\mathbb{P}\{V \le y\}$.

Proposition 4. *For a random triangle with $c = 1$, the area V has PDF*

$$f(y) = \frac{2}{\pi^2}\int_0^{2\operatorname{arcot}(4y)} \frac{4\sin\theta}{\sqrt{1-(\cos\theta-4y\sin\theta)^2}}\,\mathrm{d}\theta \tag{21}$$

$$= \frac{8}{\pi^2}\int_0^{2\operatorname{arcot}(z)} \frac{\mathrm{d}\theta}{\sqrt{1-z^2+2z\cot\theta}} \quad (z = 4y) \tag{22}$$

$$= \frac{16}{\pi^2}\int_0^{1/z} \frac{1}{1+t^2}\sqrt{\frac{t}{(t+z)(1-zt)}}\,\mathrm{d}t \quad \left(t = \tan\tfrac{1}{2}\theta\right) \tag{23}$$

$$= \frac{8}{\pi^2}\int_{(z^2-1)/2z}^{\infty} \frac{\mathrm{d}x}{(1+x^2)\sqrt{1-z^2+2zx}} \quad (x = \cot\theta \textit{ in } (22)). \tag{24}$$

Proof. We use the setting of Section 3 and the rectangle $\mathcal{R}$ there. Any triangle has area $\frac{1}{2}$(base $\times$ height), so, for the triangle in which P_0P_1 has unit length, and, therefore, height $\sin\psi$,

$$V = \frac{\sin\psi\sin|\psi-\theta|}{2\sin\theta} = \begin{cases} \dfrac{\cos(\theta-2\psi)-\cos\theta}{4\sin\theta} & \text{if } \pi > \theta > \psi > 0, \\[2ex] \dfrac{\cos\theta-\cos(2\psi-\theta)}{4\sin\theta} & \text{if } \frac{1}{2}\pi > \psi > \theta > 0. \end{cases} \tag{25}$$

Equation (25) shows that $V = 0$ along the edge of $\mathcal{R}$ where $\psi = 0$, and on the diagonal $\psi = \theta$ of the left square of $\mathcal{R}$. We have $V = \frac{1}{2}|\cot\theta|$ on the top edge $\psi = \frac{1}{2}\pi$, and $V = \infty$ on the two sides $\theta = 0, \pi$. We have $V = \frac{1}{4}\tan\frac{1}{2}\theta$ on the diagonal $\psi = \frac{1}{2}\theta$ of $\mathcal{R}$, and $V = \frac{1}{4}\sin 2\psi$ along $\theta = \frac{1}{2}\pi$.

For given $\theta \in (0, \frac{1}{2}\pi)$, as ψ increases from 0 to $\frac{1}{2}\pi$, V increases from 0 to a local maximum $\frac{1}{4}\tan\frac{1}{2}\theta$ at $\psi = \frac{1}{2}\theta$, decreases to 0 at $\theta = \psi$, and then increases again to another local maximum $\frac{1}{2}\cot\theta$ at $\psi = \frac{1}{2}\pi$. For given θ in $[\frac{1}{2}\pi, \pi)$, $V = 0$ at $\psi = 0$, increases to a maximum $\frac{1}{4}\tan\frac{1}{2}\theta$ at $\psi = \frac{1}{2}\theta$, and then decreases to $\frac{1}{2}|\cot\theta|$ at $\psi = \frac{1}{2}\pi$. Let $\theta_{\mathrm{c}} := 2\operatorname{artan}(\frac{1}{2}\sqrt{2})$ be a critical value of θ such that $\frac{1}{2}|\cot\theta|$ is greater than, equal to, or less than $\frac{1}{4}\tan\frac{1}{2}\theta$ according to whether θ is less than, equal to, or greater than θ_{c}, respectively.

For given (y, θ) in $\mathbb{R}_+ \times [0, \pi]$, there may be 0, 1, 2, or 3 roots in ψ of $V = y$, $\psi_i := \psi_i(y, \theta)$ say ($i = 1, 2, 3$). For $\theta > \psi$, $V = y$ holds if either $\cos(\theta - 2\psi_1) = \cos\theta + 4y\sin\theta$ or $\cos(2\psi_2 - \theta) = \cos\theta + 4y\sin\theta$, while, for $\theta < \psi$, $V = y$ holds only when $\cos(2\psi_3 - \theta) = \cos\theta - 4y\sin\theta$. Thus,

$$\psi_1 = \tfrac{1}{2}[\theta - \arccos(\cos\theta + 4y\sin\theta)], \qquad \psi_2 = \tfrac{1}{2}[\theta + \arccos(\cos\theta + 4y\sin\theta)],$$
$$\psi_3 = \tfrac{1}{2}[\theta + \arccos(\cos\theta - 4y\sin\theta)]. \tag{26}$$

As a check, for $\theta \in (0, \pi)$ and $y > 0$, $0 < \psi_1 < \frac{1}{2}\theta < \psi_2 < \min(\theta, \frac{1}{2}\pi)$ and $\frac{1}{2}\pi > \psi_3 > \theta$.

