Bull. Aust. Math. Soc. **86** (2012), 119–125
doi:10.1017/S0004972712000123

ON THE TOPOLOGICAL CENTRE OF $L^1(G/H)^{**}$

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(Received 29 August 2011)

Abstract

Let G be a locally compact group and H be a compact subgroup of G. Using a general criterion established by Neufang ['A unified approach to the topological centre problem for certain Banach algebras arising in abstract harmonic analysis', *Arch. Math.* **82**(2) (2004), 164–171], we show that the Banach algebra $L^1(G/H)$ is strongly Arens irregular for a large class of locally compact groups.

2010 *Mathematics subject classification*: primary 43A10; secondary 43A20.

Keywords and phrases: homogeneous space, topological centre, Arens product, factorisation property, Mazur property.

1. Introduction

In the last twenty years research on the topological centre problem has mostly centred around the Banach algebra $L^1(G)$, and has been dealt with by Lau *et al.* in the papers [5, 6, 8, 9]. They showed using different approaches that $L^1(G)$ is strongly Arens irregular, where G is a locally compact group. We recall that A is said to be Arens irregular if the topological centre of A^{**} is reduced to A itself. In [8] Neufang established a general criterion for a Banach algebra to be Arens irregular, which specifically led to the proof of strong Arens irregularity of the measure algebra $M(G)$ for a large class of locally compact groups.

Let A be a Banach algebra and κ be a cardinal number. We say that A^* has the property (F_κ) if for any family of functionals $(h_\alpha)_{\alpha\in I} \subseteq \mathrm{Ball}(A^*)$ there exist a family $(\psi_\alpha)_{\alpha\in I} \subseteq \mathrm{Ball}(A^{**})$ and a single functional $h \in A^*$ such that the factorisation formula

$$h_\alpha = h \cdot \psi_\alpha \tag{1.1}$$

holds, where '$\cdot$' is the second Arens product on A^{**} and the cardinality of I is at most κ.

Let A be a Banach algebra and $\kappa \geq \aleph_0$ be a cardinal number. A functional $f \in A^{**}$ is called w^*-κ-continuous if, for all nets $(x_\alpha)_{\alpha\in I} \subseteq \mathrm{Ball}(A^*)$ of cardinality $\aleph_0 \leq |I| \leq \kappa$ with $x_\alpha \xrightarrow{w^*} 0$, we have $f(x_\alpha) \to 0$. We say that A has the Mazur property of level κ (property (M_κ)) if every w^*-κ-continuous functional $f \in A^{**}$ is an element of A.

The following theorem is [8, Theorem 2.3].

THEOREM 1.1. *Let A be a Banach algebra satisfying (M_κ) and whose dual A^* has the property (F_κ), for some $\kappa \geq \aleph_0$. Then A is strongly Arens irregular.*

Let G be a locally compact group and H be a compact subgroup of G. Consider the homogeneous space G/H with a relatively invariant measure μ which arises from a rho-function ρ (see [1, 4, 10]). In [4, Theorem 4.4] it is shown that $L^1(G/H)$ is a Banach algebra. In this paper, for a large class of locally compact groups G, using Neufang's criterion (Theorem 1.1), we show that the Banach algebra $L^1(G/H)$ is strongly Arens irregular.

Let G be a locally compact group, H be a compact subgroup of G and μ be a relatively invariant measure which arises from a rho-function ρ on G/H. The mapping $T : L^1(G) \mapsto L^1(G/H)$ defined by

$$Tf(xH) = \int_H \frac{f(x\xi)}{\rho(x\xi)}\, d\xi \quad (\mu\text{-almost all } xH \in G/H)$$

is a surjective bounded linear operator with $\|T\| \leq 1$ (see [10]). Consider $\tilde{T}$ as the mapping from $M(G)$ to $M(G/H)$ defined by

$$\tilde{T}(\mu)(E) = \mu(q^{-1}(E)) \tag{1.2}$$

for each Borel subset $E \subseteq G/H$ and $\mu \in M(G)$, where $q : G \to G/H$ is the canonical quotient map $q(x) = xH$. Then it is easy to see that $\tilde{T}$ is onto and $M(G/H)$ is a Banach algebra endowed with the following convolution: for $\nu, \acute{\nu} \in M(G/H)$,

$$\nu * \acute{\nu} := \lambda * \acute{\lambda}(q^{-1}(E)), \tag{1.3}$$

where $\lambda, \acute{\lambda} \in M(G)$ and $\tilde{T}(\lambda) = \nu$, $\tilde{T}(\acute{\lambda}) = \acute{\nu}$ (see [10]).

