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L^p-SPECTRAL MULTIPLIERS FOR SOME ELLIPTIC SYSTEMS

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Abstract We show results on L^p-spectral multipliers for Maxwell operators with bounded measurable coefficients. We also present similar results for the Stokes operator with Hodge boundary conditions and the Lamé system. Here, we rely on resolvent estimates recently established by Mitrea and Monniaux.

Keywords: spectral multipliers; Maxwell operator; Stokes operator; Lamé system; Lipschitz domains; Hodge boundary conditions

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1. Introduction

For self-adjoint operators $A \geqslant 0$ in a Hilbert space H, the spectral theorem establishes a functional calculus for bounded Borel-measurable functions $F\colon [0,\infty) \to \mathbb{C}$. This property is crucial in countless applications in mathematical physics. In particular, in the context of nonlinear phenomena one studies differential operators and associated semigroup or resolvent operators also in spaces L^p for $p \neq 2$. In this context, the holomorphic H^∞-functional calculus, i.e. a functional calculus for bounded holomorphic functions on a complex sector symmetric to the real half-line, has turned out to be a very useful tool. But, if the operator is self-adjoint in L^2, it might have a better functional calculus in L^p for $p \neq 2$ for appropriate functions $F\colon [0,\infty) \to \mathbb{C}$.

The classical result in this field is Hörmander's spectral multiplier theorem for $A = -\Delta$ on $\mathbb{R}^D$ from 1960 (see Theorem 2.1). Various generalizations of this result have been given since then, in several directions. Quite recently, considerable progress has been made (see [**7**, **15**, **17**, **22**, **24**]) concerning operators for which the associated semigroups satisfy generalized Gaussian bounds or Davies–Gaffney estimates (see § 2 for more details). In this paper we show that these results can be applied to several elliptic systems, namely, the Stokes operator with Hodge boundary conditions, the Lamé system and the Maxwell operator.

The Maxwell operator is of great importance in the study of electrodynamics. Following the outline in [**10**, Chapter 6], we briefly explain how an interest in its spectral properties arises. The Maxwell equations

$$\operatorname{rot} \mathcal{E} + \partial_t \mathcal{H} = 0, \quad \operatorname{rot} \mathcal{H} - \varepsilon(\cdot)\partial_t \mathcal{E} = 0, \quad \operatorname{div} \mathcal{H} = 0 \quad \text{in } \Omega$$

govern the propagation of electromagnetic waves in a region $\Omega \subset \mathbb{R}^3$. Here, $\mathcal{E}\colon \Omega\times\mathbb{R} \to \mathbb{R}^3$ and $\mathcal{H}\colon \Omega \times \mathbb{R} \to \mathbb{R}^3$ denote the electric and magnetic field, respectively, whereas the matrix-valued function $\varepsilon(\cdot)\colon \Omega \to \mathbb{R}^{3\times 3}$ describes the electric permittivity. The magnetic permeability is taken to be the identity matrix, and the electric conductivity to be 0. We assume perfect conductor boundary conditions:

$$\nu \times \mathcal{E} = 0, \quad \nu \cdot \mathcal{H} = 0 \quad \text{on } \partial\Omega.$$

If the waves behave time periodically with respect to the same frequency $\omega > 0$, the ansatz $\mathcal{E}(x,t) = \mathrm{e}^{-\mathrm{i}\omega t}E(x)$ and $\mathcal{H}(x,t) = \mathrm{e}^{-\mathrm{i}\omega t}H(x)$ leads to the *time-harmonic Maxwell equations* $\operatorname{rot} E - \mathrm{i}\omega H = 0$ and $\operatorname{rot} H + \mathrm{i}\omega\varepsilon(\cdot)E = 0$. Elimination of E finally yields that

$$\begin{aligned} \operatorname{rot}\varepsilon(\cdot)^{-1}\operatorname{rot} H - \omega^2 H &= 0 \quad \text{in } \Omega, \\ \operatorname{div} H &= 0 \quad \text{in } \Omega, \\ \nu \cdot H &= 0 \quad \text{on } \partial\Omega, \\ \nu \times \varepsilon(\cdot)^{-1}\operatorname{rot} H &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

We call the operator $\operatorname{rot}\varepsilon(\cdot)^{-1}\operatorname{rot}$ the *Maxwell operator*, and we study it in the following setting. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$ and let $\varepsilon(\cdot) \in L^\infty(\Omega, \mathbb{C}^{3\times 3})$ be a matrix-valued function such that $\varepsilon(\cdot)^{-1} \in L^\infty(\Omega, \mathbb{C}^{3\times 3})$ and $\varepsilon(x) \in \mathbb{C}^{3\times 3}$ is a positive definite Hermitian matrix for almost all $x \in \Omega$. We emphasize that no additional regularity assumptions on $\varepsilon(\cdot)$ are made. This is of interest in solid state physics, e.g. for photonic crystals. Inspired by the approach in [**29**], we consider in $L^2(\Omega, \mathbb{C}^3)$ the operator A_2, which is associated with the densely defined sesquilinear form

$$\mathfrak{a}(u,v) := \int_\Omega \varepsilon(\cdot)^{-1}\operatorname{rot} u \cdot \overline{\operatorname{rot} v}\,\mathrm{d}x + \int_\Omega \operatorname{div} u\,\overline{\operatorname{div} v}\,\mathrm{d}x \quad (u, v \in \mathcal{D}(\mathfrak{a})),$$

where $\mathcal{D}(\mathfrak{a}) := \{u \in L^2(\Omega, \mathbb{C}^3)\colon \operatorname{div} u \in L^2(\Omega, \mathbb{C}),\ \operatorname{rot} u \in L^2(\Omega, \mathbb{C}^3),\ \nu \cdot u|_{\partial\Omega} = 0\}$. Here and in the following, $\nu(x)$ denotes the outer normal at a point x of the boundary $\partial\Omega$, and the operators div and rot are defined in the distributional sense (see, for example, [**2**]).

In order to apply the recent results on spectral multipliers mentioned above, it is necessary to establish generalized Gaussian estimates for the semigroup $(\mathrm{e}^{-tA_2})_{t>0}$ associated with the operator A_2 (see Theorem 5.1). We do this via Davies's perturbation method, and we thus obtain that a spectral multiplier theorem holds for A_2 (see Theorem 5.6). We define the Maxwell operator M_2 as the restriction of A_2 to the space of divergence-free vector fields. Since the Helmholtz projection and A_2 are commuting (see Lemma 5.4), many properties of A_2 can be transferred to the Maxwell operator M_2. This includes, in particular, the validity of the spectral multiplier theorem (see Theorem 5.7).

Besides the Maxwell operator, we study the Stokes operator with Hodge boundary conditions in bounded Lipschitz domains using results from [**29**]. Actually, this operator corresponds to the special case of $\varepsilon(x)$ being the identity matrix for every $x \in \Omega$. The operator A_2 then equals the *Hodge Laplacian* (observe that [**29**] also studied a Maxwell operator, but a different one to ours). Mitrea and Monniaux [**29**] proved that A_2 is then given by

$$\mathcal{D}(A_2) = \{u \in V(\Omega)\colon\ \operatorname{rot}\operatorname{rot} u \in L^2(\Omega, \mathbb{C}^3),\ \operatorname{div} u \in H^1(\Omega, \mathbb{C}),\ \nu \times \operatorname{rot} u|_{\partial\Omega} = 0\},$$
$$A_2 u = \operatorname{rot}\operatorname{rot} u - \nabla \operatorname{div} u = -\Delta u \quad \text{for } u \in \mathcal{D}(A_2),$$

and that $-A_2$ generates an analytic semigroup on $L^p(\Omega, \mathbb{C}^3)$ for all $p \in (p_\Omega, p'_\Omega)$, where $p'_\Omega > 3$ and $1/p_\Omega + 1/p'_\Omega = 1$. As a consequence, they obtained that (minus) the Stokes operator with Hodge boundary conditions, which is defined as the restriction of the Hodge Laplacian on the space of divergence-free vector fields, also generates an analytic semigroup on $L^p(\Omega, \mathbb{C}^3)$ for all $p \in (p_\Omega, p'_\Omega)$. We show that even a spectral multiplier theorem holds for the Stokes operator with Hodge boundary conditions (see Theorem 3.4). Our arguments rely on the proof of Mitrea and Monniaux, in which certain two-ball estimates for the resolvents of the Hodge Laplacian were verified. We prove that these kinds of bounds entail generalized Gaussian estimates for the corresponding semigroup operators (see Lemma 2.5) and, thus, the same reasoning as for the Maxwell operator can be used to obtain Theorem 3.4.

By using a similar approach based on [**30**], we verify generalized Gaussian estimates for the time-dependent Lamé system equipped with homogeneous Dirichlet boundary conditions. Thus, we obtain a spectral multiplier theorem for the Lamé system (see Theorem 4.1).

We mention that the generalized Gaussian estimates we establish for the elliptic systems in this paper have other consequences that have not been mentioned in the literature so far. Application of a result from [**5**] yields boundedness of H^∞-functional calculus in the stated range of L^p-spaces. Of course, this weaker assertion also follows from the results on spectral multipliers of the present paper. Due to [**6**, Corollary 1.5] (one could also use results of Arendt or Davies), the spectrum of these operators in L^p does not depend on p for the stated range of L^p-spaces. Finally, we note that, in general, pointwise Gaussian kernel estimates for all the above operators fail.

This paper has the following structure. In § 2 we present the spectral multiplier result we will use (see Theorem 2.3) and a lemma that allows us to obtain generalized Gaussian estimates from estimates on resolvent operators (see Lemma 2.5). We postpone its proof till § 6. Via this lemma and estimates from [**29**, **30**] we obtain our results on the Stokes operator with Hodge boundary conditions and the Lamé system, which are presented in §§ 3 and 4, respectively. Section 5 is devoted to generalized Gaussian estimates for A_2 with $\varepsilon(\cdot)$ as above, and to spectral multiplier results for the Maxwell operator M_2.

Throughout this paper, we make use of the following notation. For $p \in [1, \infty]$, the conjugate exponent p' is defined by $1/p + 1/p' = 1$ with the usual convention $1/\infty := 0$. In the proofs, the letters b, C denote generic positive constants that are independent of the relevant parameters involved in the estimates and may take different values at different occurrences. We often use the notation $a \lesssim b$ if there exists a constant $C > 0$ such that $a \leqslant Cb$ for two non-negative expressions a, b; $a \cong b$ stands for the validity of $a \lesssim b$ and $b \lesssim a$. Moreover, the notation $|E|$ for a Lebesgue measurable subset E of $\mathbb{R}^D$ stands for the D-dimensional Lebesgue measure of E.

2. Spectral multiplier theorems

In this section we quote and discuss results on spectral multipliers. Let (X, d, μ) be a *space of homogeneous type* in the sense of Coifman and Weiss, i.e. (X, d) is a non-empty metric space endowed with a σ-finite regular Borel measure μ with $\mu(X) > 0$ that satisfies the so-called *doubling condition*, that is, there exists a constant $C > 0$ such that, for all $x \in X$ and all $r > 0$,

$$\mu(B(x, 2r)) \leqslant C\mu(B(x, r)), \tag{2.1}$$

where $B(x, r) := \{y \in X : d(y, x) < r\}$. It is easy to see that the doubling condition (2.1) entails the *strong homogeneity property*, i.e. the existence of constants $C, D > 0$ such that, for all $x \in X$, all $r > 0$ and all $\lambda \geqslant 1$,

$$\mu(B(x, \lambda r)) \leqslant C\lambda^D \mu(B(x, r)). \tag{2.2}$$

In the following, the value D always refers to the constant in (2.2), which is also called the *dimension* of (X, d, μ). Of course, D is not uniquely determined.

