On certain Theorems in Determinants.

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(*Abstract.*)

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The object of this note is to show that, in applying the methods of hypercomplex numbers to the theory of determinants, there is, for many purposes, no gain in using a particular number system.

1. We assume as usual a basis of n linearly independent units or positional symbols $e_1, e_2, \ldots e_n$ which are associated with ordinary numbers to form the elements of an extended algebra. In what follows the elements of the extended algebra are denoted by ordinary letters, and common numbers by Greek letters. The laws of operation are then briefly :—

I. If $x = \Sigma \xi_r e_r,\ y = \Sigma \eta_r e_r$

(i) $x + y = \Sigma(\xi_r + \eta_r)e_r = y + x$

(ii) $\lambda x = \Sigma \lambda \xi_r e_r$

(iii) $x = y$ if, and only if, $\xi_r = \eta_r$ $(r = 1, 2, \ldots n)$

II. (i) $xy = \Sigma \xi_r \eta_s e_r e_s$

(ii) $x . yz = xy . z$

III. $x(y + z) = xy + xz$

$(x + y)z = xz + yz$

x, y and z being any three elements.

In linear associative algebra, $e_r e_s (r, s = 1, 2, \ldots n)$ are considered to be linearly dependent on $e_1, e_2, \ldots e_n$, but for many purposes it is convenient to regard them as a set of n^2 new positional symbols. Similarly the products of p units form n^p independent units, and so on. We arrive in this way at an extended linear associative algebra with an infinite number of units. This is, of course, practically Gibb's indeterminate product.

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2. Let $x_r = \sum_s \xi_{rs} e_s \quad (r = 1, 2, \ldots n)$

and let

$$\| x_1, x_2, \ldots x_m \| = \Sigma \pm x_{r_1} x_{r_2} \ldots x_{r_m},$$

the sign being determined as usual.

Evidently

$$\| x_1 + x_1', x_2 \ldots x_m \| = \| x_1, x_2 \ldots x_m \| + \| x_1', x_2 \ldots x_m \|$$

$$\| x_1, x_2 \ldots x_m \| = - \| x_2, x_1 \ldots x_m \|$$

and so on.

Replacing the x's by their expressions in terms of the e's, we get

$$\| x_1, x_2 \ldots x_m \| = \Sigma \xi_{1r_1} \xi_{2r_2} \ldots \xi_{mr_m} \| e_{r_1}, e_{r_2} \ldots e_{r_m} \|$$

$$= \Sigma \, | \, \xi_{1s_1}, \xi_{2s_2} \ldots \xi_{ms_m} \, | \, \| e_{s_1}, e_{s_2} \ldots e_{s_m} \|$$

where $s_1, s_2 \ldots s_m$ are arranged in order of magnitude, and the summation extends over all such arrangements of $1, 2, \ldots n$, m at a time.

In particular we may notice that

$$\| x_1, x_2 \ldots x_n \| = | \, \xi_{rs} \, | \, \| e_1, e_2 \ldots e_n \|.$$

3. Suppose now that we transform the elements $x_1, x_2, \ldots$ by a linear transformation $A = (a_{rs})$ so that x_r becomes

$$x_r' = \Sigma \xi'_{rs} e_s = A x_r$$

where
$$\xi'_{rs} = \Sigma a_{sq} \xi_{rq}.$$

Then

$$\| A x_1, A x_2, \ldots A x_m \| = \sum_s | \, \xi'_{1s_1} \ldots \xi'_{ms_m} \, | \, \| e_{s_1}, \ldots e_{s_m} \|$$

$$= \sum_s | \, \sum_q a_{s_1 q} \xi_{1q}, \ldots \, | \, \| e_{s_1}, \ldots e_{s_m} \|$$

$$= \sum_s \sum_r | \, a_{s_1 r_1}, \ldots a_{s_m r_m} \, | \, | \, \xi_{r_1 s_1}, \ldots \xi_{r_m s_m} \, | \, \| e_{s_1} \ldots e_{s_m} \|$$

$$= [A]_m (\| x_1, \ldots x_m \|)$$

where $[A]_m$ is a linear transformation whose coefficients are the minors of the determinant $|A|$ of A.

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The well known properties of $[A]_m$ follow immediately. For, if B is any other linear transformation,

$$\| ABx_1,\ ABx_2 \ldots ABx_m \| = [A]_m \| Bx_1, \ldots Bx_m \|$$
$$= [A]_m [B]_m \| x_1 \ldots x_m \|$$

so that* $[AB]_m = [A]_m[B]_m$.

In the same way it can be shown that $Ax_1Ax_2 \ldots Ax_m$ may be derived from $x_1x_2 \ldots x_m$ by a linear transformation which has invariantive properties similar to those possessed by $[A]_m$.

Many other relations may be similarly deduced, *e.g.*,

$$\sum_r \| x_1, \ldots x_{r-1},\ Ax_r,\ x_{r+1}, \ldots x_n \|$$
$$= \sum_r \| Bx_1, \ldots Bx_{r-1},\ ABx_r,\ Bx_{r+1},\ \ldots Bx_m \| \div |B|.$$

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* A full discussion of the matrices S(A) whose co-ordinates are rational functions of the co-ordinates of the matrix A, and which possess the property S(AB) = S(A)S(B), is given by I. Schur *Über eine Klasse von Matrizen die sich einer gegebenen Matrix zuordnen lassen* (Berlin, 1901), where further references to the literature will be found.