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DIFFERENTIAL INCLUSIONS AND ABSTRACT CONTROL PROBLEMS

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We prove an existence theorem for differential inclusions

$$x' \in A(t)x + F(t,x),\ x(0) = x_0$$

in Banach spaces. Here $\{A(t) : t \in [0,T]\}$ is a family of linear operators generating a continuous evolution operator $K(t,s)$. We concentrate on maps F with $F(t,\cdot)$ weakly sequentially hemi-continuous.

Moreover, we show a compactness of the set of all integral solutions of the above problem. These results are also applied to a semilinear optimal control problem. Some corollaries, important in the theory of optimal control, are given too. We extend in several ways theorems existing in the literature.

1. INTRODUCTION

This paper is concerned with the differential inclusion

$$x'(t) \in A(t)x(t) + F(t,x(t)), \quad x(0) = x_0, \tag{1}$$

where $\{A(t) : t \in [0,T] \subset R_+\}$ is a family of densely defined, closed, linear operators on a Banach space E. We shall deal with this problem in the case when $\dim E = \infty$. Our motivation is to study control problems

$$x'(t) = A(t)x(t) + f(t,x(t)), \quad u(t) \in U(t). \tag{CP}$$

As in a classical case (Ważewski, Filippov), by setting

$$F(t,x) = f(t,x,U(t))$$

we replace (CP) by (1). One can find some results dealing with the equivalence of (CP) and (1) (see [**15, 19**], for instance). In this paper we omit this question, referring to [**15**] or [**18**] (see also Section 3). For simplicity we shall deal directly with (1).

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A brief discussion about control problems can be found in [**15**] or [**14**], for instance. We shall touch on only a few aspects of the theory.

Our purpose is to weaken the continuity hypotheses on F and to generalise the compactness assumptions. Moreover, we present some corollaries of the compactness theorems, which have immediate consequences in control theory (in particular in certain optimal control problems). A comparision with previous results of this type will be given in Section 3.

2. Continuity Concepts

Throughout this paper E will denote an infinite dimensional Banach space. Denote by (E,w) the space E with its weak topology $\sigma(E,E^*)$. Let $I=[0,T]\subset R_+$ and let $B_r=\{x\in E:\|x\|\leqslant r\}$.

Recall that a multifunction $G:E\to 2^E$ with nonempty, closed values is upper semicontinuous (usc) if and only if $G^-(A):=\{x\in E:G(x)\cap A\neq\emptyset\}$ is closed whenever $A\subset E$ is closed (see [**1, 3, 18**] for instance). Taking on E its weak topology we obtain in a similar way a notion of $w-w$ upper semicontinuity ($w-w$ usc) that is, upper semicontinuity from (E,w) into (E,w) (see [**17**]). If the set $G^-(A)$ is weakly sequentially closed whenever A is weakly closed, we shall say that G is $w-w$ sequentially usc (see [**18**]). Following [**2**], we can introduce another continuity concept.

A multifunction $G:E\to 2^E$ with nonempty, closed values is called upper hemi-continuous (uhc) [weakly upper hemi-continuous, w-uhc] if and only if for each $x^*\in E^*$ and for each $\lambda\in R$ the set $\{x\in E:\sigma(x^*,G(x))<\lambda\}$ is open in E [in (E,w)] that is, $\sigma(x^*,G(\cdot))$ is an upper semicontinuous function, where $\sigma(x^*,A):=\sup\limits_{x\in A}\langle x^*,x\rangle$.

In the case of the weak topology on E we can introduce the following more general concept.

Definition 1: A multifunction $G:E\to 2^E$ is called weakly sequentially upper hemi-continuous (w-seq uhc) if and only if for each $x^*\in E^*$, $\sigma(x^*,G(\cdot)):E\to R$ is sequentially upper semicontinuous from (E,w) into R.

This "sequential" concept is, on the one hand, more general than w-uhc and on the other hand more useful, because a continuity condition is more easily verified for sequences than for nets (see [**12**, Remark 3 p.105], or [**25**, Remark 3.1.5 p.123]).

Some comparision results about different concepts of continuity can be found in [**1, 7, 23**].