There is a multitude of examples of spaces of homogeneous type. The simplest is the Euclidean space $\mathbb{R}^D$, $D \in \mathbb{N}$, equipped with the Euclidean metric and Lebesgue measure. Bounded open subsets of $\mathbb{R}^D$ with Lipschitz boundary endowed with the Euclidean metric and Lebesgue measure also form spaces of homogeneous type (with $\mu(B(x, r)) \cong r^D$). More general definitions of spaces of homogeneous type can be found in [**9**, Chapitre III.1] or in [**33**, § I.1.2].

Let A be a non-negative self-adjoint operator on the Hilbert space $L^2(X)$. If E_A denotes the resolution of the identity associated with A, the spectral theorem asserts that the operator

$$F(A) := \int_0^\infty F(\lambda)\, \mathrm{d}E_A(\lambda)$$

is well defined and acts as a bounded linear operator on $L^2(X)$ whenever $F : [0, \infty) \to \mathbb{C}$ is a bounded Borel function. Spectral multiplier theorems provide regularity assumptions on F that ensure that the operator $F(A)$ extends from $L^p(X) \cap L^2(X)$ to a bounded linear operator on $L^p(X)$ for all p ranging in some interval $I \subset (1, \infty)$ containing 2.

In 1960, Hörmander addressed this question for the Laplacian $A = -\Delta$ on $\mathbb{R}^D$ during his studies of the boundedness of Fourier multipliers on $\mathbb{R}^D$. In order to formulate his famous result, we fix once and for all a non-negative cut-off function $\omega \in C_c^\infty(0, \infty)$ such that

$$\operatorname{supp} \omega \subset (\tfrac{1}{4}, 1) \qquad \text{and} \qquad \sum_{n \in \boldsymbol{Z}} \omega(2^{-n}\lambda) = 1 \quad \text{for all } \lambda > 0.$$

Theorem 2.1 (Hörmander [20, Theorem 2.5]). *If $F : [0, \infty) \to \mathbb{C}$ is a bounded Borel function such that*

$$\sup_{n \in \boldsymbol{Z}} \|\omega F(2^n \cdot)\|_{H_2^s} < \infty$$

for some $s > D/2$, then $F(-\Delta)$ is a bounded linear operator on $L^p(\mathbb{R}^D)$ for all $p \in (1, \infty)$, and one has that

$$\|F(-\Delta)\|_{L^p \to L^p} \leqslant C_p\Big(\sup_{n \in \boldsymbol{Z}} \|\omega F(2^n \cdot)\|_{H_2^s} + |F(0)|\Big),$$

where C_p is a constant not depending on F.

Hörmander's multiplier theorem was generalized, on the one hand, to other spaces beyond $\mathbb{R}^D$ and, on the other hand, to more general operators than the Laplacian. Mauceri and Meda [**27**] and Christ [**8**] extended the result to homogeneous Laplacians on stratified nilpotent Lie groups. Further generalizations were obtained by Alexopoulos [**1**], who showed a corresponding statement for the left invariant sub-Laplacian in the setting of connected Lie groups of polynomial volume growth. This, in turn, was extended by Hebisch [**19**] to integral operators with kernels decaying polynomially away from the diagonal. The results in [**16**] due to Duong *et al.* marked an important step toward the study of more general operators. In the abstract framework of spaces of homogeneous type they investigated non-negative self-adjoint operators A on $L^2(X)$ that satisfy *pointwise Gaussian estimates*, i.e. the semigroup $(\mathrm{e}^{-tA})_{t>0}$ generated by $-A$ can be represented as integral operators,

$$\mathrm{e}^{-tA} f(x) = \int_X p_t(x, y) f(y)\, \mathrm{d}\mu(y)$$

for all $f \in L^2(X)$, $t > 0$, μ-almost everywhere (a.e.) $x \in X$, and the kernels $p_t\colon X \times X \to \mathbb{C}$ have the pointwise upper bound

$$|p_t(x, y)| \leqslant C\mu(B(x, t^{1/2}))^{-1} \exp\left(-b\frac{d(x, y)^2}{t}\right) \tag{2.3}$$

for all $t > 0$ and all $x, y \in X$, where $b, C > 0$ are constants independent of t, x, y. Under these hypotheses, the operator $F(A)$ is of weak type $(1, 1)$ whenever $F\colon [0, \infty) \to \mathbb{C}$ is a bounded Borel function such that $\sup_{n \in \boldsymbol{Z}} \|\omega F(2^n \cdot)\|_{C^s} < \infty$ for some $s > D/2$ (see [**16**, Theorem 3.1]). Consequently, $F(A)$ is bounded on $L^p(X)$ for all $p \in (1, \infty)$.

Sometimes it is not clear whether (or even not true that) a non-negative self-adjoint operator on $L^2(X)$ admits such Gaussian bounds and, thus, whether the above result is applicable. This occurs, for example, for Schrödinger operators with bad potentials (see [**32**]) or elliptic operators of higher order with bounded measurable coefficients (see [**14**]). Nevertheless, it is often possible to establish a weakened version of pointwise Gaussian estimates, so-called generalized Gaussian estimates.

Definition 2.2. Let $1 \leqslant p \leqslant 2 \leqslant q \leqslant \infty$. A non-negative self-adjoint operator A on $L^2(X)$ satisfies *generalized Gaussian (p, q)-estimates* if there exist constants $b, C > 0$ such that

$$\|\mathbf{1}_{B(x, t^{1/2})} \mathrm{e}^{-tA} \mathbf{1}_{B(y, t^{1/2})}\|_{L^p \to L^q} \leqslant C\mu(B(x, t^{1/2}))^{-(1/p - 1/q)} \exp\left(-b\frac{d(x, y)^2}{t}\right) \tag{2.4}$$

for all $t > 0$ and all $x, y \in X$. In this case, we use the shorthand notation $\mathrm{GGE}(p, q)$. If A satisfies $\mathrm{GGE}(2, 2)$, then we also say that A has *Davies–Gaffney estimates.*

Here, $\mathbf{1}_{E_1}$ denotes the characteristic function of the set E_1 and $\|\mathbf{1}_{E_1}\mathrm{e}^{-tA}\mathbf{1}_{E_2}\|_{L^p\to L^q}$ is defined via $\sup_{\|f\|_{L^p}\leqslant 1}\|\mathbf{1}_{E_1}\cdot \mathrm{e}^{-tA}(\mathbf{1}_{E_2}f)\|_{L^q}$ for all Borel sets $E_1, E_2\subset X$.

In the case $(p,q)=(1,\infty)$, this definition covers Gaussian estimates (see [**4**, Proposition 2.9]). It is known that, for the class of operators A satisfying $\mathrm{GGE}(p_0,p_0')$, where $p_0\in[1,2)$, the interval $[p_0,p_0']$ is, in general, optimal for the existence of the semigroup $(\mathrm{e}^{-tA})_{t>0}$ on $L^p(X)$ for each $p\in[p_0,p_0']$ (see, for example, [**14**, Theorem 10]).

In [**24**] we show a spectral multiplier result that also covers operators that have generalized Gaussian estimates.

Theorem 2.3 (Kunstmann and Uhl [24, Theorem 5.4]). *Assume that A is a non-negative self-adjoint operator on $L^2(X)$ satisfying generalized Gaussian (p_0,p_0')-estimates for some $p_0\in[1,2)$. Let $p\in(p_0,p_0')$ and let $s>D|1/p-1/2|$. Then, for any bounded Borel function $F\colon[0,\infty)\to\mathbb{C}$ with $\sup_{n\in\boldsymbol{Z}}\|\omega F(2^n\cdot)\|_{C^s}<\infty$, the operator $F(A)$ is bounded on $L^p(X)$. More precisely, there exists a constant $C_p>0$ such that*

$$\|F(A)\|_{L^p\to L^p}\leqslant C_p\Big(\sup_{n\in\boldsymbol{Z}}\|\omega F(2^n\cdot)\|_{C^s}+|F(0)|\Big).$$

Remark 2.4.

(1) The spectral multiplier result in [**16**] corresponds to the case $p_0=1$, i.e. to the case of Gaussian-type kernel bounds (2.3).

(2) There is an earlier version of the result due to Blunck (see [**3**, Theorem 1.1]) under stronger assumptions on the differentiability order s in the Hörmander condition.

(3) The assertion of Theorem 2.3 remains valid for vector-valued operators on $L^p(X,\mathbb{C}^n)$.

(4) There are spectral multiplier results for Hardy spaces H^1_A associated with elliptic second-order operators A (see [**15, 17**]). Via an interpolation argument already used in [**21**], this also yields the assertion of Theorem 2.3. In fact, this is the idea behind the approach we use in [**24**]. However, the result in [**24**] applies to operators satisfying generalized Gaussian estimates of *any order*, which means that essential technical tools such as the finite propagation speed of the wave equation for A, which one has for second-order operators, cannot be used.

(5) Theorem 2.3 is formulated with Hölder spaces C^s in place of Bessel potential spaces H^s_2. We refer the reader to the discussion in [**16**], where it is pointed out that, in general, one cannot replace C^s by H^s_2 without additional assumptions.

(6) Very recently, a spectral multiplier theorem under the assumption of general Gaussian estimates of second order has been shown in [**7**]. The proof does not rely on a result in Hardy spaces. It relies on results from [**3**] and makes even heavier use of finite propagation speed, which only holds for second-order estimates.

For the rest of this paper we work in the Euclidean setting, and present operators A having generalized Gaussian estimates so that Theorem 2.3 can be applied. Since an analytic semigroup $(\mathrm{e}^{-tA})_{t>0}$ and resolvents of its generator $-A$ are intimately related via integral representations, we obtain a nearly equivalent formulation of generalized Gaussian estimates if we replace in the two-ball estimate (2.4) the semigroup operators with resolvent operators of the form $\lambda(\lambda+A)^{-1}$ for $\lambda \in \rho(-A)$. To be precise, the transfer from resolvent operators to semigroup operators and vice versa reads as follows.

Lemma 2.5. *Let $\Omega \subset \mathbb{R}^D$ be a Borel set, $n \in \mathbb{N}$, and let A be a non-negative self-adjoint operator on $L^2(\Omega, \mathbb{C}^n)$. Assume that $1 \leqslant p \leqslant 2 \leqslant q \leqslant \infty$ and $m \geqslant 2$ such that $(D/m)(1/p - 1/q) < 1$.*

(a) *Fix $\theta \in (0, \pi/2)$ and suppose that there exist constants $b, C > 0$ such that, for all $x, y \in \Omega$ and all $\lambda \in \mathbb{C} \setminus \{0\}$ with $|\arg \lambda| < \pi - \theta$,*

$$\|\mathbf{1}_{B(x,|\lambda|^{-1/m})}\lambda(\lambda + A)^{-1}\mathbf{1}_{B(y,|\lambda|^{-1/m})}\|_{L^p(\Omega,\mathbb{C}^n)\to L^q(\Omega,\mathbb{C}^n)} \leqslant C|\lambda|^{(D/m)(1/p-1/q)}\mathrm{e}^{-b|\lambda|^{1/m}|x-y|}. \quad (2.5)$$

There then exist constants $b', C' > 0$ such that the semigroup operators satisfy

$$\|\mathbf{1}_{B(x,t^{1/m})}\mathrm{e}^{-tA}\mathbf{1}_{B(y,t^{1/m})}\|_{L^p(\Omega,\mathbb{C}^n)\to L^q(\Omega,\mathbb{C}^n)} \leqslant C' t^{-(D/m)(1/p-1/q)} \exp\left(-b'\left(\frac{|x-y|}{t^{1/m}}\right)^{m/(m-1)}\right) \quad (2.6)$$

for any $t > 0$ and any $x, y \in \Omega$.

