Reference

1. F. G.-M., *Exercices de Géométrie*, Éditions Jacques Gabay, Paris, (1991). This is a reprint of the 6th edition published by Mame and De Gigord (1920).

(F. G.-M. is Frère Gabriel-Marie, whose given name was Edmond Jean-Antoine Brunhes (1834-1916). He was a member of the Order of Christian Brothers of La Salle, a teaching Order, and was the Superior from 1897 to 1913. He wrote several mathematical works which, under the rules of the Order, had to appear over the initials of the religious name of the current Superior.)

Correspondence

DEAR EDITOR,

Proofs of the irrationality of e

The celebration of the tercentenary of Euler's birth prompts me to raise a question that has nagged me for several years. It is generally accepted that Euler in his 1737 paper on continued fractions (Eneström number, E71) provided all the ingredients for the first proof of the irrationality of e by establishing that it has the non-terminating simple continued fraction [2;1,2,1,1,4,1,1,6,1,1,...]; he reprised the details in his later book *Introductio in analysin infinitorum* (E101, E102) and in a later paper (E595).

My query is this. Who was the first person to give the now standard short proof of the irrationality of e, by showing that $0 < n!e - \sum_{k=0}^{n} \frac{n!}{k!} < 1$ and deducing that $n!e$ is never an integer? Certainly this proof was common currency by the time of the late 19th century classic algebra texts such as Hall and Knight's *Higher algebra* and Chrystal's *Algebra*, but I would be very interested to learn of any 18th or early 19th century sightings!

Yours sincerely,

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DEAR EDITOR,

Correction & Further Generalisation: Note 91.65 A question of balance: an application of centroids (November 2007)

I am grateful to John Silvester, King's College London, for kindly pointing out an error in my attempted affine proof of the Lemma used in the above note, as well as for other corrections and improvements. This in turn stimulated a further generalisation of the main result as given below.

Though an affine transformation sends a parallelogram to a parallelogram, it cannot transform two parallelograms into two parallelograms with corresponding sides parallel unless the original parallelograms already had their corresponding sides parallel. What I actually had in mind was to use different affine transformations on different parts of the figure, and whilst it is possible to construct a proof by this approach, it is unnecessarily complicated. (One could use the fact that, if f is an affine transformation, then the map g given by $g(u) = \frac{1}{2}(u + f(u))$ is also affine).

Fortunately the proof of the Lemma is easy by using vectors, complex numbers or coordinates.

Lemma: Given two parallelograms $ABCD$ and $IJKL$, the midpoints E, F, G, H of the segments AI, BJ, CK, DL form another parallelogram (see Figure 1). (The diagram shows a case with the parallelograms both labelled anticlockwise, but both lemma and proof work equally well in all cases.)

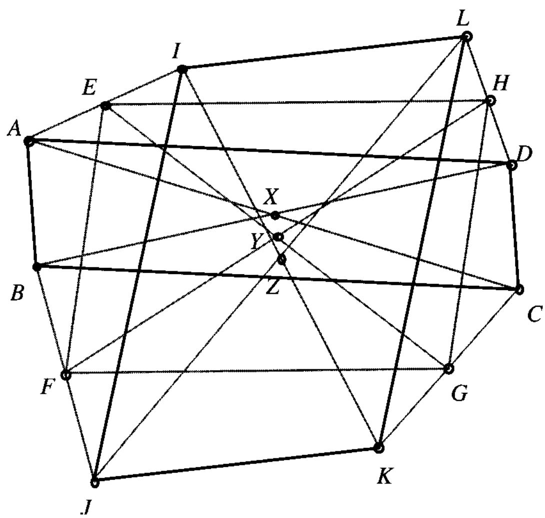

FIGURE 1

Proof: Writing $\mathbf{a} = (a_1, a_2)$ for the vector representing A, etc, the condition for $ABCD$ to be a parallelogram is $\mathbf{a} - \mathbf{b} = \mathbf{d} - \mathbf{c}$ (opposite sides, equal length and parallel) or equivalently $\mathbf{a} + \mathbf{c} = \mathbf{b} + \mathbf{d}$. Similarly, for $IJKL$, we have $\mathbf{i} + \mathbf{k} = \mathbf{j} + \mathbf{l}$, from which follows $(\mathbf{a} + \mathbf{i}) + (\mathbf{c} + \mathbf{k}) = (\mathbf{b} + \mathbf{j}) + (\mathbf{d} + \mathbf{l})$. Dividing by 2, we have the condition for the four midpoints to form a parallelogram (provided they are not collinear, or coincident, which they might be; but we will regard this as a degenerate parallelogram). In addition: the centre X of $ABCD$ is $\frac{1}{2}(\mathbf{a} + \mathbf{c})$), the centre Z of $IJKL$ is $\frac{1}{2}(\mathbf{i} + \mathbf{k})$ and the centre Y of $EFGH$ is $\frac{1}{4}[(\mathbf{a} + \mathbf{i}) + (\mathbf{c} + \mathbf{k})]$, which shows that Y is the midpoint of XZ.

The main theorem in the note can also be formulated more precisely and further generalised with the identical proof as before.

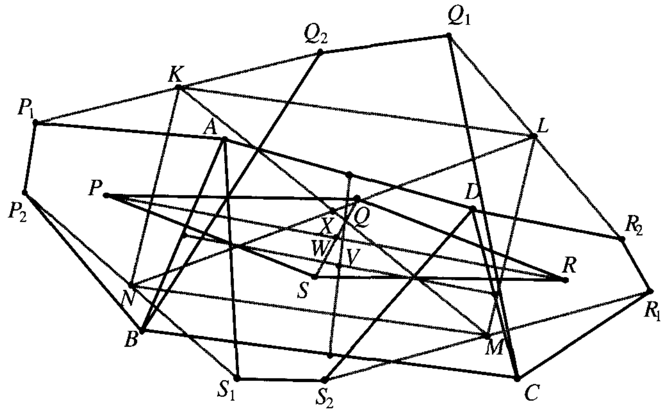

FIGURE 2

Generalised Theorem

Given four points A, B, C, D, and four directly similar quadrilaterals AP_1P_2B, CQ_1Q_2B, CR_1R_2D, AS_1S_2D with respective centroids P, Q, R, S, let K, L, M and N be the midpoints of the segments P_1Q_2, Q_1R_2, R_1S_2 and S_1P_2 respectively (or of P_1S_2, S_1R_2, R_1Q_2 and Q_1P_2), and let V, W, X be the centroids of the quadrilaterals $ABCD$, $PQRS$, $KLMN$ respectively (see Figure 2). Then:

(i) $PQRS$ is a parallelogram;

(ii) $KLMN$ is a parallelogram; and

(iii) W is the midpoint of the segment VX.

Note that the centroid of a quadrilateral is the same as the Varignon centre, the centre (or centroid) of the parallelogram formed by the midpoints of the sides of the quadrilateral. Lastly, for the centroid part of the proof of the original and generalised theorem to work correctly we need to place a mass of 2 at each of A, B, C, D, and a unit mass at the other eight places: 16 altogether, not 12. Then for the quadrilateral AP_1P_2B we take the unit masses at P_1 and P_2, and one each of the 2 at A and B; the other mass at A gets used up in AS_1S_2D and the other one at B in CQ_1Q_2B. When we now collect up, we have $2 + 2 + 2 + 2 = 8$ at V and the other 8 at X, so altogether W is the midpoint as required.

Yours sincerely,

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