The following lemmas are necessary in the proof of our main theorem:

Lemma 1. *(Convergence Theorem) Let Y be a Banach space. Assume:*

(L1) $F:E\to 2^Y$ *- w-seq uhc,*

(L2) $F(x)$ *are nonempty, closed and convex,*

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(L3) $\|F(x)\| \leqslant a(t)$ *almost everywhere on* I, $x \in C(I,E)$, $a \in L^1(I,R)$,
(L4) $(x_n) \subset C(I,E)$, $x_n(t) \xrightarrow{w} x_0(t)$ *almost everywhere on* I,
(L5) $y_n \xrightarrow{w-L^1} y_0$, $y_n, y_0 \in L^1(I,E)$,
(L6) $y_n(t) \in F(x_n(t))$, *almost all* $t \in I$.

Thus: $y_0(t) \in F(x_0(t))$ *almost everywhere on* I.

PROOF: By theorem (AB) from [**24**], letting $A = \{y_0, y_1, y_2, \dots\}$ we have the following implication:

$$y_n \xrightarrow{w-L^1} y_0 \Longrightarrow \exists v_n \operatorname{conv} \bigcup_{k \geqslant n} y_k, \quad v_n(t) \xrightarrow{w} y_0(t) \quad \text{almost everywhere on } I.$$

But $y_n(t) \in F(x_n(t))$, so

$$v_n(t) \in \operatorname{conv} \bigcup_{k \geqslant n} y_k(t) \subset \operatorname{conv} \bigcup_{k \geqslant n} F(x_k(t)) \quad \text{almost everywhere on } I.$$

Fix an arbitrary $x^* \in E^*$

Then

$$\begin{aligned} \langle x^*, v_n(t)\rangle &\leqslant \sigma\Big(x^*, \operatorname{conv} \bigcup_{k \geqslant n} F(x_k(t))\Big) = \sigma\Big(x^*, \bigcup_{k \geqslant n} F(x_k(t))\Big) \\ &= \sup_{k \geqslant n} \sigma(x^*, F(x_k(t))) \end{aligned}$$

(see [**2**] or [**3**]).

Since

$$\begin{aligned} \langle x^*, v_n(t)\rangle &\longrightarrow \langle x^*, y_0(t)\rangle \text{ almost everywhere} \\ \langle x^*, y_0(t)\rangle &\leqslant \inf_{n \in N} \sup_{k \geqslant n} \sigma(x^*, F(x_k(t))) = \overline{\lim_{n \to \infty}} \sigma(x^*, F(x_n(t))) \\ &\leqslant \sigma(x^*, F(x_0(t))) \text{ almost everywhere on } I. \end{aligned}$$

Finally

$$\langle x^*, y_0(t)\rangle \leqslant \sigma(x^*, F(x_0(t))) \text{ almost everywhere on } I$$

and by the Separation Theorem [**3**]:

$$y_0(t) \in \overline{\operatorname{conv}} F(x_0(t)) = F(x_0(t)).$$

∎

LEMMA 2. *Let* $F : E \to 2^E$ *be* w*-seq. uhc with nonempty, convex and weakly compact values, and let* $A \subset E$ *be weakly compact.*

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Then $F(A)$ is a weakly compact set in E.

PROOF: Put $U_\lambda = \{x \in A : \sigma(x^*, F(x)) \leqslant \lambda\}$, where $x^* \in E^*$, $\lambda \in R$. It is clear, that U_λ is weakly sequentially closed, but since $U_\lambda \subset A$, U_λ is weakly compact and finally by the Eberlein-Šmulian Theorem, weakly closed.

Thus $F|_A$ is w-uhc. By the theorem of Castaing [7, Theorem II.20] F is usc from (E, w) into (E, w). Now, by Berge's Theorem [7, Theorem II.25] $F(A)$ is compact in (E, w). ∎

LEMMA 3. *Let $v \in C(I, E)$, and let $F : I \times E \to 2^E$ such that:*

(i) *$F(\cdot, x)$ has a measurable selection for each $x \in E$,*
(ii) *$F(t, \cdot)$ is w-seq uhc for each $t \in I$,*
(iii) *$F(t, x)$ is nonempty, closed and convex,*
(iv) *$\|F(t, x)\| \leqslant a(t)$ almost everywhere, $a \in L^1(I, R)$.*

Then there exists at least one measurable (and intergrable) selection z_0 of $F(\cdot, v(\cdot))$.