(b) *Suppose that there exist constants $b, C > 0$ such that, for all $t > 0$ and all $x, y \in \Omega$,*

$$\|\mathbf{1}_{B(x,t^{1/m})}\mathrm{e}^{-tA}\mathbf{1}_{B(y,t^{1/m})}\|_{L^p(\Omega,\mathbb{C}^n)\to L^q(\Omega,\mathbb{C}^n)} \leqslant C t^{-(D/m)(1/p-1/q)} \exp\left(-b\left(\frac{|x-y|}{t^{1/m}}\right)^{m/(m-1)}\right).$$

Then, for any $\theta \in (0, \pi/2)$, there exist constants $b', C' > 0$ such that, for all $x, y \in \Omega$ and all $\lambda \in \mathbb{C} \setminus \{0\}$ with $|\arg \lambda| < \theta$,

$$\|\mathbf{1}_{B(x,|\lambda|^{-1/m})}\lambda(\lambda + A)^{-1}\mathbf{1}_{B(y,|\lambda|^{-1/m})}\|_{L^p(\Omega,\mathbb{C}^n)\to L^q(\Omega,\mathbb{C}^n)} \leqslant C'|\lambda|^{(D/m)(1/p-1/q)}\mathrm{e}^{-b'|\lambda|^{1/m}|x-y|}.$$

As applications of this result, whose proof is postponed to § 6, we obtain (based on results from [**29**] and [**30**]) generalized Gaussian estimates for the Hodge Laplacian and for the Lamé system with Dirichlet boundary conditions, so that these operators fall under the scope of Theorem 2.3. This is done in the next two sections. In § 5 we verify generalized Gaussian estimates directly and obtain our spectral multiplier result for the Maxwell operator.

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3. The Stokes operator with Hodge boundary conditions

We show that the spectral multiplier result, presented in Theorem 2.3, holds for the Stokes operator A with Hodge boundary conditions. Our argument is based on off-diagonal norm estimates for the resolvents of the Hodge Laplacian, which were recently established by Mitrea and Monniaux [**29**]. According to Lemma 2.5, bounds of this type entail the validity of generalized Gaussian estimates for the Hodge Laplacian, so that Theorem 2.3 can be applied. Since A is the restriction of the Hodge Laplacian to the space of divergence-free vector fields, and the Hodge Laplacian and the Helmholtz projection are commuting, we obtain a spectral multiplier theorem for the Stokes operator with Hodge boundary conditions.

First, we provide a short overview of the definitions and some basic properties of the natural function spaces needed to define the Stokes operator. We start with the specification of the underlying domain. Throughout the section, let Ω be a *bounded Lipschitz domain in* $\mathbb{R}^3$, i.e. a bounded connected open subset of $\mathbb{R}^3$ with a Lipschitz continuous boundary $\partial\Omega$. This definition allows domains with corners, but cuts or cusps are excluded. Furthermore, we remark that the unit exterior normal field $\nu\colon \partial\Omega \to \mathbb{R}^3$ can then be defined almost everywhere on the boundary $\partial\Omega$ of Ω.

We consider the differential operators divergence (div) and rotation (rot) on $L^2(\Omega, \mathbb{C}^3)$ in the distributional sense, and introduce the function space

$$V(\Omega) := \{u \in L^2(\Omega, \mathbb{C}^3)\colon \operatorname{div} u \in L^2(\Omega, \mathbb{C}),\ \operatorname{rot} u \in L^2(\Omega, \mathbb{C}^3),\ \nu \cdot u|_{\partial\Omega} = 0\} \qquad (3.1)$$

equipped with the inner product

$$(u, v)_{V(\Omega)} := (u, v)_{L^2(\Omega,\mathbb{C}^3)} + (\operatorname{div} u, \operatorname{div} v)_{L^2(\Omega,\mathbb{C})} + (\operatorname{rot} u, \operatorname{rot} v)_{L^2(\Omega,\mathbb{C}^3)}.$$

Then, $V(\Omega)$ becomes a Hilbert space that is dense in $L^2(\Omega, \mathbb{C}^3)$. Note that the boundary condition of $V(\Omega)$ means that the normal component vanishes. In general, $V(\Omega)$ is not contained in $H^1(\Omega, \mathbb{C}^3)$ (see, for example, [**2**, p. 832]). However, under additional assumptions on the domain Ω, the space $V(\Omega)$ is continuously embedded into $H^1(\Omega, \mathbb{C}^3)$. For example, this is the case if Ω has a $C^{1,1}$-boundary or if Ω is convex (see, for example, [**2**, Theorems 2.9 and 2.17]). Nevertheless, the following statement due to Mitrea *et al*. (see [**31**, p. 87]) holds for arbitrary bounded Lipschitz domains Ω in $\mathbb{R}^3$.

Fact 3.1. *The space $V(\Omega)$ is continuously embedded into $H^{1/2}(\Omega, \mathbb{C}^3)$. More precisely, there exists a constant $C > 0$ depending only on the boundary $\partial\Omega$ and on the diameter* $\operatorname{diam}(\Omega)$ *of Ω such that, for every $u \in V(\Omega)$,*

$$\|u\|_{H^{1/2}(\Omega,\mathbb{C}^3)} \leqslant C(\|u\|_{L^2(\Omega,\mathbb{C}^3)} + \|\operatorname{div} u\|_{L^2(\Omega,\mathbb{C})} + \|\operatorname{rot} u\|_{L^2(\Omega,\mathbb{C}^3)}).$$

Recall the definition of the *Hodge Laplacian B*, which is the operator associated with the densely defined sesquilinear symmetric form

$$\mathfrak{b}(u, v) := \int_\Omega \operatorname{rot} u \cdot \overline{\operatorname{rot} v} \,\mathrm{d}x + \int_\Omega \operatorname{div} u \,\overline{\operatorname{div} v} \,\mathrm{d}x \quad (u, v \in V(\Omega)).$$

Then, B is self-adjoint invertible, and $-B$ generates an analytic semigroup on $L^2(\Omega, \mathbb{C}^3)$. According to [**29**, (3.17) and (3.18)], the Hodge Laplacian B can be characterized by

$$\mathcal{D}(B) = \{u \in V(\Omega)\colon \operatorname{rot}\operatorname{rot} u \in L^2(\Omega, \mathbb{C}^3),\ \operatorname{div} u \in H^1(\Omega, \mathbb{C}),\ \nu \times \operatorname{rot} u|_{\partial\Omega} = 0\},$$
$$Bu = -\Delta u \quad \text{for } u \in \mathcal{D}(B).$$

In order to introduce the Stokes operator, we first recall some basic facts concerning the Helmholtz decomposition in $L^p(\Omega, \mathbb{C}^3)$. For $p \in (1, \infty)$ define the space of divergence-free vector fields

$$L^p_\sigma(\Omega) := \{v \in L^p(\Omega, \mathbb{C}^3)\colon \operatorname{div} v = 0,\ \nu \cdot v|_{\partial\Omega} = 0\}$$

and the space of gradients

$$G^p(\Omega) := \{\nabla g\colon g \in W^1_p(\Omega, \mathbb{C})\}.$$

Both are then closed subspaces of $L^p(\Omega, \mathbb{C}^3)$. In the case $p = 2$, the corresponding orthogonal projection $\mathbb{P}_2$ from $L^2(\Omega, \mathbb{C}^3)$ onto $L^2_\sigma(\Omega)$ is called the *Helmholtz projection*. Fabes *et al*. established a *Helmholtz decomposition in* $L^p(\Omega, \mathbb{C}^3)$, which reads as follows.

Fact 3.2 (Fabes *et al.* [18, Theorems 11.1 and 12.2]). *For every bounded Lipschitz domain Ω in $\mathbb{R}^3$ there exists $\varepsilon > 0$ such that $\mathbb{P}_2$ extends to a bounded linear operator $\mathbb{P}_p$ from $L^p(\Omega, \mathbb{C}^3)$ onto $L^p_\sigma(\Omega)$ for all $p \in (3/2 - \varepsilon, 3 + \varepsilon)$. In this range, one has an L^p-Helmholtz decomposition*

$$L^p(\Omega, \mathbb{C}^3) = L^p_\sigma(\Omega) \oplus G^p(\Omega) \tag{3.2}$$

as a topological direct sum. The operator $\mathbb{P}_p$ is then called the L^p-Helmholtz projection.

In the class of bounded Lipschitz domains, this result is sharp in the sense that, for any $p \notin [3/2, 3]$, there exists a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$ for which the L^p-Helmholtz decomposition (3.2) fails.

If, however, Ω has a regular boundary $\partial\Omega \in C^1$, then the result is even true for all $p \in (1, \infty)$.

Definition 3.3. The *Stokes operator A with Hodge boundary conditions* on $L^2_\sigma(\Omega)$ is defined via $A := \mathbb{P}_2 B$ with the domain $\mathcal{D}(A) := \mathbb{P}_2 \mathcal{D}(B)$.

Starting from norm estimates of annular type on $L^p(\Omega, \mathbb{C}^3)$ with $p = 2$ for resolvents of the Hodge Laplacian B, Mitrea and Monniaux developed an iterative bootstrap argument (see [**29**, Lemma 5.1]) that allows us to incrementally increase the value of p to $p^* := \frac{3}{2}p$ (due to Sobolev embeddings), as long as $p < q_\Omega$, where q_Ω denotes the critical index for the well-posedness of the Poisson-type problem for the Hodge Laplacian (see [**29**, (1.9)]). In the present situation of a bounded Lipschitz domain Ω in $\mathbb{R}^3$, it is known that $q_\Omega > 3$ (see [**28**]). Mitrea and Monniaux (see [**29**, § 6]) showed that for any $\theta \in (0, \pi)$ there exist $q \in (3, \infty]$ and constants $b, C > 0$ such that, for all $j \in \mathbb{N}$, $x \in \Omega$, and $\lambda \in \mathbb{C} \setminus \{0\}$ with $|\arg\lambda| < \pi - \theta$,

$$\|\mathbf{1}_{B(x,|\lambda|^{-1/2})}\lambda(\lambda + B)^{-1}\mathbf{1}_{B(x,2^{j+1}|\lambda|^{-1/2})\setminus B(x,2^{j-1}|\lambda|^{-1/2})}\|_{L^2(\Omega,\mathbb{C}^3)\to L^q(\Omega,\mathbb{C}^3)} \leqslant C|\lambda|^{(3/2)(1/2-1/q)}\mathrm{e}^{-b2^j}. \tag{3.3}$$

Due to Lemma 2.5 together with [**6**, Proposition 2.1], the validity of those estimates for resolvent operators ensures generalized Gaussian $(2, q)$-estimates for the Hodge Laplacian B. Similarly to in § 5, Theorem 2.3 entails the boundedness of spectral multipliers, at first for the Hodge Laplacian B and then, by restriction, for the Stokes operator A with Hodge boundary conditions because the Hodge Laplacian and the Helmholtz projection are commuting (see Lemma 5.4 or [**29**, Lemma 3.7]). This leads to the following statement.

Theorem 3.4. *Assume that (3.3) holds for some $q \in (3, \infty]$ and that there is an L^q-Helmholtz decomposition. Fix $p \in (q', q)$ and take $s > 3|1/p - 1/2|$. Then, for every bounded Borel function $F\colon [0, \infty) \to \mathbb{C}$ with $\sup_{n\in\boldsymbol{Z}} \|\omega F(2^n\cdot)\|_{C^s} < \infty$, the operator $F(A)$ is bounded on $L^p_\sigma(\Omega)$ and there exists a constant $C > 0$ such that*

$$\|F(A)\|_{L^p_\sigma(\Omega)\to L^p_\sigma(\Omega)} \leqslant C\Big(\sup_{n\in\boldsymbol{Z}} \|\omega F(2^n\cdot)\|_{C^s} + |F(0)|\Big).$$

4. The Lamé system

Recently, Mitrea and Monniaux [**30**] studied properties of the Lamé system that appears in the linearization of the compressible Navier–Stokes equations. They showed analyticity of the semigroup generated by the Lamé operator and maximal regularity for the time-dependent Lamé system equipped with homogeneous Dirichlet boundary conditions. Their approach is essentially based on off-diagonal estimates for the resolvents of the Lamé operator. But, according to Lemma 2.5, the latter are basically equivalent to generalized Gaussian estimates, and this leads to further consequences for the Lamé system.

