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ON GROUPS WITH A FINITE NUMBER OF NORMALISERS

MOHAMMAD ZARRIN

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Abstract

Groups having exactly one normaliser are well known. They are the Dedekind groups. All finite groups having exactly two normalisers were classified by Pérez-Ramos ['Groups with two normalizers', *Arch. Math.* **50** (1988), 199–203], and Camp-Mora ['Locally finite groups with two normalizers', *Comm. Algebra* **28** (2000), 5475–5480] generalised that result to locally finite groups. Then Tota ['Groups with a finite number of normalizer subgroups', *Comm. Algebra* **32** (2004), 4667–4674] investigated properties (such as solubility) of arbitrary groups with two, three and four normalisers. In this paper we prove that every finite group with at most 20 normalisers is soluble. Also we characterise all nonabelian simple (not necessarily finite) groups with at most 57 normalisers.

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1. Introduction and results

We say that a group G is an $\mathfrak{N}_n$-group ($\mathfrak{N}_n^c$-group, respectively) if it has exactly n normalisers of subgroups (normalisers of cyclic subgroups, respectively).

The groups belonging to $\mathfrak{N}_1$ are the Dedekind groups, which are well known. Pérez-Ramos [13] characterised finite groups belonging to $\mathfrak{N}_2$, and then Camp-Mora [5] generalised this result to locally finite groups. Subsequently Tota [15] investigated the behaviour of normaliser subgroups of a group on the structure of the group itself and gave some properties of arbitrary groups with two, three and four normalisers. More precisely, among other things, her results can be described in the following way.

- A group G has a finite number of normalisers if and only if G is central-by-finite.
- Let G be an $\mathfrak{N}_3$-group. Then G is nilpotent of class at most three.
- Let G be an $\mathfrak{N}_4$-group. Then G is soluble of derived length two.
- Let G be an $\mathfrak{N}_4$-group. If G is not locally finite, then G is nilpotent of class at most three.

In particular, she showed that every $\mathfrak{N}_n$-group with $n \leq 4$ is soluble of derived length at most two.

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Observing that the characterisation of such groups is possibly very hard, the above writers restricted themselves to studying $\mathfrak{N}_n$-groups with small values of n. On the other hand, it is easy to see that

$$\mathfrak{N}_n \subseteq \mathfrak{N}^c_m,$$

for some $m \le n$. So to investigate $\mathfrak{N}_n$-groups it suffices to study $\mathfrak{N}^c_n$-groups. The aim of this paper is to investigate $\mathfrak{N}^c_n$-groups. We obtain a solubility criterion for $\mathfrak{N}^c_n$-groups ($\mathfrak{N}_n$-groups) in terms of n. In fact we prove that every finite group with at most 20 normalisers of cyclic subgroups (normalisers of subgroups, respectively) is soluble, while we conjecture that the best bound must be 21 (26, respectively).

The main result of this paper is the following theorem.

Theorem A. *Let G be a finite $\mathfrak{N}^c_n$-group ($\mathfrak{N}_n$-group) with $n \le 20$. Then G is a soluble group.*

We also investigate nonabelian simple $\mathfrak{N}^c_n$-groups with $n \le 57$.

Theorem B. *Let G be a nonabelian simple $\mathfrak{N}^c_n$-group with $n \le 57$. Then $G \cong A_5$.*

We write A_n and D_{2m}, respectively, to denote the alternating groups on n letters and the dihedral group of order $2m$ and $m > 1$. For any prime power q, we denote by $L_2(q)$, $\mathrm{PGL}(2, q)$, $\mathrm{SL}(2, q)$ and $\mathrm{GL}(2, q)$, the projective special linear group, the projective general linear group, the special linear group and the general linear group of degree two over the finite field of size q, respectively.

2. Properties of groups with a finite number of normalisers

In 1980 Polovickiĭ [12] proved that if a group has finitely many normalisers of abelian subgroups, then its centre has finite index. This result suggests that the behaviour of normalisers has a strong influence on the structure of the group. In fact Tota [15] (see Theorem 2.1 and also the remark after Theorem 2.2) has proved that a group G has finitely many normalisers of cyclic subgroups if and only if G is a central-by-finite group if and only if G has finitely many normalisers of subgroups. Therefore we can summarise the latter results in the following theorem. (Note that every group with n normalisers of abelian subgroups is also a group with m normalisers of cyclic subgroups, for some $m \le n$.)