PROOF: Take a sequence of simple functions v_n, such that $v_n \to v$ uniformly on I. Thus by (i) there exists a measurable selection z_k such that $z_k(\cdot) \in F(\cdot, v_k(\cdot))$.

Put $G(t) = \overline{\text{conv}}\{z_k(t) : k \geqslant 1\}$. Since $w_k(\cdot)$ is measurable, $\{z_k(\cdot) : k \geqslant 1\}$ is measurable and hence $G(\cdot)$ is measurable (see [7]).

Moreover $G(t) \subset \overline{\text{conv}} F(t, V(t))$, where $V(t) = \{v_k(t) : k \geqslant 1\}$. But (v_k) is a convergent sequence, so $V(t)$ is relatively compact. By Lemma 2 and using Mazur's lemma we have that the values $\overline{\text{conv}} F(t, V(t))$ are weakly compact.

Note that $z_n(t) \in G(t)$, $n \geqslant 1$, almost everywhere $t \in I$.

Our multifunction G is measurable and integrably bounded with weakly compact values, so S^1_G is weakly compact in $L^1(I, E)$.

Here and subsequently, S^1_G denotes the set of all integrable selections of G. We subtract a subsequence (z_{n_k}) of (z_n) such that

$$z_{n_k} \xrightarrow{w-L^1} z_0 \in S^1_G.$$

By the Convergence Theorem (Lemma 1): $z_0(t) \in F(t, v(t))$ almost everywhere and $\|z_0(t)\| \leqslant a(t)$ almost everywhere, so $z_0 \in S^1_{F(\cdot, v(\cdot))}$. ∎

3. MAIN RESULTS

We begin by recalling some indispensable definitions.

A function $\omega : I \times R_+ \to R_+$ is said to be a Kamke function if it satisfies the Carathéodory conditions, $\omega(t, 0) = 0$ and $u(t) \equiv 0$ is the only absolutely continuous function satisfying:

$$u(t) \leqslant \int_0^t \omega(s, u(s))ds, \quad u(0) = 0, \quad t \in I$$

(see [26], for instance).

For completness, recall the following

DEFINITION 2: Given a bounded subset $A \subset E$, we define the Kuratowski [Hausdorff] measure of noncompactness (mnc) $\alpha(A)$ [$\beta(A)$] as follows:

$$\alpha(A) = \inf\{\varepsilon > 0 : A \text{ admits a finite covers by sets of diameter } \leqslant \varepsilon\}$$
$$[\beta(A) = \inf\{\varepsilon > 0 : A \text{ can be covered by finitely many balls of radius } \leqslant \varepsilon\}]$$

(see [4, 8, 12]). For the properties of α and β, see [4], for instance.

Put $\Delta = \{(t,s) : 0 \leqslant s \leqslant t \leqslant T\}$, and let $L(E)$ denote the algebra of all continuous, linear operators from E into E (see [16]). Let $\{A(t) : t \in I\}$ be a family of densely defined, closed, linear operators on E.

In this paper we study (1) and its mild solutions, that is, integral solutions of the Cauchy problem $x'(t) = A(t)x(t) + f(t, x(t))$, $f(t, x(t)) \in F(t, x(t))$ almost everywhere, $x(0) = x_0$.

We shall look for continuous solutions to the integral equation

$$x(t) = K(t,s)x_0 + \int_0^t K(t,s)f(s,x(s))ds,$$

where $K(\cdot,\cdot) : \Delta \to L(E)$ is a fundamental solution, that is $K(t,s)x_0$ is a solution of

$$\begin{cases} x' = A(t)x \\ x(s) = x_0 \end{cases}$$

(see [22]).

It is clear that each solution in the sense of Carathéodory is an integral solution and that the first concept is convenient for solving (1) (see [14, 22]).

A continuous function $x : I \to E$ is called an integral solution if there exists a function $f \in L^1(I, E)$ such that $f(t) \in F(t, x(t))$ almost everywhere on I, and for each $t \in I$

$$x(t) = K(t,s)x_0 + \int_0^t K(t,s)f(s)ds.$$

Now, we are in a position to state our main result.