We first describe the setting of [**30**]. Although our results apply in the general framework of [**30**] as well, we restrict ourselves to the three-dimensional case. This restriction serves only to introduce less notation. Furthermore, we consider complex-valued functions. Let Ω be a bounded open subset of $\mathbb{R}^3$ such that the *interior ball condition* holds, i.e. there exists a positive constant c such that, for all $x \in \Omega$ and all $r \in (0, \frac{1}{2}\operatorname{diam}(\Omega))$,

$$|B(x, r)| \geqslant cr^3.$$

This condition ensures that Ω becomes a space of homogeneous type when Ω is equipped with the three-dimensional Lebesgue measure and the Euclidean distance. For example, any bounded Lipschitz domain in $\mathbb{R}^3$, or domains satisfying an interior corkscrew condition, satisfy the interior ball condition.

Fix $\mu, \mu' \in \mathbb{R}$ with $\mu > 0$ and $\mu + \mu' > 0$. We consider the sesquilinear form $\mathfrak{l}$ defined by

$$\mathfrak{l}(u, v) := \mu \int_\Omega \operatorname{rot} u \cdot \overline{\operatorname{rot} v}\, \mathrm{d}x + (\mu + \mu') \int_\Omega \operatorname{div} u\, \overline{\operatorname{div} v}\, \mathrm{d}x$$

for $u, v \in H^1_0(\Omega, \mathbb{C}^3)$, where $H^1_0(\Omega, \mathbb{C}^3)$ denotes the closure of the test function space $C^\infty_c(\Omega, \mathbb{C}^3)$ with respect to the norm of the Sobolev space $H^1(\Omega, \mathbb{C}^3)$. It is then easy to see that the form $\mathfrak{l}$ is closed, continuous, symmetric and coercive. Therefore, the operator L associated with the form $\mathfrak{l}$ is self-adjoint on $L^2(\Omega,\mathbb{C}^3)$, and $-L$ generates a bounded analytic semigroup on $L^2(\Omega,\mathbb{C}^3)$. In [**30**, § 1.1] it was checked that L is given by

$$\mathcal{D}(L)=\{u\in H_0^1(\Omega,\mathbb{C}^3)\colon \mu\Delta u+\mu'\nabla\operatorname{div}u\in L^2(\Omega,\mathbb{C}^3)\},$$
$$Lu=-\mu\Delta u-\mu'\nabla\operatorname{div}u\quad\text{for } u\in\mathcal{D}(L).$$

The operator L is called the *Lamé operator with Dirichlet boundary conditions*. In [**30**, § 2], Mitrea and Monniaux adapt their approach of [**29**] to the Lamé operator L, and establish the following statement. For any fixed angle $\theta\in(0,\pi)$ there exist $q\in(2,\infty]$ and constants $b,C>0$ such that, for all $j\in\mathbb{N}$, $x\in\Omega$ and $\lambda\in\mathbb{C}\setminus\{0\}$ with $|\arg\lambda|<\pi-\theta$,

$$\|\mathbf{1}_{B(x,|\lambda|^{-1/2})}\lambda(\lambda+L)^{-1}\mathbf{1}_{B(x,2^{j+1}|\lambda|^{-1/2})\setminus B(x,2^{j-1}|\lambda|^{-1/2})}\|_{L^2(\Omega,\mathbb{C}^3)\to L^q(\Omega,\mathbb{C}^3)} \leqslant C|\lambda|^{(3/2)(1/2-1/q)}\mathrm{e}^{-b2^j}. \tag{4.1}$$

Since this guarantees the validity of (2.5), Lemma 2.5 yields generalized Gaussian $(2,q)$-estimates for the Lamé operator L.

As remarked in [**30**, Remark 1.5], the estimate (4.1) is always valid for $q=6$ due to the Sobolev embedding $H^1(\Omega,\mathbb{C}^3)\hookrightarrow L^6(\Omega,\mathbb{C}^3)$. If the Poisson problem for the Lamé operator (see [**30**, (1.15)]) is well posed in $L^6(\Omega,\mathbb{C}^3)$, then, according to [**30**, Lemma 2.2], (4.1) also holds for $q^*=\infty$. It turns out that the largest value $q_0\in(2,\infty]$, for which the iterative method of Mitrea and Monniaux delivers (4.1) and, thus, generalized Gaussian $(2,q_0)$-estimates for L, depends on the well-posedness of the Poisson problem for the Lamé operator, and this is deeply connected to the regularity properties of the boundary $\partial\Omega$. Only for certain domains Ω is the exact characterization of q_0 known. We refer the reader to [**30**, Theorem 4.1] for a discussion of this topic, and only mention that if Ω is a bounded Lipschitz domain in $\mathbb{R}^3$, then one can even prove (4.1) for $q=\infty$ (see [**30**, Remark 1.6]), i.e. L actually satisfies pointwise Gaussian estimates. However, in general, (4.1) with $q=\infty$ does not hold. All in all, the Lamé operator L fulfils generalized Gaussian (q_0',q_0)-estimates for some $q_0\in[6,\infty]$. Therefore, Theorem 2.3 applies for L and gives the following result.

Theorem 4.1. *Fix $p\in(q_0',q_0)$. Suppose that $s>3|1/p-1/2|$. Then, for every bounded Borel function $F\colon[0,\infty)\to\mathbb{C}$ with $\sup_{n\in\boldsymbol{Z}}\|\omega F(2^n\cdot)\|_{C^s}<\infty$, the operator $F(L)$ is bounded on $L^p(\Omega,\mathbb{C}^3)$, and there exists a constant $C>0$ such that*

$$\|F(L)\|_{L^p(\Omega,\mathbb{C}^3)\to L^p(\Omega,\mathbb{C}^3)}\leqslant C\Big(\sup_{n\in\boldsymbol{Z}}\|\omega F(2^n\cdot)\|_{C^s}+|F(0)|\Big).$$

5. The Maxwell operator

Let $\Omega\subset\mathbb{R}^3$ be a bounded Lipschitz domain and let $V(\Omega)$ denote the function space introduced in (3.1). The definition of the Maxwell operator on $L^2(\Omega,\mathbb{C}^3)$ is given in a quite general framework without any regularity assumptions on the coefficient matrix. As a first step, we introduce a form $\mathfrak{a}$ with the form domain $V(\Omega)$, and establish generalized Gaussian estimates for the corresponding semigroup $(\mathrm{e}^{-tA_2})_{t>0}$ on $L^2(\Omega,\mathbb{C}^3)$ by using Davies's perturbation method (see Theorem 5.1). To the best of our knowledge, this procedure has never before been elaborated in this context. The Maxwell operator is then defined as the restriction of A_2 on the subspace of divergence-free vector fields.

Fix, once and for all, a matrix-valued function $\varepsilon(\cdot) \in L^\infty(\Omega,\mathbb{C}^{3\times 3})$ taking values in the set of positive definite Hermitian matrices. Assume additionally that $\varepsilon(\cdot)^{-1} \in L^\infty(\Omega,\mathbb{C}^{3\times 3})$. As an immediate consequence, we deduce that, for almost every $x \in \Omega$, the matrix $\varepsilon(x)^{-1}$ is also Hermitian, and that $\varepsilon(\cdot)^{-1}$ fulfils the following *uniform ellipticity condition*:

$$\varepsilon(x)^{-1}\xi \cdot \overline{\xi} \geqslant \varepsilon_0|\xi|^2 \tag{5.1}$$

for all $\xi \in \mathbb{C}^3$ and almost all $x \in \Omega$, where the constant $\varepsilon_0 > 0$ is independent of ξ and x. We consider the densely defined sesquilinear form

$$\mathfrak{a}(u,v) := \int_\Omega \varepsilon(\cdot)^{-1}\operatorname{rot} u \cdot \overline{\operatorname{rot} v}\,\mathrm{d}x + \int_\Omega \operatorname{div} u\,\overline{\operatorname{div} v}\,\mathrm{d}x \quad (u,v \in \mathcal{D}(\mathfrak{a}))$$

with the form domain $\mathcal{D}(\mathfrak{a}) := V(\Omega)$. Due to the properties of the coefficient matrix $\varepsilon(\cdot)^{-1}$, the form $\mathfrak{a}$ is continuous and coercive in the sense that there exist constants $C_1 \geqslant 0$, $C_2 > 0$ such that, for all $u \in V(\Omega)$,

$$\operatorname{Re}\mathfrak{a}(u,u) + C_1\|u\|^2_{L^2(\Omega,\mathbb{C}^3)} \geqslant C_2\|u\|^2_{V(\Omega)} \tag{5.2}$$

(in fact one can take $C_1 = C_2 = \min\{\varepsilon_0, 1\}$). Moreover, the form is symmetric and satisfies $\operatorname{Re}\mathfrak{a}(u,u) \geqslant 0$ for all $u \in V(\Omega)$. The *operator A_2 associated with the form $\mathfrak{a}$* is defined via $u \in \mathcal{D}(A_2)$, $A_2u = f$ if and only if $u \in V(\Omega)$ and $\mathfrak{a}(u,v) = (f,v)_{L^2(\Omega,\mathbb{C}^3)}$ for all $v \in V(\Omega)$. Then, A_2 is self-adjoint and $-A_2$ generates a bounded analytic semigroup $(\mathrm{e}^{-tA_2})_{t>0}$ acting on $L^2(\Omega,\mathbb{C}^3)$ (see, for example, [**12**, p. 450]).

Theorem 5.1. *The operator A_2 associated with the form $\mathfrak{a}$ has generalized Gaussian $(3/2,3)$-estimates.*

Proof. We have only to show that A_2 fulfils generalized Gaussian $(2,3)$-estimates. Due to the self-adjointness of A_2, generalized Gaussian $(3/2,2)$-estimates follow by dualization, and the claimed generalized Gaussian $(3/2,3)$-estimates follow by composition and the semigroup law. We divide the proof into several steps. The first three steps are devoted to the proof of Davies–Gaffney estimates for the operator families $(\mathrm{e}^{-tA_2})_{t>0}$, $\{t^{1/2}\operatorname{div}\mathrm{e}^{-tA_2} : t>0\}$ and $\{t^{1/2}\operatorname{rot}\mathrm{e}^{-tA_2} : t>0\}$. In order to derive these bounds, we use Davies's perturbation method. It consists in studying the 'twisted' forms

$$\mathfrak{a}_{\varrho\phi}(u,v) := \mathfrak{a}(\mathrm{e}^{\varrho\phi}u, \mathrm{e}^{-\varrho\phi}v) \quad (u,v \in V(\Omega)),$$

where $\varrho \in \mathbb{R}$ and $\phi \in \mathcal{E} := \{\phi \in C_c^\infty(\bar{\Omega},\mathbb{R}) : \|\partial_j\phi\|_\infty \leqslant 1 \text{ for all } j \in \{1,2,3\}\}$. Observe that multiplication with a function of the form $\mathrm{e}^{\varrho\phi}$ leaves the space $V(\Omega)$ invariant, and hence the form $\mathfrak{a}_{\varrho\phi}$ is well defined. In the remaining two steps we deduce generalized Gaussian $(2,3)$-estimates for A_2 by combining the Davies–Gaffney estimates and the Sobolev embedding theorem. In the following, we use the shorthand notation $\|\cdot\|_{p\to q}$ for the norm $\|\cdot\|_{L^p(\Omega,\mathbb{C}^3)\to L^q(\Omega,\mathbb{C}^3)}$.