For $\theta < \frac{1}{2}\pi$, when $y < \frac{1}{4}\tan\frac{1}{2}\theta$, there are at least two roots, ψ_1 and ψ_2; ψ_3 is another root when $y < \frac{1}{2}\cot\theta$. When $\theta < \theta_{\mathrm{c}}$ and $\frac{1}{4}\tan\frac{1}{2}\theta < y < \frac{1}{2}\cot\theta$, there is one root, ψ_3, and no roots for $y > \frac{1}{2}\cot\theta$. When $\frac{1}{2}\pi > \theta > \theta_{\mathrm{c}}$, there are no roots for $y > \frac{1}{4}\tan\frac{1}{2}\theta$ and otherwise 3 or 2 roots according to whether y is less than or greater than $\frac{1}{2}\cot\theta$.

For $\theta \geq \frac{1}{2}\pi$, when $y < \frac{1}{2}|\cot\theta|$, there are two roots, ψ_1 and ψ_2; when $\frac{1}{2}|\cot\theta| < y < \frac{1}{4}\tan\frac{1}{2}\theta$, there is one root, ψ_1; and there are no roots when $y > \frac{1}{4}\tan\frac{1}{2}\theta$.

Continuing as in Section 3,

$$\tfrac{1}{2}\pi F(y \mid \theta) := \tfrac{1}{2}\pi\, \mathbb{P}\{V \le y \mid \theta\} = \begin{cases} \psi_3 - \psi_2 + \psi_1 & \text{if three roots of } V = y, \\ \tfrac{1}{2}\pi - \psi_2 + \psi_1 & \text{if two roots,} \\ \psi_3 & \text{if one root and } \theta < \tfrac{1}{2}\pi, \\ \psi_1 & \text{if one root and } \theta > \tfrac{1}{2}\pi, \\ \tfrac{1}{2}\pi & \text{if no roots.} \end{cases} \tag{27}$$

Define $h_\pm(y \mid \theta) = (2\sin\theta)/\sqrt{1 - (\cos\theta \pm 4y\sin\theta)^2}$. Combining (27) with the relations for ψ_i given in (26) yields

$$\tfrac{1}{2}\pi\, f(y \mid \theta) := \frac{\mathrm{d}F(y \mid \theta)}{(2/\pi)\,\mathrm{d}y} = \begin{cases} 0 & \text{if no root of } V = y \text{ on } L_\theta, \\ h_+(y \mid \theta) & \text{if one root and } \theta > \tfrac{1}{2}\pi, \\ h_-(y \mid \theta) & \text{if one root and } \theta < \tfrac{1}{2}\pi, \\ 2h_+(y \mid \theta) & \text{if two roots,} \\ 2h_+(y \mid \theta) + h_-(y \mid \theta) & \text{if three roots.} \end{cases}$$

Let $\tfrac{1}{2}\pi^2 f(y)\,\mathrm{d}y = \tfrac{1}{2}\pi^2\,\mathbb{P}\{V \in (y, y + \mathrm{d}y)\}$. Then

$$\begin{aligned} \frac{1}{2}\pi^2 f(y) &= \int_0^{2\,\mathrm{artan}(4y)} h_-\,\mathrm{d}\theta + \int_{2\,\mathrm{artan}(4y)}^{\mathrm{arcot}(2y)} [2h_+ + h_-]\,\mathrm{d}\theta + \int_{\mathrm{arcot}(2y)}^{\pi-\mathrm{arcot}(2y)} 2h_+\,\mathrm{d}\theta \\ &\quad + \int_{\pi-\mathrm{arcot}(2y)}^{\pi} h_+\,\mathrm{d}\theta. \end{aligned}$$

For $0 < \theta < \pi$, $h_+(y, \theta) = h_-(y, \pi - \theta)$, so the last integral here equals $\int_0^{\mathrm{arcot}(2y)} h_-\,\mathrm{d}\theta$. More generally, for $0 < \alpha < \tfrac{1}{2}\pi$, $\int_\alpha^{\pi/2} h_+\,\mathrm{d}\theta = \int_{\pi/2}^{\pi-\alpha} h_-\,\mathrm{d}\theta$. This yields the condensed form $\tfrac{1}{2}\pi^2 f(y) = \int_{2\,\mathrm{artan}(4y)}^{\pi} 2h_+\,\mathrm{d}\theta$.