We shall assume in the sequel:

(A1) $\{A(t) : t \in I\}$ is a generator of a fundamental solution

$K(\cdot,\cdot) : \Delta \to L(E)$ such that

(1^0) $K(s,s) = Id$, $s \in I$,

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(2^0) $K(t,s)K(s,r) = K(t,r)$, $(t,s),(s,r) \in \Delta$,
(3^0) $K : \Delta \to L(E)$ is strongly continuous,
(4^0) $\|K(t,s)\| \leqslant M < \infty$, $(t,s) \in \Delta$,
(5^0) $K(\cdot,s) : I \to L(E)$ is uniformly continuous ($s \in I$),

(F1) $F(t,x)$ is nonempty, closed and convex,
(F2) $F(\cdot,x)$ has a measurable selection, (for each $x \in E$),
(F3) $F(t,\cdot)$ is w-seq uhc, (for each $t \in I$),
(F4) $\|F(t,x)\| \leqslant k(t) \cdot (1 + \|x\|)$, $k \in L^1(I,R)$, $x \in E$, almost everywhere on I,
(F5) for each bounded $B \subset E$

$$\lim_{\tau \to 0+} \mu(F(I_{t,\tau} \times B)) \leqslant \omega(t,\mu(B)) \text{ almost everywhere on } I,$$

where
(1^o) $I_{t,\tau} = [t - \tau, t] \cap I$,
(2^o) ω is such that $M \cdot \omega$ is a Kamke function,
(3^o) μ is either the Kuratowski *mnc* or the Hausdorff *mnc*.

Theorem 1. *Under the assumptions (A1), (F1)-(F5), for each $x_0 \in E$ there exists at least one integral solution for the problem (1). Moreover, for each $x_0 \in E$ the set $S(x_0)$ of all integral solutions for (1) is compact.*

Proof: We have the following "a priori" estimate: if $x(\cdot)$ is an integral solution of (1), then

$$x(t) = K(t,0)x_0 + \int_0^t K(t,s)f(s)ds, \quad t \in I, \quad f \in S^1_{f(\cdot,x(\cdot))}.$$

Then

$$\begin{aligned}\|x(t)\| &\leqslant \|K(t,0)x_0\| + \int_0^t \|K(t,s)f(s)\|\,ds \\ &\leqslant M \cdot \|x_0\| + \int_0^t M \cdot k(s) \cdot (1 + \|x(s)\|)ds \\ &\leqslant M \cdot \|x_0\| + M \cdot \|k\|_1 + \int_0^t M \cdot k(s)ds,\end{aligned}$$

and by Gronwall's lemma

$$\|x(t)\| \leqslant M \cdot (\|x_0\| + \|k\|_1) \cdot \exp(M \cdot \|k\|_1).$$

Denote the right-hand side of the above inequality by N and put $m(t) = k(t) \cdot (1 + N)$.

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Create a new multifunction

$$\widetilde{F}(t,x) = \begin{cases} F(t,x), & t \in I,\ x \in B_N, \\ F(t,(N \cdot x)/\|x\|), & t \in I,\ x \notin B_N. \end{cases}$$

As a superposition of F and a retraction r onto B_N, $\widetilde{F}$ satisfies (F5).

Note that the continuity of K implies that for each $g \in L^1(I,E)$ the function $y(\cdot)$ given by

$$y(t) = K(t,0)x_0 + \int_0^t K(t,s)g(s)ds$$

is continuous on I.

Now, we can define a multifunction $R : B_N \to 2^{C(I,E)}$ by the following formula:

$$R(x)(t) = K(t,0)x_0 + \int_0^t K(t,s)\widetilde{F}(s,x(s))ds$$

First of all we remark, that for almost all $t \in I$ and $\tau \geqslant 0$ we have

$$\mu\Big(\widetilde{F}(t,x)\Big) \leqslant \mu\Big(\widetilde{F}(I_{t,\tau} \times \{x\})\Big)$$

hence

$$\mu\Big(\widetilde{F}(t,x)\Big) \leqslant \omega(t,0) = 0.$$

We see that $\widetilde{F}(t,x)$ is compact for almost all $t \in I$.