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Step 1. We claim that for each $\gamma \in (0,1)$ there exists a constant $\omega_0 \geqslant 0$ such that, for all $u \in V(\Omega)$, $\varrho \in \mathbb{R}$ and $\phi \in \mathcal{E}$,

$$|\mathfrak{a}_{\varrho\phi}(u,u) - \mathfrak{a}(u,u)| \leqslant \gamma \mathfrak{a}(u,u) + \omega_0 \varrho^2 \|u\|_2^2. \tag{5.3}$$

After expanding $\mathfrak{a}_{\varrho\phi}(u,u)$ with the help of the product rules for div and rot, we get, for any $u \in V(\Omega)$, $\varrho \in \mathbb{R}$ and $\phi \in \mathcal{E}$, that

$$\begin{aligned}|\mathfrak{a}_{\varrho\phi}(u,u) - \mathfrak{a}(u,u)| \leqslant |\varrho| \int_\Omega |\varepsilon(\cdot)^{-1}(\nabla\phi \times u) \cdot \overline{\operatorname{rot} u}| \,\mathrm{d}x + |\varrho| \int_\Omega |(\nabla\phi \cdot u)\,\overline{\operatorname{div} u}| \,\mathrm{d}x \\ + |\varrho| \int_\Omega |\varepsilon(\cdot)^{-1} \operatorname{rot} u \cdot (\nabla\phi \times \bar{u})| \,\mathrm{d}x + |\varrho| \int_\Omega |\operatorname{div} u (\nabla\phi \cdot \bar{u})| \,\mathrm{d}x \\ + \varrho^2 \int_\Omega |\varepsilon(\cdot)^{-1}(\nabla\phi \times u) \cdot (\nabla\phi \times \bar{u})| \,\mathrm{d}x + \varrho^2 \int_\Omega |\nabla\phi \cdot u|^2 \,\mathrm{d}x.\end{aligned}$$

We analyse each of the summands on the right-hand side separately. Let $\delta > 0$, to be chosen later. By applying the Cauchy–Schwarz inequality, by using the elementary inequality $ab \leqslant \delta a^2 + b^2/4\delta$, which is valid for any real numbers a, b, and by recalling the properties of ϕ, we can estimate the first term in the following way:

$$\begin{aligned}|\varrho| \int_\Omega |\varepsilon(\cdot)^{-1}(\nabla\phi \times u) \cdot \overline{\operatorname{rot} u}| \,\mathrm{d}x &\leqslant |\varrho| \int_\Omega \|\varepsilon(\cdot)^{-1}\|_\infty |\nabla\phi|\,|u|\,|\operatorname{rot} u| \,\mathrm{d}x \\ &\leqslant \|\varepsilon(\cdot)^{-1}\|_\infty \sqrt{3} \|\nabla\phi\|_\infty \int_\Omega |\operatorname{rot} u|\,|\varrho|\,|u| \,\mathrm{d}x \\ &\leqslant \sqrt{3}\|\varepsilon(\cdot)^{-1}\|_\infty \biggl(\delta \|\operatorname{rot} u\|_2^2 + \frac{1}{4\delta}\varrho^2 \|u\|_2^2\biggr).\end{aligned}$$

The second term is bounded by

$$\begin{aligned}|\varrho| \int_\Omega |(\nabla\phi \cdot u)\,\overline{\operatorname{div} u}| \,\mathrm{d}x &\leqslant |\varrho| \int_\Omega |\nabla\phi|\,|u|\,|\operatorname{div} u| \,\mathrm{d}x \\ &\leqslant \sqrt{3} \int_\Omega |\varrho|\,|u|\,|\operatorname{div} u| \,\mathrm{d}x \\ &\leqslant \sqrt{3}\biggl(\delta \|\operatorname{div} u\|_2^2 + \frac{1}{4\delta}\varrho^2 \|u\|_2^2\biggr).\end{aligned}$$

The third term can be treated analogously to the first term, yielding

$$|\varrho| \int_\Omega |\varepsilon(\cdot)^{-1} \operatorname{rot} u \cdot (\nabla\phi \times \bar{u})| \,\mathrm{d}x \leqslant \sqrt{3}\|\varepsilon(\cdot)^{-1}\|_\infty \biggl(\delta \|\operatorname{rot} u\|_2^2 + \frac{1}{4\delta}\varrho^2 \|u\|_2^2\biggr).$$

The estimate for the fourth term follows in a similar manner as that for the second term, yielding

$$|\varrho| \int_\Omega |\operatorname{div} u (\nabla\phi \cdot \bar{u})| \,\mathrm{d}x \leqslant \sqrt{3}\biggl(\delta \|\operatorname{div} u\|_2^2 + \frac{1}{4\delta}\varrho^2 \|u\|_2^2\biggr).$$

Treating the fifth term gives that

$$\begin{aligned}\varrho^2\int_\Omega |\varepsilon(\cdot)^{-1}(\nabla\phi\times u)\cdot(\nabla\phi\times\bar{u})|\,\mathrm{d}x &\leqslant \varrho^2\int_\Omega |\varepsilon(\cdot)^{-1}(\nabla\phi\times u)|\,|\nabla\phi\times\bar{u}|\,\mathrm{d}x\\ &\leqslant \varrho^2\int_\Omega \sqrt{3}\|\varepsilon(\cdot)^{-1}\|_\infty|u|\sqrt{3}|u|\,\mathrm{d}x\\ &= 3\|\varepsilon(\cdot)^{-1}\|_\infty\varrho^2\|u\|_2^2,\end{aligned}$$

whereas the sixth term is bounded by

$$\varrho^2\int_\Omega|\nabla\phi\cdot u|^2\,\mathrm{d}x\leqslant\varrho^2\int_\Omega|\nabla\phi|^2|u|^2\,\mathrm{d}x\leqslant 3\varrho^2\|u\|_2^2.$$

Putting all these estimates together, we finally end up with

$$\begin{aligned}|\mathfrak{a}_{\varrho\phi}(u,u)-\mathfrak{a}(u,u)| &\leqslant (2\sqrt{3}\|\varepsilon(\cdot)^{-1}\|_\infty+2\sqrt{3})\delta(\|\mathrm{rot}\,u\|_2^2+\|\mathrm{div}\,u\|_2^2)\\ &\quad+\left((2\sqrt{3}\|\varepsilon(\cdot)^{-1}\|_\infty+2\sqrt{3})\frac{1}{4\delta}+3\|\varepsilon(\cdot)^{-1}\|_\infty+3\right)\varrho^2\|u\|_2^2.\end{aligned}$$

The ellipticity property (5.1) of the coefficient matrix $\varepsilon(\cdot)^{-1}$ yields, for each $u\in V(\Omega)$, that

$$\mathfrak{a}(u,u)\geqslant\min\{\varepsilon_0,1\}(\|\mathrm{rot}\,u\|_2^2+\|\mathrm{div}\,u\|_2^2). \tag{5.4}$$

Now let $\gamma\in(0,1)$ be arbitrary. Take $\delta>0$ such that

$$\gamma=(2\sqrt{3}\|\varepsilon(\cdot)^{-1}\|_\infty+2\sqrt{3})\delta/\min\{\varepsilon_0,1\}.$$

We then deduce, for each $u\in V(\Omega)$, $\varrho\in\mathbb{R}$ and $\phi\in\mathcal{E}$, that

$$|\mathfrak{a}_{\varrho\phi}(u,u)-\mathfrak{a}(u,u)|\leqslant\gamma\mathfrak{a}(u,u)+\omega_0\varrho^2\|u\|_2^2,$$

with some constant $\omega_0\geqslant 0$ depending exclusively on γ, ε_0, $\|\varepsilon(\cdot)^{-1}\|_\infty$. This leads to (5.3).

Step 2. Due to (5.3), if $\omega>\omega_0$, we can write, for any $u\in V(\Omega)$, $\varrho\in\mathbb{R}$ and $\phi\in\mathcal{E}$,

$$\mathrm{Re}\,\mathfrak{a}_{\varrho\phi}(u,u)\geqslant\mathfrak{a}(u,u)-|\mathfrak{a}(u,u)-\mathfrak{a}_{\varrho\phi}(u,u)|\geqslant(1-\gamma)\mathfrak{a}(u,u)-\omega\varrho^2\|u\|_2^2.$$

By recalling (5.4), we have thus shown that the form $\mathfrak{a}_{\varrho\phi\omega}:=\mathfrak{a}_{\varrho\phi}+\omega\varrho^2$ is coercive in the sense of (5.2) with $C_1=C_2=(1-\gamma)\min\{\varepsilon_0,1\}$. This entails that the operator $A_{\varrho\phi\omega}$ associated with the form $\mathfrak{a}_{\varrho\phi\omega}$ is sectorial of some angle $\theta_0\in(0,\pi/2)$. Therefore, $-A_{\varrho\phi\omega}$ generates a bounded analytic semigroup $(\mathrm{e}^{-tA_{\varrho\phi\omega}})_{t>0}$ on $L^2(\Omega,\mathbb{C}^3)$, and, additionally,

$$\|\mathrm{e}^{-zA_{\varrho\phi\omega}}\|_{2\to2}\leqslant 1 \tag{5.5}$$

for all $z\in\mathbb{C}\setminus\{0\}$ with $|\arg z|\leqslant\theta_0$. In view of [**26**, Lemma 3.2], this yields, for any $\varrho\in\mathbb{R}$, $\phi\in\mathcal{E}$ and $z\in\mathbb{C}\setminus\{0\}$ with $|\arg z|\leqslant\theta_0$, that

$$\|\mathrm{e}^{-\varrho\phi}\mathrm{e}^{-zA_2}\mathrm{e}^{\varrho\phi}\|_{2\to2}\leqslant\mathrm{e}^{\omega\varrho^2\,\mathrm{Re}\,z} \tag{5.6}$$

and, thus, by a similar reasoning as in the proof of [**25**, Proposition 8.22], the operator A_2 satisfies Davies–Gaffney estimates. Additionally, we have, for each $\varrho \in \mathbb{R}$, $\phi \in \mathcal{E}$ and $t > 0$, that

$$\|A_{\varrho\phi\omega}\mathrm{e}^{-tA_{\varrho\phi\omega}}\|_{2\to 2} \leqslant \frac{1}{t\sin\theta_0}.$$

Indeed, this estimate follows easily from Cauchy's formula and (5.5):

$$\begin{aligned}\|A_{\varrho\phi\omega}\mathrm{e}^{-tA_{\varrho\phi\omega}}\|_{2\to 2} &= \left\|\frac{1}{2\pi i}\int_{|z-t|=t\sin\theta_0}\frac{1}{(z-t)^2}\mathrm{e}^{-zA_{\varrho\phi\omega}}\,\mathrm{d}z\right\|_{2\to 2}\\ &\leqslant \frac{1}{2\pi}2\pi t\sin\theta_0\frac{1}{(t\sin\theta_0)^2}\\ &= \frac{1}{t\sin\theta_0}.\end{aligned}$$

Step 3. Our next task consists in verifying Davies–Gaffney estimates for the operator families $\{t^{1/2}\operatorname{div}\mathrm{e}^{-tA_2}: t>0\}$ and $\{t^{1/2}\operatorname{rot}\mathrm{e}^{-tA_2}: t>0\}$.