For $\tfrac{1}{4}\tan\tfrac{1}{2}\theta_{\mathrm{c}} < y < \tfrac{1}{4}$,

$$\begin{aligned} \frac{1}{2}\pi^2 f(y) &= \int_0^{\mathrm{arcot}(2y)} h_-\,\mathrm{d}\theta + \int_{2\,\mathrm{artan}(4y)}^{\pi-\mathrm{arcot}(2y)} 2h_+\,\mathrm{d}\theta + \int_{\pi-\mathrm{arcot}(2y)}^{\pi} h_+\,\mathrm{d}\theta \\ &= \int_0^{\mathrm{arcot}(2y)} 2h_-\,\mathrm{d}\theta + \int_{\mathrm{arcot}(2y)}^{\pi/2} 2h_-\,\mathrm{d}\theta + \int_{2\,\mathrm{artan}(4y)}^{\pi/2} 2h_+\,\mathrm{d}\theta \\ &= \int_{2\,\mathrm{artan}(4y)}^{\pi} 2h_+\,\mathrm{d}\theta \\ &= \int_0^{2\,\mathrm{arcot}(4y)} 2h_-\,\mathrm{d}\theta. \end{aligned}$$

Finally, for $\tfrac{1}{4} < y < \infty$,

$$\begin{aligned} \frac{1}{2}\pi^2 f(y) &= \int_0^{\mathrm{arcot}(2y)} h_-\,\mathrm{d}\theta + \int_{2\,\mathrm{artan}(4y)}^{\pi-\mathrm{arcot}(2y)} 2h_+\,\mathrm{d}\theta + \int_{\pi-\mathrm{arcot}(2y)}^{\pi} h_+\,\mathrm{d}\theta \\ &= \int_0^{\mathrm{arcot}(2y)} 2h_-\,\mathrm{d}\theta + \int_{\mathrm{arcot}(2y)}^{2\,\mathrm{arcot}(4y)} 2h_-\,\mathrm{d}\theta \\ &= \int_0^{2\,\mathrm{arcot}(4y)} 2h_-\,\mathrm{d}\theta. \end{aligned}$$

This establishes (21) in our final result; the indicated substitutions show the rest.

We have used both MAPLE and MATHEMATICA® symbolic manipulation packages to express this PDF f for the area V in terms of elliptic integrals of the third kind, but the manipulations are delicate. We found the following expression after substituting $t = u^2$ in (23) and further manipulation:

$$\tfrac{1}{16}\pi^2 f(y) = \nu_1\big(\Pi\big(\tfrac{1}{2}\pi, \nu_1, \nu_1\big) - \Pi\big(\tfrac{1}{2}\pi, -\nu_1, \nu_1\big)\big).$$

Here

$$\Pi(\varphi, n, k) = \int_0^{\sin\varphi} \frac{(1 + nx^2)^{-1}\,\mathrm{d}x}{\sqrt{(1 - x^2)(1 - k^2x^2)}}$$

and $\nu_1 = 1/(4iy)$. Since Figure 1(d) is based on numerical computation via Proposition 4, we see little point in further elaboration here.

Remark 1. The tail $\mathbb{P}\{V > y\}$ for larger y can be approximated as follows starting from (23). This relation implies that

$$\begin{aligned}\frac{\pi^2}{16} f(y) &= \int_0^{1/z} \frac{1}{1 + t^2}\sqrt{\frac{t}{(t + z)(1 - zt)}}\,\mathrm{d}t\\ &= h(\varepsilon)\int_0^{1/z}\sqrt{\frac{t}{1 - zt}}\,\mathrm{d}t\\ &= h(\varepsilon)\int_0^{1}\sqrt{\frac{u/z}{1 - u}}\,\frac{\mathrm{d}u}{z} \qquad (t = zu),\end{aligned}$$

where, for $\varepsilon = 1/z$, $h(\varepsilon)$ is bounded by $1/\sqrt{z}$ and $1/[\sqrt{z}(1 + \varepsilon^2)\sqrt{1 + \varepsilon}\,]$. Substituting $u = \sin^2\theta$ into $\int_0^1 \sqrt{u/(1 - u)}\,\mathrm{d}u$ and recalling that $z = 4y$ shows that $f(y) = [1/(2\pi y^2)] \times [1 + o(1)]$ as $y \to \infty$.

Remark 2. We found (23) to be the simplest relation to use in the numerical computations. It can be used to check that $\int_0^\infty f(y)\,\mathrm{d}y = 1$ via a substitution $x = \sqrt{(t + z)/(1 - tz)}$ after forcing the square root of the denominator into a partial fraction decomposition much as in the proofs of Propositions 1 and 2. The details are left to the reader.