Theorem 2.1. *For any group G, the following statements are equivalent.*

(a) *G has finitely many normalisers of cyclic subgroups.*
(b) *G is a central-by-finite group.*
(c) *G has finitely many normalisers of subgroups.*
(d) *G has finitely many normalisers of abelian subgroups.*

For the proofs of the main theorems we need the following lemmas.

Lemma 2.2. *Suppose that G is an $\mathfrak{N}^c_n$-group and $N \trianglelefteq K \le G$. Then:*

(1) *K is an $\mathfrak{N}^c_m$-group for some $m \le n$;*
(2) *K/N is an $\mathfrak{N}^c_r$-group for some $r \le n$.*

Proof. (1) This follows from the fact that $N_K(\langle x\rangle) = N_G(\langle x\rangle) \cap K$, for any $x \in K$.
(2) This is straightforward. □

Lemma 2.3. *Let D_{2m} be an $\mathfrak{N}_t^c$-group. Then:*

(1) $t = (m/2) + 1$ *if $m \neq 2$ is even;*
(2) $t = m + 1$ *if m is odd.*

Proof. This is straightforward. □

According to Lemma 2.3, it follows that, for any positive integer n, there exists a finite $\mathfrak{N}_n^c$-group.

The following is a key lemma for some of our results.

Lemma 2.4. *Let G be a group, and let x and y be elements of G such that $N_G(\langle x\rangle) = N_G(\langle y\rangle)$. Then the subgroup $H = \langle x, y\rangle$ is a nilpotent group of class at most two.*

Proof. Since $N_G(\langle x\rangle) = N_G(\langle y\rangle)$, we have $y^{-1}xy = x^i$ and $x^{-1}yx = y^j$ for some $i, j \in \mathbb{N}$. This implies that $x^{-1}y^{-1}xy \in \langle x\rangle \cap \langle y\rangle$ and so $H' \leq Z(H)$. It follows that H is a nilpotent group of class at most two. □

Finally, we show that a semisimple $\mathfrak{N}_n^c$-group has order bounded by a function of n. (Recall that a group G is semisimple if G has no nontrivial normal abelian subgroups.)

Proposition 2.5. *Let G be a semisimple $\mathfrak{N}_n^c$-group ($\mathfrak{N}_n$-group). Then G is finite and*

$$|G| \leq (n-1)!.$$

Proof. The group G acts on the set

$$\mathcal{N} := \{N_G(\langle x\rangle) \mid x \in G\} \setminus \{G\}$$

by conjugation. By assumption, $|\mathcal{N}| = n - 1$. Put

$$N'(G) = \bigcap_{x \in G} N_G(N_G(\langle x\rangle)).$$

The subgroup $N'(G)$ is the kernel of this action and so

$$G/N'(G) \hookrightarrow S_{n-1}. \tag{$*$}$$

By definition of $N'(G)$, it follows that $N_{N'(G)}(\langle x\rangle) \lhd N'(G)$ for any element $x \in N'(G)$, and so

$$\langle x\rangle \lhd N_{N'(G)}(\langle x\rangle) \lhd N'(G).$$

That is, every cyclic subgroup of $N'(G)$ is 2-subnormal. Therefore $N'(G)$ is a 2-Engel group, and so (as is well known; see [11]) $N'(G)$ is nilpotent of class at most three. Now, as G is a semisimple group, we can obtain that $N'(G) = 1$. It follows from ($*$) that G is a finite group and $|G| \leq (n-1)!$, as required. □

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3. Soluble $\mathfrak{N}_n$-groups

In this section we prove Theorem A.

First, by Lemma 2.4, we give some interesting relations between $\mathfrak{N}_n^c$-groups and $(\mathcal{N}, n)$-groups that were considered by Lennox and Wiegold in 1981 [10]. Let $n \in \mathbb{N}$. We say that a group G satisfies condition $(\mathcal{N}, n)$ (or that G is an $(\mathcal{N}, n)$-group) whenever in every subset of G with $n+1$ elements there exist distinct elements x, y such that $\langle x, y\rangle$ is nilpotent.