Equip $L^1(G/H)^{**}$ with the second Arens product denoted by '$\cdot$' as follows: for $m, n \in L^1(G/H)^{**}$, $\eta \in L^1(G/H)^*$, $\varphi, \gamma \in L^1(G/H)$,

$$\begin{aligned}
\langle m \cdot n, \eta \rangle &= \langle n, \eta \cdot m \rangle \\
\langle \eta \cdot m, \gamma \rangle &= \langle m, \gamma \cdot \eta \rangle \\
\langle \gamma \cdot \eta, \varphi \rangle &= \langle \eta, \varphi * \gamma \rangle,
\end{aligned}$$

where '$*$' is the convolution of $L^1(G/H)$ (see [4]).

2. Main result

Throughout this paper we assume that G is a locally compact group and H is a compact subgroup of G. Denote by $\kappa(G)$ and $b(G)$ the compact covering number of G and the least cardinality of an open basis at the neutral element of G, respectively (see [8]). We show that $L^1(G/H)^*$ has the factorisation property of level $\kappa(G)$ and $L^1(G/H)$ satisfies the Mazur property of level $\kappa(G)$, for a large class of locally compact groups G. Theorem 1.1 will then imply that in this case $L^1(G/H)$ is strongly Arens irregular. Indeed, the main result of this paper is the following theorem.

https://doi.org/10.1017/S0004972712000123 Published online by Cambridge University Press

Theorem 2.1. *Let G be a locally compact noncompact group and H be a compact subgroup of G. Assume that $\kappa(G) \geq 2^{b(G)}$. Then $L^1(G/H)^*$ has the property $(F_{\kappa(G)})$ and $L^1(G/H)$ satisfies $(M_{\kappa(G)})$. In particular, $L^1(G/H)$ is strongly Arens irregular.*

To prove Theorem 2.1, we first discuss the factorisation property. To begin, we establish the following lemmata. Denote by L_y the left translation operator, defined by $L_y\gamma(xH) = \gamma(y^{-1}xH)$, $x, y \in G$ (see [4]).

Consider

$$\delta_{yH}(xH) = \begin{cases} 1 & xH = yH, \\ 0 & xH \neq yH. \end{cases}$$

Denote by $\hat{\delta}_{yH}$ the image of δ_{yH} under the canonical mapping $\hat{\ } : M(G/H) \to M(G/H)^{**}$.

Lemma 2.2. *Let G be a locally compact group and H be a compact subgroup of G. Consider G/H as the homogeneous space with relatively invariant measure μ which arises from a rho-function ρ. Then for $\gamma \in L^1(G/H)^*$,*

$$\gamma \cdot \hat{\delta}_{yH} = \frac{\rho(y)}{\rho(e)} L_{y^{-1}}\gamma, \tag{2.1}$$

where e is the identity element of G, $y \in G$.

Proof. Let $\gamma \in L^1(G/H)^*$, $\eta \in L^1(G/H)$. Then

$$\begin{aligned} \langle L_{y^{-1}}\gamma, \eta\rangle &= \int_{G/H} L_{y^{-1}}\gamma(xH)\eta(xH)\, d\mu(xH) \\ &= \int_{G/H} \gamma(yxH)\eta(xH)\, d\mu(xH) \\ &= \int_{G/H} \gamma(xH)L_y\eta(xH)\frac{\rho(y^{-1})}{\rho(e)}\, d\mu(xH) \\ &= \int_{G/H} \gamma(xH)L_y\eta(xH)\frac{\rho(e)}{\rho(y)}\, d\mu(xH), \end{aligned}$$

where the last equality follows from the identity (see [4])

$$\rho(xy) = \frac{\rho(x)\rho(y)}{\rho(e)}.$$

Therefore,

$$\frac{\rho(y)}{\rho(e)}\langle L_{y^{-1}}\gamma, \eta\rangle = \int_{G/H} \gamma(xH)L_y\eta(xH)\, d\mu(xH). \tag{2.2}$$

On the other hand,

$$\begin{aligned} \langle \gamma \cdot \hat{\delta}_{yH}, \eta\rangle &= \langle \hat{\delta}_{yH}, \eta\gamma\rangle \\ &= \langle \eta\gamma, \delta_{yH}\rangle \\ &= \langle \gamma, \delta_{yH} \cdot \eta\rangle \\ &= \gamma(\delta_{yH} * \eta), \end{aligned} \tag{2.3}$$