Remark 3. MATHEMATICA produces formulae for (21) involving both complete and incomplete elliptic integrals of the third kind. Power series expansions for $f(\cdot)$ can be developed, writing $g(z)$ for $\frac{1}{2}\pi^2 f(\frac{1}{4}z)$ (cf. (22)–(23)); the simplest cases are for $z \downarrow 0$ and $z \uparrow 1$, when from the right-hand sides of (22) and (24) we obtain $f(0+) = 8/\pi = 2.5465$ and

$$\begin{aligned}g(1) &= \frac{4}{\sqrt{2}}\int_0^\infty \frac{\mathrm{d}x}{(1 + x^2)\sqrt{x}}\,\mathrm{d}x\\ &= \lim_{a\uparrow 1}\int_0^a \frac{2\sqrt{2}}{1 + x^2}(x^{1/2} + x^{-1/2})\,\mathrm{d}x\\ &= 2\sqrt{2}\lim_{a\uparrow 1}\int_0^a \sum_{n=0}^\infty (-)^n(x^{2n+1/2} + x^{2n-1/2})\,\mathrm{d}x\\ &= 4\sqrt{2}\sum_{n=0}^\infty\bigg[\frac{1}{(4n + 1/2)(4n + 5/2)} + \frac{1}{(4n + 3/2)(4n + 7/2)}\bigg],\end{aligned}$$

where we have combined the steps of taking the limit $a \uparrow 1$ and compressing the sums of alternating series.

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Table 1: Cumulative distribution functions $\mathbb{P}\{\cdot \le z\}$ for Z_0, Z_1, Z_2, and V.

z	Z_0	Z_1	Z_2	V
0.25	0.102 040	0.0	0.204 080	0.379 937
0.50	0.208 854	0.0	0.417 709	0.564 280
0.75	0.328 530	0.079 996	0.577 064	0.669 148
1.00	0.500 000	0.333 333	0.666 667	0.734 927
1.50	0.713 661	0.662 288	0.765 034	0.811 446
2.00	0.791 146	0.764 027	0.818 265	0.854 075
2.50	0.834 823	0.817 912	0.851 734	0.881 097
3.00	0.863 166	0.851 579	0.874 754	0.899 719
4.00	0.897 959	0.891 524	0.904 393	0.923 676
5.00	0.918 578	0.914 483	0.922 672	0.938 411
10.0	0.959 426	0.958 411	0.960 442	0.968 682
Median	1.0	1.1453	0.608 42	0.394 70

In general, we must distinguish the cases in which z is greater than or less than 1 in (23), and within each of those cases, we must also distinguish whether $|(z^2 - 1)/2z|$ is greater than or less than 1. However, we do not think it worth presenting the details here.

6. Discussion

The conditional probability argument used to establish most of the results could in principle be used to find $\mathbb{P}\{Z_1 \le z_1,\ Z_2 \le z_2\}$ (in the notation of Sections 3 and 4), but with more than half a dozen cases shown in (11) and (12), and a similar number analogously for conditional probabilities for Z_2, and with the break points in these definitions not all coinciding for Z_1 and Z_2, the algebra becomes heavier and more detailed. Nevertheless, an expression would be of interest because, as the triangle inequality shows,

$$\mathbb{P}\{0 \le Z_1 - Z_2 \le 1\} = 1,$$

so the joint density function is concentrated in a band of unit width. This same bound and the argument in the proof of Proposition 2 justifies the assertion that

$$\lim_{z\to\infty} \frac{\mathbb{P}\{Z_i > z\}}{\mathbb{P}\{Z_0 > z\}} = 1, \qquad i = 1, 2,$$

i.e. the tail behaviour of the distributions of all three Z_i, $i = 0, 1, 2$, is the same, as can also be seen from Table 1.

A referee suggested an alternative possible route that we have not pursued, namely, the use of the order statistics θ_1, θ_2, and θ_3 of three independent and identically distributed directions on $[0, \pi]$, and reference to standard integral tables in place of the integrals which we evaluated explicitly.

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References

DALEY, D. J., EBERT, S. AND LAST, G. (2014). Two lilypond systems of finite line-segments. Submitted.

KENDALL, D. G. (1974). An introduction to stochastic geometry. In *Stochastic Geometry*, eds E. F. Harding and D. G. Kendall, John Wiley, London, pp. 3–9.

KENDALL, M. G. AND MORAN, P. A. P. (1963). *Geometric Probability*. Hafner Publishing, New York.

MATHAI, A. M. (1999). *An Introduction to Geometrical Probability: Distributional Aspects with Applications*. Gordon and Breach, Amsterdam.

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