PROPOSITION 3.1. *Let n be a positive integer and G be an $\mathfrak{N}_n^c$-group (not necessarily finite). Then G satisfies condition $(\mathcal{N}, n)$.*

PROOF. Suppose, for a contradiction, that G does not satisfy condition $(\mathcal{N}, n)$. Then there exists a set $X = \{a_1, a_2, \ldots, a_{n+1}\}$ of G such that $\langle a_i, a_j\rangle$ is not nilpotent, for every $1 \le i \ne j \le n+1$. It follows, by Lemma 2.4, that G is an $\mathfrak{N}_t^c$-group with $t \ge n+1$, a contradiction. □

It it easy to see that the dihedral group of order eight, D_8, is an $(\mathcal{N}, 1)$-group and that it is also, by Lemma 2.3, an $\mathfrak{N}_3^c$-group. This example shows that the converse of Proposition 3.1 is not true.

REMARK 3.2. Proposition 3.1 holds whenever G is an $\mathfrak{N}_n$-group since, if $G \in \mathfrak{N}_n$, then $G \in \mathfrak{N}_m^c$ for some $m \le n$. Hence, according to Proposition 3.1, G satisfies condition $(\mathcal{N}, m)$. On the other hand, it is easy to see that $(\mathcal{N}, r) \subseteq (\mathcal{N}, s)$ for all $r \le s$. Thus G satisfies condition $(\mathcal{N}, n)$.

PROOF OF THEOREM A. This follows from Proposition 3.1 and the main result of [6]. □

In particular, we conjecture that the best bound must be 21. Some results (see case (1) of Lemmas 3.3 and 3.4, below) show that every $\mathfrak{N}_{21}^c$-group is an $(\mathcal{N}, 20)$-group, and so, by [6], such groups are soluble.

An element x of G is called right n-Engel if $[x,_n y] = 1$ for all $y \in G$, where $[x, y] = x^{-1}y^{-1}xy = x^{-1}x^y$ and $[x,_{m+1} y] = [[x,_m y], y]$ for all positive integers m. We denote by $R_n(G)$, the set of all right n-Engel elements of G. The subset corresponding to $R_n(G)$ which can be similarly defined is $L_n(G)$, the set of all left n-Engel elements of G, where an element x of G is called left n-Engel if $[y,_n x] = 1$ for all $y \in G$.

LEMMA 3.3. *Let n be a positive integer and G be an $\mathfrak{N}_n^c$-group (not necessarily finite). Then:*

(1) *if $L_t(G) = 1$ for some $t \ge 2$, then G satisfies condition $(\mathcal{N}, n-1)$;*
(2) *if G is a semisimple group, then G satisfies condition $(\mathcal{N}, n-1)$.*

PROOF. (1) Suppose, for a contradiction, that G does not satisfy condition $(\mathcal{N}, n-1)$. Then there exists a set $X = \{a_1, a_2, \ldots, a_n\}$ of G such that $\langle a_i, a_j\rangle$ is not nilpotent, for every $1 \le i \ne j \le n$. We claim that $N_G(\langle a_i\rangle) \ne G$ for every $i \in \{1, \ldots, n\}$. To show this, suppose that there exists an element $a_i \in X$ such that $N_G(\langle a_i\rangle) = G$. This implies that $[g,_2 a_i] = 1$ for every $g \in G$. That is, $a_i \in L_2(G) \subseteq L_t(G) = 1$, which gives a

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contradiction. On the other hand since $N_G(\langle e\rangle) = G$, where e is the trivial element of G, we obtain that G is an $\mathfrak{N}_k^c$-group with $k \geq n + 1$, a contradiction.

(2) It can easily be seen that

$$L_2(G) = \{a \in G \mid \langle a\rangle^G \text{ is abelian}\},$$

where $\langle a\rangle^G$ denotes the normal closure of a in G. As G is semisimple, so $L_2(G) = 1$ and so, by part (2), the result follows. □

Lemma 3.4. *Let G be an $\mathfrak{N}_n^c$-group and $R_t(G) = L_t(G)$, for some $t \geq 2$. If G satisfies the maximal condition on its subgroups or G is a finitely generated soluble group, then G satisfies condition $(\mathcal{N}, n - 1)$.*

Proof. By [14, Theorem 12.3.7] and the main result of [4], $R(G) = Z^*(G)$, where $Z^*(G)$ is the hypercentre of G. Now, by an argument similar to the proof of part (1) of Lemma 3.3, the result follows. (Note that $R_t(G) \subseteq R(G)$ and if $a \in Z^*(G)$ then $\langle a, b\rangle$ is nilpotent, for all $b \in G$.) □

We say that a group G is an $\mathfrak{N}_n^a$-group if it has exactly n normalisers of abelian subgroups (see also [12]). Since every group with n normalisers of abelian subgroups is also a group with m normalisers of cyclic subgroups, for some $m \leq n$, Theorem A gives the following corollary.