By Lemma 3, for each continuous function $v \in C(I,E)$ there exists a measurable selection u such that $u(t) \in \widetilde{F}(t,v(t))$ almost everywhere, and by (F4) $u \in S^1_{\widetilde{F}(\cdot,v(\cdot))}$.

Thus $R(x) \neq \emptyset$ (for each $x \in B_N$). It is clear that the values of R are closed and convex (because F has closed, convex values). Let

$$W = \{f \in L^1(I,E) : \|f(t)\| \leqslant m(t) \text{ almost everywhere on } I\},$$

$$G' = \{x \in C(I,E) : x(t) = K(t,0)x_0 + \int_0^t K(t,s)f(s)ds, t \in I, f \in W\}.$$

So W is uniformly integrable in $L^1(I,E)$ and since $K(\cdot,s)$ is uniformly continuous, G' is a equicontinuous subset of $C(I,E)$. Then $G := \overline{\text{conv}}G'$ is nonempty, closed, convex, bounded and equicontinuous in $C(I,E)$. But $S^1_{\widetilde{F}(\cdot,x(\cdot))} \subset W$ for each $x \in G$.

Indeed, for $g \in S^1_{\widetilde{F}(\cdot,x(\cdot))}$, $\|g(t)\| \leqslant \Big\|\widetilde{F}(t,x(t))\Big\| \leqslant k(t) \cdot (1+N)$ and from the obvious equality $\|A\| = \|\overline{\text{conv}}A\|$ for arbitrary set A we have our estimate $\|g(t)\| \leqslant m(t)$ almost everywhere on I.

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Now, we are in a position to show that R has a closed graph.

Let $(x_n, y_n) \in GrR$, $(x_n, y_n) \to (x, y)$ in $C(I, E)$.

Thus $y_n(t) = K(t,0)x_0 + \int_0^t K(t,s)f_n(s)ds$, $f_n \in S^1_{\widetilde{F}(\cdot, x_n(\cdot))}$, $t \in I$.

But

$$\mu(\{f_n(t) : n \geqslant 1\}) \leqslant \mu\Big(\{\widetilde{F}(t, x_n(t)) : n \geqslant 1\}\Big)$$
$$\leqslant \omega(t, \mu\{x_n(t) : n \geqslant 1\}) \quad \text{almost everywhere.}$$

Since x_n is convergent, $\{x_n(t) : n \geqslant 1\}$ is relatively compact in E, hence $\mu(\{x_n(t) : n \geqslant 1\}) = 0$ and finally $\mu(\{f_n(t) : n \geqslant 1\}) = 0$ almost everywhere on I.

By redefining (if necessary) a new multifunction H, on the set of measure zero:

$$H(t) = \overline{\text{conv}}\{f_n(t) : n \geqslant 1\}$$

we can say that $H(t)$ is nonempty, closed, convex and compact.

Thus S^1_H is nonempty, convex and weakly compact in $L^1(I, E)$ (see [**16, 21**]). By the Eberlein-Šmulian Theorem there exists a subsequence (f_{n_k}) of (f_n) such that

$$f_{n_k} \xrightarrow{w-L^1} f, \quad f \in S^1_H.$$

Since $x_n \to x$ in $C(I, E)$ and $f_{n_k} \xrightarrow{w-L^1} f$, by our Convergence Theorem we obtain that $f(t) \in F(t, x(t))$ almost everywhere on I. Thus y_n tends weakly to $K(t,0)x_0 + \int_0^t K(t,s)f(s)ds$, hence $y(t) = K(t,0)x_0 + \int_0^t K(t,s)f(s)ds$, $f \in S^1_{\widetilde{F}(\cdot, x(\cdot))}$ and $(x, y) \in GrR$. As in [**12**] we define a sequence of sets: $K_0 = G$, $K_{n+1} = \overline{\text{conv}}R(K_n)$, $n \geqslant 0$ and put $K_\infty = \bigcap_{n=1}^{\infty} K_n$.

Then:

(i) K_n is nonempty, closed and convex,

(ii) we can prove (by induction), that (K_n) is a nonincreasing sequence of sets.

Set $a_n(t) = \mu(R(K_n)(t))$. The set $\{a_n : n \in N\}$ is equicontinuous (because $R(K_n) \subset G$). So by the properties of μ a_n is absolutely continuous and moreover, it is clear that $a_n(0) = 0$.