For arbitrary $f \in C_c^\infty(\Omega,\mathbb{C}^3)$, $\varrho\in\mathbb{R}$, $\phi\in\mathcal{E}$, $\omega>\omega_0$ and $t>0$, define $v(t) := \mathrm{e}^{-tA_{\varrho\phi\omega}}f$. Then, $v(t)$ belongs to $\mathcal{D}(A_{\varrho\phi\omega})$ and, due to (5.4) and the estimates in Step 2, we obtain that

$$\begin{aligned}\|\operatorname{rot} v(t)\|_2^2 + \|\operatorname{div} v(t)\|_2^2 &\leqslant \frac{1}{\min\{\varepsilon_0,1\}}\mathfrak{a}(v(t),v(t))\\ &\leqslant \frac{1}{(1-\gamma)\min\{\varepsilon_0,1\}}\operatorname{Re}\mathfrak{a}_{\varrho\phi\omega}(v(t),v(t))\\ &\leqslant \frac{1}{(1-\gamma)\min\{\varepsilon_0,1\}}|(A_{\varrho\phi\omega}v(t),v(t))_{L^2(\Omega,\mathbb{C}^3)}|\\ &\leqslant \frac{1}{(1-\gamma)\min\{\varepsilon_0,1\}}\|A_{\varrho\phi\omega}v(t)\|_2\|v(t)\|_2\\ &\leqslant \frac{1}{(1-\gamma)\min\{\varepsilon_0,1\}\sin\theta_0}t^{-1}\|f\|_2^2.\end{aligned}$$

As the space of test functions $C_c^\infty(\Omega,\mathbb{C}^3)$ is dense in $L^2(\Omega,\mathbb{C}^3)$, we conclude that

$$\|\operatorname{div}\mathrm{e}^{-tA_{\varrho\phi}}\|_{2\to 2} \leqslant \frac{1}{\sqrt{(1-\gamma)\min\{\varepsilon_0,1\}\sin\theta_0}}t^{-1/2}\mathrm{e}^{\omega\varrho^2 t}$$

and

$$\|\operatorname{rot}\mathrm{e}^{-tA_{\varrho\phi}}\|_{2\to 2} \leqslant \frac{1}{\sqrt{(1-\gamma)\min\{\varepsilon_0,1\}\sin\theta_0}}t^{-1/2}\mathrm{e}^{\omega\varrho^2 t} \tag{5.7}$$

for all $\varrho\in\mathbb{R}$, $\phi\in\mathcal{E}$, $\omega>\omega_0$ and $t>0$.

In order to obtain weighted norm estimates for $t^{1/2}\operatorname{rot}\mathrm{e}^{-tA_2}$, we have to interchange rot and multiplication by $\mathrm{e}^{-\varrho\phi}$. To this end, we represent $\mathrm{e}^{-\varrho\phi}\operatorname{rot} h$ in terms of $\operatorname{rot}(\mathrm{e}^{-\varrho\phi}h)$ and apply this representation to $h := \mathrm{e}^{-tA_2}\mathrm{e}^{\varrho\phi}f$. By using the product rule for rot we obtain that

$$\mathrm{e}^{-\varrho\phi}\operatorname{rot} h = \operatorname{rot}(\mathrm{e}^{-\varrho\phi}h) + \varrho\nabla\phi\times(\mathrm{e}^{-\varrho\phi}h).$$

The L^2-norm of the first term on the right-hand side can be estimated using (5.7), whereas for the second term we use $\|\nabla\phi\|_\infty \leqslant \sqrt{3}$, the elementary fact that $|\varrho| \leqslant C_\delta t^{-1/2}\mathrm{e}^{\delta\varrho^2 t}$, for arbitrary $\delta > 0$ and some constant $C_\delta > 0$ depending only on δ, and (5.6), to obtain

$$\begin{aligned}\|\mathrm{e}^{-\varrho\phi}\operatorname{rot}\mathrm{e}^{-tA_2}\mathrm{e}^{\varrho\phi}f\|_2 &= \|\mathrm{e}^{-\varrho\phi}\operatorname{rot} h\|_2\\ &\leqslant \|\operatorname{rot}(\mathrm{e}^{-\varrho\phi}h)\|_2 + |\varrho|\|\nabla\phi\|_\infty\|\mathrm{e}^{-\varrho\phi}h\|_2\\ &\lesssim t^{-1/2}\mathrm{e}^{(\omega+\delta)\varrho^2 t}\|f\|_2,\end{aligned}$$

which yields that

$$\|\mathrm{e}^{-\varrho\phi}t^{1/2}\operatorname{rot}\mathrm{e}^{-tA_2}\mathrm{e}^{\varrho\phi}\|_{2\to 2} \lesssim \mathrm{e}^{(\omega+\delta)\varrho^2 t}.$$

By adapting the arguments given in the proof of [**25**, Proposition 8.22], we see that the family of operators $\{t^{1/2}\operatorname{rot}\mathrm{e}^{-tA_2} \colon t > 0\}$ satisfies Davies–Gaffney estimates. Similar reasoning shows that $\{t^{1/2}\operatorname{div}\mathrm{e}^{-tA_2} \colon t > 0\}$ has the same property.

Step 4. Let Ω_0 be a bounded Lipschitz domain in $\mathbb{R}^3$. In view of Fact 3.1 and the Sobolev embedding $H^{1/2}(\Omega_0, \mathbb{C}^3) \hookrightarrow L^{p^*}(\Omega_0, \mathbb{C}^3)$ for $p^* := 3 \cdot 2/(3-1) = 3$, we find a constant $C > 0$ depending only on $\partial\Omega_0$ and $\operatorname{diam}(\Omega_0)$ such that, for every $u \in V(\Omega_0)$,

$$\|u\|_{L^3(\Omega_0,\mathbb{C}^3)} \leqslant C(\|u\|_{L^2(\Omega_0,\mathbb{C}^3)} + \|\operatorname{div} u\|_{L^2(\Omega_0,\mathbb{C})} + \|\operatorname{rot} u\|_{L^2(\Omega_0,\mathbb{C}^3)}). \tag{5.8}$$

With the help of the rescaling procedure used in [**29**, p. 3145], we obtain, for all $w \in V(\Omega_0)$,

$$\|w\|_{L^3(\Omega_0,\mathbb{C}^3)} \leqslant CR^{-1/2}(\|w\|_{L^2(\Omega_0,\mathbb{C}^3)} + R\|\operatorname{div} w\|_{L^2(\Omega_0,\mathbb{C})} + R\|\operatorname{rot} w\|_{L^2(\Omega_0,\mathbb{C}^3)}), \tag{5.9}$$

where $R := \operatorname{diam}(\Omega_0)$ and the constant C depends exclusively on the Lipschitz character of Ω_0.

Step 5. The desired generalized Gaussian $(2, 3)$-estimates for A_2 follow by combining the Davies–Gaffney estimates from Steps 2 and 3 with the inequality (5.9). Similar reasoning was applied in [**29**, § 5].

Let $t > 0$, let $x, y \in \Omega$ and let $f \in C_c^\infty(\Omega, \mathbb{C}^3)$ with $\operatorname{supp} f \subset B(y, t^{1/2})$ be arbitrary. Set $\Omega_0 := B(x, 2t^{1/2}) \subset \Omega$ and choose a cut-off function $\eta \in C_c^\infty(\Omega_0, \mathbb{R})$ such that

$$0 \leqslant \eta \leqslant 1, \quad \eta = 1 \text{ on } B(x, t^{1/2}) \quad\text{and}\quad \|\nabla\eta\|_\infty \leqslant t^{-1/2}.$$

First, we remark that

$$\begin{aligned}\|\operatorname{div}(\eta\mathrm{e}^{-tA_2}f)\|_{L^2(\Omega_0,\mathbb{C})} &\leqslant \|\eta\operatorname{div}(\mathrm{e}^{-tA_2}f)\|_{L^2(\Omega_0,\mathbb{C})} + \|\nabla\eta\cdot\mathrm{e}^{-tA_2}f\|_{L^2(\Omega_0,\mathbb{C})}\\ &\lesssim \|\operatorname{div}(\mathrm{e}^{-tA_2}f)\|_{L^2(\Omega_0,\mathbb{C})} + t^{-1/2}\|\mathrm{e}^{-tA_2}f\|_{L^2(\Omega_0,\mathbb{C}^3)}\end{aligned}$$

and, similarly,

$$\|\operatorname{rot}(\eta\mathrm{e}^{-tA_2}f)\|_{L^2(\Omega_0,\mathbb{C}^3)} \lesssim \|\operatorname{rot}(\mathrm{e}^{-tA_2}f)\|_{L^2(\Omega_0,\mathbb{C}^3)} + t^{-1/2}\|\mathrm{e}^{-tA_2}f\|_{L^2(\Omega_0,\mathbb{C}^3)}.$$

Since $\nu \cdot (\eta \mathrm{e}^{-tA_2} f)|_{\partial\Omega_0} = 0$ and the Lipschitz character of Ω_0 is controlled by that of Ω, we may use (5.9) and arrive at

$$\begin{aligned}
\|\mathrm{e}^{-tA_2} f\|_{L^3(B(x,t^{1/2}),\mathbb{C}^3)} &\leqslant \|\eta \mathrm{e}^{-tA_2} f\|_{L^3(\Omega_0,\mathbb{C}^3)} \\
&\lesssim t^{-1/4}(\|\eta \mathrm{e}^{-tA_2} f\|_{L^2(\Omega_0,\mathbb{C}^3)} + t^{1/2}\|\mathrm{div}(\eta \mathrm{e}^{-tA_2} f)\|_{L^2(\Omega_0,\mathbb{C})} \\
&\qquad\qquad + t^{1/2}\|\mathrm{rot}(\eta \mathrm{e}^{-tA_2} f)\|_{L^2(\Omega_0,\mathbb{C}^3)}) \\
&\lesssim t^{-1/4}(3\|\mathrm{e}^{-tA_2} f\|_{L^2(\Omega_0,\mathbb{C}^3)} + t^{1/2}\|\mathrm{div}(\mathrm{e}^{-tA_2} f)\|_{L^2(\Omega_0,\mathbb{C})} \\
&\qquad\qquad + t^{1/2}\|\mathrm{rot}(\mathrm{e}^{-tA_2} f)\|_{L^2(\Omega_0,\mathbb{C}^3)}) \\
&\lesssim t^{-(3/2)(1/2-1/3)} \exp\left(-b\frac{|x-y|^2}{t}\right)\|f\|_{L^2(B(y,t^{1/2}),\mathbb{C}^3)}, \quad (5.10)
\end{aligned}$$

where the implicit constants are independent of f, t, x, y, and the last inequality is due to Davies–Gaffney estimates for $(\mathrm{e}^{-tA_2})_{t>0}$, $\{t^{1/2}\,\mathrm{div}\,\mathrm{e}^{-tA_2} : t > 0\}$, $\{t^{1/2}\,\mathrm{rot}\,\mathrm{e}^{-tA_2} : t > 0\}$. Finally, by density, we deduce generalized Gaussian $(2,3)$-estimates for A_2. □

As noted at the beginning of § 3, $V(\Omega)$ has better embedding properties if Ω is convex or if its boundary is of class $C^{1,1}$. In these cases, the space $V(\Omega)$ continuously embeds into $H^1(\Omega,\mathbb{C}^3)$, which in turn continuously embeds into $L^6(\Omega,\mathbb{C}^3)$. Hence, in this situation one can take the $L^6(\Omega,\mathbb{C}^3)$-norm on the left-hand side of (5.8). Observe that this automatically gives the desired exponent of t in (5.10) (see, for example, [**23**, proof of Theorem 3.1]), and thus the rescaling argument in Step 4 is not needed. Summing up, the following statement holds.

Corollary 5.2. *In the situation of Theorem 5.1, suppose additionally that the domain Ω is convex or has a $C^{1,1}$-boundary. The operator A_2 associated with the form $\mathfrak{a}$ then satisfies generalized Gaussian $(6/5,6)$-estimates.*

Since A_2 satisfies generalized Gaussian (p_0,p_0')-estimates for some $p_0 \in [1,3/2]$, the semigroup generated by $-A_2$ can be extended to a bounded analytic semigroup on $L^p(\Omega,\mathbb{C}^3)$ for every $p \in [p_0,p_0']$ with $p \neq \infty$. For the rest of this section, we denote its generator by $-A_p$.

For convenience, we introduce the following abbreviation.

Notation 5.3. We denote by I_Ω the largest subinterval of the real line containing 2 such that for each $p \in I_\Omega$ the semigroup $(\mathrm{e}^{-tA_2})_{t>0}$ extends to a bounded analytic semigroup on $L^p(\Omega,\mathbb{C}^3)$, and such that there exists an L^p-Helmholtz decomposition.