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where '$*$' in the last equality is the convolution in $M(G/H)$ defined as in (1.3). To continue the calculations in (2.3), note that since $L^1(G/H)$ is an ideal of $M(G/H)$, $\delta_{yH} * \eta \in L^1(G/H)$. It is easy to see that

$$\delta_{yH} = \tilde{T}(\delta_y),$$

where $\tilde{T}$ is as in (1.2). Now choose $g \in L^1(G)$ such that $\eta = Tg$. Then

$$\begin{aligned}
\delta_{yH} * \eta &= \tilde{T}(\delta_y) * \tilde{T}(g) \\
&= \tilde{T}(\delta_y * g) \\
&= T(\delta_y * g) \\
&= \int_H \frac{\delta_y * g(x\xi)}{\rho(x\xi)}\, d\xi \\
&= \int_H \int_G \frac{g(z^{-1}x\xi)\, d\delta_y(z)}{\rho(x\xi)}\, d\xi \\
&= \int_H \frac{g(y^{-1}x\xi)}{\rho(x\xi)}\, d\xi \\
&= T(L_y g)(xH) \\
&= L_y T g(xH) \\
&= L_y \eta(xH),
\end{aligned}$$

where in the above equalities we have used the fact that the restriction $\tilde{T}\,|_{L^1(G)}$ to $L^1(G)$ equals T and $\tilde{T}$ is a homomorphism. Thus (2.3) becomes

$$\langle \gamma \cdot \hat{\delta}_{yH}, \eta \rangle = \int_{G/H} \gamma(xH) L_y \eta(xH)\, d\mu(xH). \tag{2.4}$$

Comparing (2.2) and (2.4), we conclude (2.1). □

The following lemma is a generalisation of [6, Lemma 3] to the setting of G/H (see also [3, Lemma 2.1]).

Lemma 2.3. *Let G be a locally compact noncompact group and H be a compact subgroup of G. Then there exist a family of compact subsets $(K_\alpha)_{\alpha\in I}$ of G/H, indexed by I, and a family $(y_\alpha)_{\alpha\in I} \subseteq G$ such that $K_\alpha^\circ \neq \emptyset$, $\bigcup_{\alpha\in I} K_\alpha^\circ = G/H$, $(K_\alpha)_{\alpha\in I}$ is closed under finite unions and $(y_\alpha K_\alpha)_{\alpha\in I}$ are pairwise disjoint.*

Proof. Let $(K_\alpha)_{\alpha\in I}$ be a family of compact subsets with $K_\alpha^\circ \neq \emptyset$ such that $\bigcup_{\alpha\in I} K_\alpha^\circ = G/H$, and assume that I has minimal cardinality among all such families. By taking finite unions of such sets we may assume that $(K_\alpha)_{\alpha\in I}$ is closed under finite unions. Consider compact subsets E_α in G so that $K_\alpha = q(E_\alpha)$, where q is the canonical

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quotient map (see [1, Lemma 2.46]). Also assume that I is well ordered in such a way that each nontrivial order segment $\{i \in I, i \leq j\}$, $j \in I$, of I has smaller cardinality than I. We proceed by transfinite induction. Assume that for $\gamma < \alpha$, η_γ is chosen. Then for any $\gamma < \alpha$, $\eta_\gamma q(E_\gamma)q(E_\alpha^{-1})$ is compact, but by minimality of I, the union of these sets does not cover G/H. So we can choose $\eta_\alpha \in G/H - \bigcup_{\gamma \in I} \eta_\gamma q(E_\gamma)q(E_\alpha^{-1})$. Thus for each $\gamma < \alpha$, we have $\eta_\alpha \notin \eta_\gamma q(E_\gamma)q(E_\alpha^{-1})$. That is, $\eta_\alpha q(E_\alpha) \cap \eta_\gamma q(E_\gamma) = \emptyset$. Now choose representatives y_α of the cosets η_α such that $(y_\alpha K_\alpha)_{\alpha \in I}$ are pairwise disjoint. □

In the following proposition (with a proof similar to that for $L^1(G)$ [7]) we show that $L^1(G/H)^*$ has the property (F_κ), where κ is the least cardinality of a covering of G/H by compact subsets, which is, due to compactness of H, equal to the compact covering number $\kappa(G)$ of G.

Proposition 2.4. *Let G be a locally compact noncompact group, H be a compact subgroup of G and κ $(= \kappa(G))$ be the least cardinality of a covering of G/H by compact subsets. Then $L^1(G/H)^*$ has the property (F_κ).*

Proof. Let κ be the least cardinality of a covering of G/H by compact subsets and write $(K_\alpha)_{\alpha \in I}$ for the corresponding family of compact sets. Set

$$\tilde{I} = I \times I,\ \tilde{\alpha} = (\alpha, i) \in \tilde{I},\ K_{\tilde{\alpha}} = K_{(\alpha,i)} := K_\alpha.$$

Then $(K_{\tilde{\alpha}})_{\tilde{\alpha} \in \tilde{I}}$ is a covering of G/H with the same properties as the original one. Let $(y_{\tilde{\alpha}} K_{\tilde{\alpha}})$ be as in Lemma 2.3, that is,

$$(y_{\tilde{\alpha}} K_{\tilde{\alpha}}) \cap (y_{\tilde{\beta}} K_{\tilde{\beta}}) = \emptyset, \quad \tilde{\alpha} \neq \tilde{\beta} \in \tilde{I}. \tag{2.5}$$