For $0 < t - \tau < t \leqslant T$ we have $(n \geqslant 1)$

$$a_n(t) - a_n(t - \tau) \leqslant \mu\Big(\Big\{\int_{t-\tau}^t K(t,s)\omega(s)ds : \omega \in S_{\widetilde{F}(\cdot, K_n(\cdot))}\Big\}\Big)$$

and by the mean value theorem

$$\int_{t-\tau}^t K(t,s)\omega(s)ds \in \tau \cdot \overline{\text{conv}}\{K(t,s)\widetilde{F}(s, K_n(s)) : s \in [t - \tau, t]\}.$$

Hence

$$a_n(t) - a_n(t-\tau) \leqslant \tau \cdot \mu\{K(t,s)\widetilde{F}(s, K_n(s)) : s \in J_{t,\tau}\}.$$

Since $a_n(\cdot)$ is a real-valued absolutely continuous function, a_n is differentiable almost everywhere. Thus by the properties of μ,

$$\begin{aligned} a_n'(t) &\leqslant \lim_{\tau\to 0+} \mu\{K(t,s)\widetilde{F}(s, K_n(s)) : s \in J_{t,\tau}\} \\ &\leqslant M \cdot \lim_{\tau\to 0+} \mu\Big(\widetilde{F}(J_{t,\tau} \times K_n(J_{t,\tau}))\Big) \\ &\leqslant M \cdot \omega(t, \mu(K_n(t))) \quad \text{almost everywhere} \end{aligned}$$

But $\mu(K_n(t)) = \mu(\text{conv } R(K_{n-1}(t))) = \mu(R(K_{n-1}(t))) = a_{n-1}(t)$.

Thus

$$a_n'(t) \leqslant M \cdot \omega(t, a_{n-1}(t)) \quad \text{almost everywhere,}$$

and

(2) $$a_n(t) \leqslant M \cdot \int_0^t \omega(s, a_{n-1}(s))ds, \quad t \in I.$$

Since it is decreasing and bounded by 0, the sequence (a_n) is convergent, so by (2), (a_n) converges to 0.

By the lemma of Kuratowski (see [4]), K_∞ is nonempty, convex and compact (more precisely $\mu_C(K_\infty) = 0$).

It should be noted that $R : K_\infty \to 2^{K_\infty}$ by definition of K_∞. Arguing as in Lemma 2, we see that by the Kakutani Fixed Point Theorem ([**1**, **3**], for instance) there exists an $x_1 \in K_\infty$ such that $x_1 \in R(x_1)$.

But

$$\|x_1(t)\| \leqslant \|R(x_1(t))\| \leqslant M \cdot \|x_0\| + \int_0^t M \cdot k(s)(1 + \|x(s)\|)ds.$$

Hence by Gronwall's lemma

$$\|x_1(t)\| \leqslant N$$

and

$$\widetilde{F}(t, x_1(t)) = F(t, x_1(t)).$$

Thus x_1 is an integral solution for (1).

Since $S(x_0) \subset R(S(x_0)) \subset K_\infty$, we see that $S(x_0)$ is relatively compact. By Lemma 1, $S(x_0)$ is closed, so finally $S(x_0)$ is compact. $\square$

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REMARKS. (1°) The last assertion in the proof implies that in (F5) it is worthwhile to replace the strong measure of noncompactness μ by a weak one [**10**] and our assertions are still true. Moreover, it is possible to replace μ by arbitrary measure of weak noncompactness with a suitable set of properties (see [**9**]).

In particular, it is necessary to assume:

$$(K \in L(E),\ W \subset E \text{ bounded } \Longrightarrow \mu(KW) \leqslant \|K\| \cdot \mu(W)).$$

Thus a class of so-called (P, B, p)-measures of weak noncompactness is good enough (see [**9**]).

(2°) As in the above remark, μ in (F5) may be replaced by another measure of strong noncompactness (see [**8**]). As claimed in [**13**], the difference even between the cases $\mu = \alpha$ and $\mu = \beta$ may be essential [**13**, Remark (iii)].

(3°) Criteria for the existence of measurable selections of $F(\cdot, x)$ are available and moreover this kind of assumption is more useful than "$F(\cdot, x)$ is measurable" (see [**11**]).