In view of the foregoing statements, the length of I_Ω is intimately related to regularity properties of the boundary $\partial\Omega$, and the interval $[3/2,3]$ is always contained in I_Ω.

As we immediately see, the operators A_2 and $\mathbb{P}_2$ are commuting. This relies on the fact that $\mathbb{P}_2$ leaves the domain $V(\Omega)$ of the form $\mathfrak{a}$ invariant, which is essentially due to the boundary condition of $V(\Omega)$. We remark that this property stands in contrast to the situation of Dirichlet boundary conditions ($\nu \cdot v|_{\partial\Omega} = 0$ and $\nu \times v|_{\partial\Omega} = 0$). This situation is implicitly mentioned in [**11**, Chapter 4].

Lemma 5.4. *For any $p \in I_\Omega$, the operator A_p and the Helmholtz projection $\mathbb{P}_p$ are commuting, i.e. $\mathbb{P}_p(\mathcal{D}(A_p))$ is contained in $\mathcal{D}(A_p)$ and it holds, for all $u \in \mathcal{D}(A_p)$, that*

$$\mathbb{P}_p A_p u = A_p \mathbb{P}_p u.$$

Proof. We first treat the case $p = 2$. The statement for arbitrary $p \in I_\Omega$ then follows by density and consistency.

We claim that $\mathbb{P}_2 \colon V(\Omega) \to V(\Omega)$. Indeed, let $u \in V(\Omega)$. By the definition of $\mathbb{P}_2$, it is evident that $\operatorname{div}(\mathbb{P}_2 u) = 0$ as well as that $\nu \cdot (\mathbb{P}_2 u)|_{\partial\Omega} = 0$. In order to check that $\operatorname{rot}(\mathbb{P}_2 u) \in L^2(\Omega, \mathbb{C}^3)$, we write $\mathbb{P}_2 u = u - \nabla g$ for some $g \in W_2^1(\Omega, \mathbb{C})$ and note that it suffices to show that $\operatorname{rot}(\nabla g) = 0$. This can be easily verified via the distributional definitions of rot and ∇, which transfer the assertion to the level of test functions, where it is elementary. In particular, we have just computed $\operatorname{rot}(\mathbb{P}_2 u) = \operatorname{rot} u$ for every $u \in V(\Omega)$.

Now consider $u \in \mathcal{D}(A_2)$. We get, for each $v \in V(\Omega)$, that

$$(\mathbb{P}_2 A_2 u, v)_{L^2(\Omega,\mathbb{C}^3)} = (A_2 u, \mathbb{P}_2 v)_{L^2(\Omega,\mathbb{C}^3)} = \mathfrak{a}(u, \mathbb{P}_2 v) = \mathfrak{a}(\mathbb{P}_2 u, v),$$

where the last equality is obtained with the help of $\operatorname{rot}(\mathbb{P}_2 u) = \operatorname{rot} u$. This means that $\mathbb{P}_2 u \in \mathcal{D}(A_2)$ and $\mathbb{P}_2 A_2 u = A_2 \mathbb{P}_2 u$.

Let $p \in I_\Omega$. Observe that A_p and $\mathbb{P}_p$ are commuting if and only if resolvents of A_p commute with $\mathbb{P}_p$ on $L^p(\Omega, \mathbb{C}^3)$. In particular, we have seen above that $\mathbb{P}_p(\lambda + A_p)^{-1} = (\lambda + A_p)^{-1}\mathbb{P}_p$ on $L^p(\Omega, \mathbb{C}^3) \cap L^2(\Omega, \mathbb{C}^3)$ for any $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$. Since $-A_2$ and $-A_p$ are both generators of bounded analytic semigroups, their resolvent sets include the right-half complex plane, and their resolvents are consistent. Hence, by the density of $L^p(\Omega, \mathbb{C}^3) \cap L^2(\Omega, \mathbb{C}^3)$ in $L^p(\Omega, \mathbb{C}^3)$ and by the boundedness of resolvent operators, the equality $\mathbb{P}_p(\lambda + A_p)^{-1} = (\lambda + A_p)^{-1}\mathbb{P}_p$ extends to the whole space $L^p(\Omega, \mathbb{C}^3)$. This yields the lemma. □

We are now ready to introduce the Maxwell operator.

Definition 5.5. For $p \in I_\Omega$ we define the *Maxwell operator* M_p on $L^p_\sigma(\Omega)$ by setting

$$\mathcal{D}(M_p) := \mathbb{P}_p \mathcal{D}(A_p) = \mathcal{D}(A_p) \cap L^p_\sigma(\Omega),$$
$$M_p u := A_p u \quad \text{for } u \in \mathcal{D}(M_p).$$

Since A_2 satisfies generalized Gaussian $(3/2, 3)$-estimates (see Theorem 5.1), Theorem 2.3 yields the following result.

Theorem 5.6. *Let $p \in (3/2, 3)$. Suppose that $s > 3|1/p - 1/2|$. Then, for every bounded Borel function $F \colon [0,\infty) \to \mathbb{C}$ with $\sup_{n\in\boldsymbol{Z}} \|\omega F(2^n\cdot)\|_{C^s} < \infty$, the operator $F(A_2)$ is bounded on $L^p(\Omega, \mathbb{C}^3)$, and there exists a constant $C_p > 0$ such that*

$$\|F(A_2)\|_{L^p(\Omega,\mathbb{C}^3)\to L^p(\Omega,\mathbb{C}^3)} \leqslant C_p\Big(\sup_{n\in\boldsymbol{Z}} \|\omega F(2^n\cdot)\|_{C^s} + |F(0)|\Big).$$

As A_p and $\mathbb{P}_p$ are commuting, the functional calculus for A_2 on $L^p(\Omega, \mathbb{C}^3)$ and the Helmholtz projection $\mathbb{P}_p$ are also commuting. Therefore, we deduce a spectral multiplier theorem for the Maxwell operator by restricting $F(A_2)$ to the space of divergence-free vector fields.

Theorem 5.7. *Let $p \in (3/2, 3)$. Suppose that $s > 3|1/p - 1/2|$. Then, for every bounded Borel function $F\colon [0,\infty) \to \mathbb{C}$ with $\sup_{n\in\boldsymbol{Z}} \|\omega F(2^n\cdot)\|_{C^s} < \infty$, the operator $F(M_2)$ is bounded on $L^p_\sigma(\Omega)$, and there exists a constant $C_p > 0$ such that*

$$\|F(M_2)\|_{L^p_\sigma(\Omega)\to L^p_\sigma(\Omega)} \leqslant C_p\Big(\sup_{n\in\boldsymbol{Z}} \|\omega F(2^n\cdot)\|_{C^s} + |F(0)|\Big).$$

Remark 5.8. (1) If Ω is convex or has a $C^{1,1}$-boundary, the assertion of Theorem 5.6 even holds for any $p \in (6/5, 6)$, because A_2 then satisfies generalized Gaussian $(6/5, 6)$-estimates (see Corollary 5.2). Also, the assertion of Theorem 5.7 then holds for $p \in (6/5, 6)$, for which the Helmholtz projection is bounded in $L^p(\Omega, \mathbb{C}^3)$.

(2) The situation on the whole space $\Omega = \mathbb{R}^3$ is more comfortable, because no boundary terms occur. In particular, the form $\mathfrak{a}$ is better suited concerning partial integration. Note that the range of values $p_0 \in [1, 2)$, for which our method gives generalized Gaussian (p_0, p_0')-estimates for A_2, then depends only on the regularity of the coefficient matrix $\varepsilon(\cdot)$. In the case of smooth coefficients one can even prove pointwise Gaussian estimates for A_2.

(3) Also, for Ω Lipschitz, there is the possibility of obtaining a larger range of $p_0 \in [1, 2)$ for generalized Gaussian (p_0, p_0')-estimates under additional regularity assumptions on the coefficient matrix $\epsilon(\cdot)$ (in a similar way as in [**29**] for $\varepsilon(\cdot) = I$). A starting point for $\varepsilon(\cdot) \in C^{1+\gamma}$ [**31**], for example, would be a description of the domain $\mathcal{D}(A_2)$ of A_2 (similar to the one for $\mathcal{D}(B)$ before Definition 3.3) as

$$\{u \in V(\Omega)\colon \operatorname{rot} \varepsilon(\cdot)^{-1} \operatorname{rot} u \in L^2(\Omega, \mathbb{C}^3),\ \operatorname{div} u \in H^1(\Omega, \mathbb{C}),\ \nu \times \varepsilon(\cdot)^{-1} \operatorname{rot} u|_{\partial\Omega} = 0\},$$
$$A_2 u = \operatorname{rot} \varepsilon(\cdot)^{-1} \operatorname{rot} u - \nabla \operatorname{div} u \quad \text{for } u \in \mathcal{D}(A_2).$$

A modification of Step 2 gives Davies–Gaffney estimates for $\{tA_2 \mathrm{e}^{-tA_2} : t > 0\}$. As in [**29**], the idea is to use the estimate (5.9) (valid also for $w \in L^2(\Omega, \mathbb{C}^3)$ with $\operatorname{rot} w \in L^2(\Omega, \mathbb{C}^3)$, $\operatorname{div} w \in L^2(\Omega, \mathbb{C})$ and $\nu \times w|_{\partial\Omega} = 0$; see [**31**, p. 87]) for $w := \varepsilon(\cdot)^{-1} \operatorname{rot} u$ to obtain estimates on $\operatorname{rot} \mathrm{e}^{-tA_2} f$ in $L^3(\Omega, \mathbb{C}^3)$. For $\operatorname{div} u$ one can use the usual embedding of $H^1(\Omega, \mathbb{C})$ to get estimates on $\operatorname{div} \mathrm{e}^{-tA_2} f$ in $L^3(\Omega, \mathbb{C})$. Via

$$\|u\|_{B^{1/p}_{pp}(\Omega, \mathbb{C}^3)} \leqslant C(\|u\|_{L^p(\Omega,\mathbb{C}^3)} + \|\operatorname{div} u\|_{L^p(\Omega,\mathbb{C})} + \|\operatorname{rot} u\|_{L^p(\Omega,\mathbb{C}^3)})$$

(see [**31**, p. 87]) for $p = 3$ and $B^{1/3}_{33}(\Omega, \mathbb{C}^3) \hookrightarrow L^{9/2}(\Omega, \mathbb{C}^3)$ one finally obtains generalized Gaussian $(9/7, 9/2)$-estimates for the semigroup. However, for the above-mentioned application of the estimate (5.9) to $w := \varepsilon(\cdot)^{-1} \operatorname{rot} u$ one has to check for $\operatorname{div} w \in L^2(\Omega, \mathbb{C})$. This is possible if $\varepsilon(\cdot)$ is scalar valued, but it does not seem to be obvious for a matrix-valued $\varepsilon(\cdot)$. Precise descriptions of the domain of A_p for $p \neq 2$ could help to further extend the scale of p_0.

6. Proof of Lemma 2.5

In this section we give a detailed proof of the result stated in Lemma 2.5.

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Proof of Lemma 2.5. As noted in [**5**, pp. 934–935], one can assume that $\Omega = \mathbb{R}^D$. Otherwise, instead of an operator $T\colon L^p(\Omega,\mathbb{C}^n) \to L^q(\Omega,\mathbb{C}^n)$, one considers the extended operator $\tilde{T}\colon L^p(\mathbb{R}^D,\mathbb{C}^n) \to L^q(\mathbb{R}^D,\mathbb{C}^n)$ defined by

$$\tilde{T}u(x) := \begin{cases} T(\mathbf{1}_\Omega u)(x) & \text{for } x \in \Omega, \\ 0 & \text{for } x \notin \Omega \end{cases} \qquad (u \in L^p(\mathbb{R}^D,\mathbb{C}^n),\ x \in \mathbb{R}^D).$$

It is then straightforward to check that $\|\widetilde{T}\|_{L^p(\mathbb{R}^D,\mathbb{C}^n)\to L^q(\mathbb{R}^D,\mathbb{C}^n)} = \|T\|_{L^p(\Omega,\mathbb{C}^n)\to L^q(\Omega,\mathbb{C}^n)}$. In the following, for short, we write $\|\cdot\|_{p\to q}$ for the norm $\|\cdot\|_{L^p(\mathbb{R}^D,\mathbb{C}^n)\to L^q(\mathbb{R}^D,\mathbb{C}^n)}$.