Corollary 3.5. *Let G be a finite group. If G has k normalisers of abelian subgroups such that $k \leq 20$, then G is a soluble group.*

In view of the above results and as $A_5 \in \mathfrak{N}_{27} \cap \mathfrak{N}_{27}^a$ (for example, by GAP [7]) and also $A_5 \in \mathfrak{N}_{22}^c$, we can give the following conjectures.

Conjecture 3.6. Every finite $\mathfrak{N}_n$-group (or $\mathfrak{N}_n^a$-group) with $n \leq 26$ is soluble.

Conjecture 3.7. Every finite $\mathfrak{N}_n^c$-group with $n \leq 21$ is soluble.

4. Nonabelian simple $\mathfrak{N}_n$-groups

In this section, we prove Theorem B. We need some preliminary lemmas. Finding the number of normalisers of cyclic subgroups of a group G, specifically for simple groups, itself is of independent interest as a pure combinatorial problem. Here we give a lower bound for the number of normalisers of cyclic subgroups, for some famous groups that we need for proofs of some main results.

A set C of vertices of a graph Γ whose induced subgraph is a complete subgraph is called a clique and the maximum size (if it exists) of a clique in a graph is called the clique number of the graph and is denoted by $\omega(\Gamma)$. Let G be a group. Following [1], we shall use the notation $\mathfrak{N}_G$ to denote the nonnilpotent graph as follows: take G as the vertex set and let two vertices be adjacent if they generate a nonnilpotent subgroup. Here, for an $\mathfrak{N}_n^c$-group G, we give some interesting relations between n and $\omega(\mathfrak{N}_G)$.

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Lemma 4.1. *Let G be an $\mathfrak{N}_n^c$-group. Then:*

(1) $\omega(\mathfrak{N}_G) \leq n$;

(2) *if G is a semisimple group, then* $\omega(\mathfrak{N}_G) + 1 \leq n$.

Proof. (1) The statement follows from Lemma 2.4.

(2) Let $\{a_1, \ldots, a_k\}$ be a clique of the graph $\mathfrak{N}_G$. As in the proof of part (2) of Lemma 3.3, it follows that $N_G(\langle a_i \rangle) \neq G$ for all $i \in \{1, \ldots, k\}$. This completes the proof, since $N_G(\langle e \rangle) = G$. □

Proposition 4.2. *If, for any prime power order q with $q > 5$, $L_2(q) \in \mathfrak{N}_s^c$ and* $\mathrm{PGL}(2, q) \in \mathfrak{N}_t^c$, *where $s, t \in \mathbb{N}$, then*

$$q^2 + q + 2 \leq s, t.$$

Proof. It follows from part (2) of Lemma 4.1 and [16, Proposition 4.2] that $q^2 + q + 2 \leq s$. Now, since

$$L_2(q) \cong \frac{Z\mathrm{SL}(2, q)}{Z} \leq \mathrm{PGL}(2, q)$$

where Z is the centre of $\mathrm{GL}(2, q)$, we can obtain, by Lemma 2.2, that $q^2 + q + 2 \leq t$, as required. □

Remark 4.3. According to Proposition 4.2, the group $L_2(7)$ is an $\mathfrak{N}_s^c$-group for some $58 \leq s$. But it is easy to see (for example, by GAP [7]) that $s = 58$.

Lemma 4.4. *Let G be a finite group, and p a prime divisor of the order of G. Suppose that all pairs of distinct Sylow p-subgroups of G intersect trivially. Let P and Q be different Sylow p-subgroups of G. If $x \in P$ and $y \in Q$, then $N_G(\langle x \rangle) \neq N_G(\langle y \rangle)$.*

Proof. Suppose, for a contradiction, that $N_G(\langle x \rangle) = N_G(\langle y \rangle)$. According to Lemma 2.4, the subgroup $\langle x, y \rangle$ is a nilpotent group. This implies that the subgroup $\langle x, y \rangle$ is a p-group so $\langle x, y \rangle \leq P$ or $\langle x, y \rangle \leq Q$. It follows that $P \cap Q \neq 1$, which is contrary to the hypothesis. Thus $N_G(\langle x \rangle) \neq N_G(\langle y \rangle)$. This completes the proof. □

Let G be a finite group, p a prime divisor of the order of G. We denote by $\nu_p(G)$ the number of Sylow p-subgroups of G such that every two distinct Sylow p-subgroups of G have trivial intersection. The following lemma is key to some of our results.