(4°) Important examples of mappings satisfying Pianigianni's condition (F5) and comparisions between (F5) and others noncompactness conditions can be found in [**4, 12**], for instance.

We shall prove some Corollaries.

PROPOSITION 1. *The multifunction $S(\cdot) : E \to 2^{C(I,E)}$ is usc from $(E, \|\cdot\|)$ into $(C(J,E), \|\cdot\|_c)$.*

PROOF: Fix an arbitrary closed set $A \subset C(I, E)$.

We shall show that $L = \{x \in E : S(x) \cap A \neq \emptyset\}$ is closed.

Let $(x_n) \subset L$, $x_n \xrightarrow{\|\cdot\|} x_0$. Then $\sup_n \|x_n\| < \infty$ and

$$x_n \in L \Longrightarrow S(x_n) \cap A \neq \emptyset \Longrightarrow \exists y_n \in S(x_n) \cap A$$

$$y_n(t) = K(t,0)x_n + \int_0^t K(t,s) f_n(s)ds, \quad f_n \in S^1_{\widetilde{F}(\cdot, y_n(\cdot))}.$$

Since (x_n) is a convergent sequence, $\mu\{x_n : n \geqslant 1\} = 0$ and hence

$$\begin{aligned} \mu(\{y_n(t) : n \geqslant 1\}) &\leqslant \|K(t,0)\| \cdot \mu(\{x_n : n \geqslant 1\}) \\ &\quad + \mu\Big(\{\int_0^t K(t,s) f_n(s)ds : n \geqslant 1\}\Big) \\ &\leqslant \int_0^t M \cdot \mu(\{f_n(s) : n \geqslant 1\}) \\ &\leqslant \int_0^t M \cdot \mu\Big(\{\widetilde{F}(s, y_n(s)) : n \geqslant 1\}\Big) ds. \end{aligned} \tag{3}$$

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Arguing as in the proof of Theorem 1 (with $a_n = \mu(\{y_n(t) : n \geqslant 1\})$) we obtain $\mu(\{y_n(t) : n \geqslant 1\}) = 0$ for $t \in I$.

The set $\{y_n : n \geqslant 1\}$ is strongly equicontinuous, so by the Arzela - Ascoli theorem there exists a subsequence $y_{n_k} \to y_0$ in $C(I, E)$.

Now

$$\begin{aligned}\mu(\{f_{n_k}(t) : n \geqslant 1\}) &\leqslant \mu\Big(\{\widetilde{F}(t, y_{n_k}(t)) : n_k \geqslant 1\}\Big)\\ &\leqslant M \cdot \omega(t, 0) = 0\end{aligned}$$

and by Lemma 1 $f_0(t) \in \widetilde{F}(t, y_0(t))$ almost everywhere

Thus $y_0(t) = K(t,0)x_0 + \int_0^t K(t,s)f_0(s)ds$ and $y_0 \in S(x_0) \cap A$, so $S(x_0) \cap A \neq \emptyset$. We see that L is sequentially closed and $S(\cdot)$ is usc. $\square$

And now, we can formulate some corollaries.

Corollary 1. *The mapping $P_t : E \to 2^E$ given by*

$$P_t(x) = \{u(t) : u \in S(x)\}$$

is usc and has compact values.

Corollary 2. *The mapping $R_x : I \to 2^E$ given by*

$$R_x(t) = \{u(t) : u \in S(x)\}$$

is usc and has compact values.

We obtain these corollaries by using the Arzela-Ascoli theorem and from the continuity of the function $e_t(x) = x(t)$. Similarly, we have

Corollary 3. *The reachable set*

$$R_x = \bigcup_{t \in I} R_x(t)$$

is compact in E.

The last two corollaries are well-known in the theory of optimal control (the proofs are analogous to those in [**19**]).

Corollary 4. *For each fixed $x_0 \in E$ and each $y_0 \in R_{x_0}$ there exists a solution $u \in S(x_0)$ for which y_0 is attainable at a minimum time t.*

Corollary 5. *For each fixed $x_0 \in E$ a mapping $T : R_{x_0} \to I$ given by $T(z) = \inf\{t \in I : z \in R_{x_0}(t)\}$ is lsc.*

An important consequence of our Theorem 1 is the following.