Define a partial ordering on $\tilde{I}$ by setting, for $(\alpha, i), (\beta, j) \in \tilde{I}$,

$$(\alpha, i) \leq (\beta, j) \Longleftrightarrow K_{(\alpha,i)} \subseteq K_{(\beta,j)},$$

and on I by

$$\alpha \leq \beta \Longleftrightarrow K_\alpha \subseteq K_\beta.$$

Define

$$\acute{\psi}_j := w^* - \lim_\beta \hat{\delta}_{y_{(\beta,j)}H}$$

and let ψ_j be an arbitrary Hahn–Banach extension of $\acute{\psi}_j$ to $L^\infty(G/H)^*$. Let $(\eta_i)_{i \in I} \subseteq \mathrm{Ball}(L^1(G/H)^*)$. Put

$$\eta := \sum_{(\alpha,i) \in I \times I} L_{y_{(\alpha,i)}}(\chi_{K_{(\alpha,i)}} \eta_i) \frac{\rho(e)}{\rho(y_{(\alpha,i)})}.$$

Using Lemma 2.2, for (α, i), (β, j), $(\gamma, k) \in I \times I$, where $(\gamma, k) \leq (\beta, j)$,

$$\begin{aligned} \eta \cdot \psi_j &= w^* - \lim_\beta \sum_{(\alpha,i) \in I \times I} \frac{\rho(e)}{\rho(y_{(\alpha,i)})} L_{y_{(\alpha,i)}}(\chi_{K_{(\alpha,i)}} \eta_i) \cdot \hat{\delta}_{y_{(\beta,j)}H} \\ &= w^* - \lim_\beta \sum_{(\alpha,i) \in I \times I} \frac{\rho(e)}{\rho(y_{(\alpha,i)})} \frac{\rho(y_{(\beta,j)})}{\rho(e)} L_{y^{-1}_{(\beta,j)}} L_{y_{(\alpha,i)}}(\chi_{K_{(\alpha,i)}} \eta_i). \end{aligned} \tag{2.6}$$

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Using (2.5),

$$
\begin{aligned}
&\frac{\rho(y_{(\beta,i)})}{\rho(y_{(\alpha,i)})}\chi_{K_{(\gamma,k)}}L_{y^{-1}_{(\beta,j)}}L_{y_{(\alpha,i)}}(\chi_{K_{(\alpha,i)}}\eta_i)\\
&\quad=\frac{\rho(y_{(\beta,j)})}{\rho(y_{(\alpha,i)})}\chi_{K_{(\gamma,k)}}\chi_{K_{(\beta,j)}}L_{y^{-1}_{(\beta,j)}}L_{y_{(\alpha,i)}}(\chi_{K_{(\alpha,i)}}\eta_i)\\
&\quad=\frac{\rho(y_{(\beta,j)})}{\rho(y_{(\alpha,i)})}\chi_{K_{(\gamma,k)}}[L_{y^{-1}_{(\beta,j)}}(L_{y_{(\beta,j)}}\chi_{K_{(\beta,j)}})(L_{y_{(\alpha,i)}}(\chi_{K_{(\alpha,i)}}\eta_i))]\\
&\quad=\frac{\rho(y_{(\beta,j)})}{\rho(y_{(\alpha,i)})}\delta_{(\alpha,i)(\beta,j)}\chi_{K_{(\gamma,k)}}\eta_j.
\end{aligned}
\tag{2.7}
$$

Now (2.6) and (2.7) imply that, for $j \in I$ and $(\gamma, k) \in I \times I$,

$$
\begin{aligned}
\chi_{K_{(\gamma,k)}}(\eta.\psi_j) &= w^* - \lim_{\beta}\sum_{(\alpha,i)\in I\times I}\frac{\rho(y_{(\beta,j)})}{\rho(y_{(\alpha,i)})}\delta_{(\alpha,i)(\beta,j)}\chi_{K_{(\gamma,k)}}\eta_j\\
&= \chi_{K_{(\gamma,k)}}\eta_j,
\end{aligned}
$$

which completes the proof. □

Finally, we discuss the Mazur property of $L^1(G/H)$. Let G be a locally compact noncompact group, for which $k(G) \geq 2^{b(G)}$. Let H be a compact subgroup of G. Then $M(G)$ has the Mazur property of level $k(G)$ (see [8]). Since $L^1(G/H)$ is an ideal of $M(G/H)$, and $M(G/H)$ is a linear subspace of $M(G)$ [10], $L^1(G/H)$ is a linear subspace of $M(G)$. Thus by [2, Remark 1.5] we conclude that $M(G/H)$ has the Mazur property of level $k(G)$.

Now Theorem 2.1 is a consequence of the above argument on the Mazur property of $L^1(G/H)$ together with Proposition 2.4 and Theorem 1.1.

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