Lemma 4.5. *Let G be a finite $\mathfrak{N}_n^c$-group, and p a prime divisor of the order of G. Then $\nu_p(G) + 1 \leq n$.*

Proof. This follows from Lemma 4.4. □

We are now ready to conclude the proof of Theorem B.

Proof of Theorem B. By Proposition 2.5, G is finite. Suppose, to the contrary, that there exists a nonabelian finite simple $\mathfrak{N}_n^c$-group, G, not isomorphic to A_5 and of the least possible order such that $n \leq 57$. By [3, Proposition 3], [2, Theorem 1] and Lemma 2.2, it is enough to consider the following groups: $L_2(2^m)$, $m = 4$ or an odd

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prime; $L_2(3^m)$, $L_2(5^m)$, m a prime; $L_2(m)$, m a prime and $7 \le m$; $L_3(3)$; $L_3(5)$; $U_3(4)$ (the projective special unitary group of degree three over the finite field of order 4^2); or $Sz(2^p)$, p an odd prime.

If G is isomorphic to $L_2(m)$, m a prime and $7 \le m$ ($L_2(2^m)$, $m = 4$ or an odd prime, respectively), then by Proposition 4.2, we can see that $n \ge 58$ ($n \ge 74$, respectively), a contradiction. If G is isomorphic to $L_2(3^m)$ ($L_2(5^m)$, respectively), m a prime, then again, by Proposition 4.2, we can see that $n \ge 92$ ($n \ge 652$, respectively), a contradiction. Thus among the projective special linear groups, we only need to investigate the groups $L_3(3)$ and $L_3(5)$. If $G \cong L_3(3)$, then $|G| = 2^4 \times 3^3 \times 13$ and $\nu_{13}(G) = 144$. So $n \ge 145$, by Lemma 4.5, a contradiction. If $G \cong L_3(5)$, then $|G| = 5^3 \times 2^5 \times 3 \times 31$ and $\nu_{31}(G) = 4000$. Thus $n \ge 4001$, by Lemma 4.5, a contradiction. Therefore, we must consider the groups $U_3(4)$ and $Sz(2^m)$, m an odd prime. If $G \cong U_3(4)$, then it has order $2^6 \times 3 \times 5^2 \times 13$ (see Theorem 10.12(d) of Chapter II in [8]). So $\nu_{13}(G) = 1 + 13k$ for some $k > 0$ and since $\nu_{13}(G)$ divides $|G|$, we have $\nu_{13}(G) = 1600$ and so $n \ge 1601$, by Lemma 4.5, a contradiction. If $G \cong Sz(2^m)$ (m an odd prime), then it follows from Theorem 3.10 (and its proof) of Chapter XI in [9] that $\nu_2(G) \ge 2^{2m} + 1 \ge 65$. Hence $n \ge 66$, by Lemma 4.5, a contradiction. This completes the proof. □

Theorem B has the following consequence.

Theorem 4.6. *Let G be a nonabelian simple $\mathfrak{N}_n$-group with $n \le 57$. Then $G \cong A_5$.*

Proof. Since $\mathfrak{N}_n \subseteq \mathfrak{N}^c_m$ for some $m \le n$, it follows, by Theorem B, that $G \cong A_5$. On the other hand, we can obtain that $A_5 \in \mathfrak{N}_{27}$. This gives the result. □

We have a nice characterisation for A_5.

Corollary 4.7. *The only simple group with 27 (22, respectively) normalisers (normalisers of cyclic subgroups, respectively) is A_5.*

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MOHAMMAD ZARRIN, Department of Mathematics,
University of Kurdistan, PO Box 416, Sanandaj, Iran
and
School of Mathematics, Institute for Research in Fundamental Sciences (IPM),
PO Box: 19395-5746, Tehran, Iran
e-mail: m.zarrin@uok.ac.ir, zarrin78@gmail.com