Theorem 2. *Let $K \subset E$ be compact and let $\varphi : E \to R$ be lsc. Then the problem*

$$(\mathrm{OP}) \qquad \begin{cases} x'(t) \in A(t)x(t) + F(t, x(t)) \\ x(0) \in K \end{cases}$$
$$\textit{minimise} \quad \varphi(x(T))$$

has an optimal solution, that is, there exist $y_0 \in K$ and $\tilde{x} \in S(y_0)$ such that

$$\varphi(\tilde{x}(T)) = \inf\{\varphi(x(T)) : x(\cdot) \text{ is a solution of } (OP) \\ \text{with } x(0) = y_0,\ y_0 \in K\}.$$

Proof: Since $x(T) \in P_T(K)$ and $P_T(\cdot)$ is usc with compact values $P_T(K)$ is compact. Thus φ attains its infinimum a_0 on $P_T(K)$.

Consequently, there exists $a_1 \in P_T(K)$ such that $a_0 = \varphi(a_1)$.

Then we have:

$$a_1 \in P_T(K) \Longrightarrow \exists y_0 \in K\ (a_1 \in P_T(y_0)).$$

Therefore $a_1 \in R_{y_0}(T)$ and hence $\exists \tilde{x} \in S(y_0)\,(a_1 = \tilde{x}(T))$. Finally $a_0 = \varphi(\tilde{x}(T))$, that is, there exists $y_0 \in K$, $\tilde{x} \in S(y_0)$ such that

$$\varphi(\tilde{x}(T)) = \inf\{\varphi(x(T)) : x(\cdot) \text{ is a solution of (OP)} \\ \text{with } x(0) = y_0, \quad y_0 \in K\},$$

$\tilde{x}(\cdot)$ is an optimal solution for (OP). ∎

Our results generalise many previous theorems. In the important case $A(t) \equiv 0$, we have that $K(t,s) = Id$ and a mild solution is, in fact, a Carathéodory one. Then, as special cases, we obtain (among others) the existence theorems of Deimling [**11**] (in addition we have more general continuity assumptions and a larger class of measures of noncompactness), [**12**, Theorem 9.2] (as above with $\omega(t,\rho) = k(t) \cdot \rho$, $k \in L^1(I,R)$) or Papageorgiou [**20**, Theorem 3.5'] ($F(t,\cdot)$-continuous, F satisfies Tonelli's condition with β). See also Tolstonogov [**26**, Theorem 2.5.4.] and Deimling ([**12**, Remarks p.124] and references given there. In some cases we have compactness results as well [**12**], for instance).

If $A(t) \not\equiv 0$, then many results of this kind are generalised too. For example, we extend [**20, 22**] and at least partially the results of Frankowska [**14**, Theorem 2.7.] (in this case $F(t,\cdot)$ - $k(t)$-Lipschitz, $A(t) \equiv A$, $K(t,s) = S(t-s)$, $S(\cdot)$ compact or uniformly continuous, or F-compact) and Cannarsa-Frankowska [**6, 5**, Lemma 5.4, Remark 5.5, Corollary 5.6].

It is worthwhile to note that if conditions (A1)(1^o)-(A1)(4^o) are satisfied, then the well-known case: "$K(t,s)$-compact for $t > s$" implies (A1)(5^o) (see [**20**, Proposition 2.1]). Hence, previous results with the above assumption are generalised as well (see [**17**]).

Again, recall that the equivalence for (1) and (CP) is considered, for instance, in [**15, 14, 19, 6**] or [**25**]. The (CP) problem is also considered in a direct form [**15**], for instance). Our Theorem 2 and Corollaries are the only examples of applications of an abstract consideration.

To sumarise the discussion, recall our consideration for suitable measures of weak noncompactness (see Remarks) instead of α or β:

PROPOSITION 2. *Under the assumptions (A1), (F1)-(F4) and (F5) with a measure of weak noncompactness we have that:*

(i) *the set $S(x_0)$ of all integral solutions for* (1) *is nonempty and weakly compact in $C(I,E)$,*

(ii) *$S(\cdot) : E \to 2^{C(I,E)}$ is $w - w$ sequentially usc.*

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