For the proof of (a), fix $t > 0$ and $x, y \in \mathbb{R}^D$. In order to verify (2.6), we use weighted norm estimates for the resolvent operators similar to those of Davies's perturbation method presented in § 5, and an integral representation for the semigroup operators based on the Cauchy formula.

Set $h\colon \mathbb{R} \to \mathbb{R}$, $h(\tau) := \beta\tau$ for some positive constant β. One then gets, for the Legendre transform $h^\#\colon \mathbb{R} \to [-h(0),\infty]$ of h,

$$h^\#(\sigma) := \sup_{\tau\geqslant 0}(\sigma\tau - h(\tau)) = \sup_{\tau\geqslant 0}(\sigma-\beta)\tau = \begin{cases} 0 & \text{for } \sigma \leqslant \beta, \\ \infty & \text{for } \sigma > \beta. \end{cases} \tag{6.1}$$

As in the proof of Theorem 5.1, let $\mathcal{E}$ denote the space of all functions $\phi \in C_c^\infty(\mathbb{R}^D,\mathbb{R})$ such that $\|\partial_j\phi\|_\infty \leqslant 1$ for any $j \in \{1,2,\dots,D\}$. Then,

$$d_{\mathcal{E}}(x,y) := \sup\{\phi(x) - \phi(y)\colon \phi \in \mathcal{E}\}$$

defines a metric on $\mathbb{R}^D$ that is actually equivalent to the Euclidean distance (see, for example, [**13**, Lemma 4]). Therefore, [**6**, Theorem 1.2] is applicable and gives that (2.5) is equivalent to

$$\Big\| \mathrm{e}^{-\varrho\phi} v_{|\lambda|^{-1/m}}^{1/p-1/q} \lambda(\lambda+A)^{-1}\mathrm{e}^{\varrho\phi} \Big\|_{p\to q} \lesssim \mathrm{e}^{h^\#(\varrho|\lambda|^{-1/m})},$$

where $v_{|\lambda|^{-1/m}}(x) := |B(x,|\lambda|^{-1/m})| \cong |\lambda|^{-D/m}$, and, consequently,

$$\|\mathrm{e}^{-\varrho\phi}\lambda(\lambda+A)^{-1}\mathrm{e}^{\varrho\phi}\|_{p\to q} \lesssim |\lambda|^{(D/m)(1/p-1/q)}\mathrm{e}^{h^\#(\varrho|\lambda|^{-1/m})}$$

for any $\lambda \in \mathbb{C}\setminus\{0\}$ with $|\arg\lambda| < \pi-\theta$, $\varrho \geqslant 0$ and any $\phi \in \mathcal{E}$. By exploiting (6.1), we have, for any $\lambda \in \mathbb{C}\setminus\{0\}$ with $|\arg\lambda| < \pi-\theta$, $0 \leqslant \varrho \leqslant \beta|\lambda|^{1/m}$ and $\phi \in \mathcal{E}$, that

$$\|\mathrm{e}^{-\varrho\phi}\lambda(\lambda+A)^{-1}\mathrm{e}^{\varrho\phi}\|_{p\to q} \lesssim |\lambda|^{(D/m)(1/p-1/q)}. \tag{6.2}$$

Based on the Cauchy integral formula, one can represent the semigroup operator e^{-tA} in terms of resolvent operators as

$$\mathrm{e}^{-tA} = \frac{1}{2\pi\mathrm{i}}\int_\Gamma \mathrm{e}^{t\lambda}(\lambda+A)^{-1}\,\mathrm{d}\lambda,$$

where Γ is, as usual, a piecewise smooth curve in $\Sigma_{\pi-\theta}$ going from $\infty\mathrm{e}^{-\mathrm{i}(\pi-\theta')}$ to $\infty\mathrm{e}^{\mathrm{i}(\pi-\theta')}$ for some $\theta' \in (\theta,\pi/2)$. Define $\eta := \frac{1}{2}(\pi-\theta+\frac{1}{2}\pi) = \frac{3}{4}\pi - \frac{1}{2}\theta$ and

$\omega_\varrho := |\sin\eta|^{-1}\beta^{-m}\varrho^m$ for $\varrho \geqslant 0$, with β the constant in the definition of the function h. We consider shifted versions of e^{-tA} and establish a bound on $\|\mathrm{e}^{-\varrho\phi}\mathrm{e}^{-\omega_\varrho t}\mathrm{e}^{-tA}\mathrm{e}^{\varrho\phi}\|_{p\to q}$ for any $\varrho \geqslant 0$ and $\phi \in \mathcal{E}$ by using the above integral representation for e^{-tA} with the anticlockwise-oriented integration path $\Gamma = \Gamma_{t^{-1},\eta} + \omega_\varrho$, where

$$\Gamma_{t^{-1},\eta} := -(-\infty, -t^{-1}]\mathrm{e}^{-\mathrm{i}\eta} \cup t^{-1}\mathrm{e}^{\mathrm{i}[-\eta,\eta]} \cup [t^{-1},\infty)\mathrm{e}^{\mathrm{i}\eta}.$$

It holds, for each $\varrho \geqslant 0$ and $\phi \in \mathcal{E}$, that

$$\begin{aligned}\|\mathrm{e}^{-\varrho\phi}\mathrm{e}^{-\omega_\varrho t}\mathrm{e}^{-tA}\mathrm{e}^{\varrho\phi}\|_{p\to q} &\leqslant \int_{\Gamma_{t^{-1},\eta}+\omega_\varrho} \mathrm{e}^{t(\operatorname{Re}\lambda-\omega_\varrho)}\|\mathrm{e}^{-\varrho\phi}(\lambda+A)^{-1}\mathrm{e}^{\varrho\phi}\|_{p\to q}|\mathrm{d}\lambda| \\ &= \int_{\Gamma_{t^{-1},\eta}} \frac{\mathrm{e}^{t\operatorname{Re}\zeta}}{|\zeta+\omega_\varrho|}\|\mathrm{e}^{-\varrho\phi}(\zeta+\omega_\varrho)(\zeta+\omega_\varrho+A)^{-1}\mathrm{e}^{\varrho\phi}\|_{p\to q}|\mathrm{d}\zeta|.\end{aligned}$$

For every $\zeta \in \Gamma_{t^{-1},\eta}$ we can bound the operator norm with the help of (6.2) when the condition $\varrho \leqslant \beta|\zeta+\omega_\varrho|^{1/m}$ is valid. A simple geometric argument gives that $|\zeta+\omega_\varrho| \geqslant |\sin\eta|\omega_\varrho$, and thus (6.2) surely applies for $\varrho \leqslant \beta|\sin\eta|^{1/m}\omega_\varrho^{1/m}$. But, due to the definition of ω_ϱ, this requirement imposes no restrictions on ϱ. Therefore, we can continue our estimation by applying (6.2) and the elementary fact that $|\zeta+\omega_\varrho| \cong |\zeta|+\omega_\varrho$, to further obtain that

$$\begin{aligned}\|\mathrm{e}^{-\varrho\phi}\mathrm{e}^{-\omega_\varrho t}\mathrm{e}^{-tA}\mathrm{e}^{\varrho\phi}\|_{p\to q} &\leqslant \int_{\Gamma_{t^{-1},\eta}+\omega_\varrho} \mathrm{e}^{t(\operatorname{Re}\lambda-\omega_\varrho)}\|\mathrm{e}^{-\varrho\phi}(\lambda+A)^{-1}\mathrm{e}^{\varrho\phi}\|_{p\to q}|\mathrm{d}\lambda| \\ &= \int_{\Gamma_{t^{-1},\eta}} \frac{\mathrm{e}^{t\operatorname{Re}\zeta}}{|\zeta+\omega_\varrho|}\|\mathrm{e}^{-\varrho\phi}(\zeta+\omega_\varrho)(\zeta+\omega_\varrho+A)^{-1}\mathrm{e}^{\varrho\phi}\|_{p\to q}|\mathrm{d}\zeta| \\ &\lesssim \int_{\Gamma_{t^{-1},\eta}} \frac{\mathrm{e}^{t\operatorname{Re}\zeta}}{|\zeta|+\omega_\varrho}(|\zeta|+\omega_\varrho)^{(D/m)(1/p-1/q)}|\mathrm{d}\zeta| \\ &\leqslant \int_{\Gamma_{t^{-1},\eta}} \mathrm{e}^{t\operatorname{Re}\zeta}|\zeta|^{(D/m)(1/p-1/q)-1}|\mathrm{d}\zeta|.\end{aligned}$$

Here, we made use of the condition $(D/m)(1/p-1/q) < 1$. We next estimate the integral on each of the three segments of the integration path $\Gamma_{t^{-1},\eta}$ separately. We begin with a bound for the integral on the half-ray $[t^{-1},\infty)\mathrm{e}^{\mathrm{i}\eta}$,

$$\begin{aligned}&\int_{[t^{-1},\infty)\mathrm{e}^{\mathrm{i}\eta}} \mathrm{e}^{t\operatorname{Re}\zeta}|\zeta|^{(D/m)(1/p-1/q)-1}|\mathrm{d}\zeta| \\ &\qquad= \int_{t^{-1}}^{\infty} \mathrm{e}^{tu\cos\eta}u^{(D/m)(1/p-1/q)-1}\,\mathrm{d}u \\ &\qquad= t^{-(D/m)(1/p-1/q)}\int_1^\infty \mathrm{e}^{v\cos\eta}v^{(D/m)(1/p-1/q)-1}\,\mathrm{d}v \\ &\qquad\lesssim t^{-(D/m)(1/p-1/q)},\end{aligned}$$

where the last step is due to $\cos\eta < 0$. The integral on the half-ray $-(-\infty,-t^{-1}]\mathrm{e}^{-\mathrm{i}\eta}$ can be treated in the same manner. A bound for the remaining integral over the circular

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arc $t^{-1}e^{i[-\eta,\eta]}$ is obtained by using the canonical parametrization $\zeta(\alpha) = t^{-1}e^{i\alpha}$ for $\alpha \in [-\eta, \eta]$,

$$\int_{t^{-1}e^{i[-\eta,\eta]}} e^{t\,\mathrm{Re}\,\zeta}|\zeta|^{(D/m)(1/p-1/q)-1}|d\zeta| = t^{-(D/m)(1/p-1/q)} \int_{-\eta}^{\eta} e^{\cos\alpha}\,d\alpha \lesssim t^{-(D/m)(1/p-1/q)}.$$

Putting things together, we have shown that, for all $\varrho \geqslant 0$, $\phi \in \mathcal{E}$ and $t > 0$,

$$\|e^{-\varrho\phi}e^{-\omega_\varrho t}e^{-tA}e^{\varrho\phi}\|_{p\to q} \lesssim t^{-(D/m)(1/p-1/q)}$$

and, recalling that $\omega_\varrho = |\sin\eta|^{-1}\beta^{-m}\varrho^m$,

$$\|e^{-\varrho\phi}e^{-tA}e^{\varrho\phi}\|_{p\to q} \lesssim t^{-(D/m)(1/p-1/q)}e^{|\sin\eta|^{-1}\beta^{-m}\varrho^m t}.$$

By similar arguments as in the proof of [**25**, Proposition 8.22], this entails the desired two-ball estimate (2.6).

The proof of (b) is similar to that of (a), and is therefore omitted. □

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