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On Semisimple Hopf Algebras of Dimension pq^n

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Abstract. Let p, q be prime numbers with $p^2 < q$, $n \in \mathbb{N}$, and H a semisimple Hopf algebra of dimension pq^n over an algebraically closed field of characteristic 0. This paper proves that H must possess one of the following two structures: (1) H is semisolvable; (2) H is a Radford biproduct $R\#kG$, where kG is the group algebra of group G of order p and R is a semisimple Yetter–Drinfeld Hopf algebra in ${}^{kG}_{kG}\mathcal{YD}$ of dimension q^n.

1 Introduction

Let H be a semisimple Hopf algebra of dimension $p^m q^n$, where p, q are prime numbers and m, n are nonnegative integers. Then H is solvable. That is, the fusion category rep(H) of its finite-dimensional representations is solvable in the sense of the paper [4]. Solvability is a categorical notion, meaning that the category rep(H) can be obtained from the category of finite-dimensional vector spaces by means of successive cyclic group extensions or equivariantizations. Nevertheless, the explicit classification, up to isomorphism, of semisimple Hopf algebras of dimension $p^m q^n$ is not known up to now. The only known examples that are completely classified are those of dimension p, p^2, p^3, pq, and pq^2 [2, 4, 6, 7, 17].

In this paper, we shall investigate the classification of semisimple Hopf algebras of dimension pq^n, where p, q are prime numbers with $p^2 < q$ and n is a nonnegative integer.

The notion of upper or lower semisolvability for finite-dimensional Hopf algebras was introduced by Montgomery and Witherspoon [8], as a generalization of the notion of solvability for finite groups. In particular, if a finite-dimensional semisimple Hopf algebra A is upper or lower semisolvable, then A can be constructed by successive extensions from trivial Hopf algebras. A Hopf algebra is called trivial if it is a group algebra or a dual of a group algebra.

Let A be a semisimple Hopf algebra and let ${}^{A}_{A}\mathcal{YD}$ denote the braided category of Yetter–Drinfeld modules over A. Let R be a semisimple Yetter–Drinfeld Hopf algebra in ${}^{A}_{A}\mathcal{YD}$ (see [16], for example). As observed by Radford (see [13, Theorem 1]), the Yetter–Drinfeld condition assures that $R \otimes A$ becomes a Hopf algebra with additional

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structures. This Hopf algebra is called the *Radford biproduct* of R and A. We denote this Hopf algebra by $R\#A$.

Let H be the semisimple Hopf algebras of dimension pq^n mentioned above. Our main work proves that either H or H^* contains a nontrivial central group-like element, or $|G(H)|$ and $|G(H^*)|$ have the same prime factor p. We then prove, by induction on n, that if H or H^* contains a nontrivial central group-like element, then it is upper or lower semisolvable. If $|G(H)|$ and $|G(H^*)|$ have the same prime factor p, then it follows from [10, Lemma 4.1.9] that H is a Radford biproduct $R\#kG$, where kG is the group algebra of group G of order p, and R is a semisimple Yetter–Drinfeld Hopf algebra in ${}^{kG}_{kG}\mathcal{YD}$ of dimension q^n.

The definitions and some of the basic properties of semisolvability and Drinfeld double are recalled in Section 2. Some useful lemmas are also contained in this section. Our main result is given in Section 3.

Throughout this paper, all modules and comodules are left modules and left comodules, and, moreover, they are finite-dimensional over an algebraically closed field k of characteristic 0, and $\otimes$, dim mean $\otimes_k$, $\dim_k$, respectively. Our reference for the theory of Hopf algebras is [9].

2 Preliminaries

In this paper, we always assume that all Hopf algebras involved are finite-dimensional semisimple, although some results may hold for non-semisimple ones.

Let H be a semisimple Hopf algebra over k. We define two actions of H^* on H as

$$f \rightharpoonup h = \sum f(h_2)h_1 \quad \text{and} \quad h \leftharpoonup f = \sum f(h_1)h_2$$

for all $f \in H^*, h \in H$.

Then the Drinfeld double $D(H)$ of H is defined as follows: $D(H)$ has $H^{*\,\mathrm{cop}} \otimes H$ as its underlying vector space; the multiplication of $D(H)$ is given by

$$(g \otimes h)(f \otimes l) = \sum g\big(h_1 \rightharpoonup f \leftharpoonup S^{-1}(h_3)\big) \otimes h_2 l,$$

and the coalgebra structure of $D(H)$ is the usual tensor product of coalgebras. It follows from [9] that $D(H)$ is also semisimple.

The main result in [3] proves that if V is a simple $D(H)$-module, then the dimension of V divides the dimension of H.

Let ${}^H_H\mathcal{YD}$ denote the category of (left-left) Yetter–Drinfeld modules over H. Objects of this category are vector spaces V endowed with an H-coaction $\rho\colon V \to H\otimes V$ and an H-action $\cdot : H \otimes V \to V$ that satisfy the compatibility condition

$$\rho(h \cdot v) = h_1 v_{-1} S(h_3) \otimes h_2 \cdot v_0,$$

for all $v \in V, h \in H$. Morphisms of this category are H-linear and colinear maps.

Majid proved ([5, Proposition 2.1]) that the Yetter–Drinfeld category ${}^H_H\mathcal{YD}$ can be identified with the category ${}_{D(H)}\mathcal{M}$ of left modules over the Drinfeld double $D(H)$.

In view of Majid's result and the main result in [3], the dimension of every simple Yetter–Drinfeld H-module divides $\dim H$.

It is well known that H becomes a Yetter–Drinfeld H-module with respect to the left adjoint action $\mathrm{ad}_l\colon H \otimes H \to H, (\mathrm{ad}_l\, h)(a) = h_1 a S(h_2)$ and the left regular

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coaction $\Delta\colon H \to H \otimes H$. Furthermore, the Yetter–Drinfeld submodule $V \subseteq H$ is exactly the left coideal V of H such that $h_1 V S(h_2) \subseteq V$, for all $h \in H$. Following this observation, we have the following result.

Lemma 2.1 *The* 1*-dimensional Yetter–Drinfeld H-submodule is exactly the span of a central group-like element of H.*

Let $\pi\colon H \to K$ be a Hopf algebra map. Then

$$H^{\mathrm{co}\,\pi} = \{h \in H : (\mathrm{id} \otimes \pi)\Delta(h) = h \otimes 1\}$$

is a left coideal subalgebra of H. We call $H^{\mathrm{co}\,\pi}$ the subspace of coinvariant of π. The left coideal subalgebra $H^{\mathrm{co}\,\pi}$ is stable under the left adjoint action of H and $\dim H = \dim H^{\mathrm{co}\,\pi} \dim \pi(H)$. See [15] for details.

Lemma 2.2 *Let $\pi\colon H \to K$ be a Hopf algebra map. Then $H^{\mathrm{co}\,\pi}$ is a Yetter–Drinfeld submodule of H.*

In fact, the lemma above follows from the fact that $H^{\mathrm{co}\,\pi}$ is a left coideal of H and is stable under the left adjoint action.

A semisimple Hopf algebra H is called lower semisolvable if there exists a chain of Hopf subalgebras

$$H_{n+1} = k \subseteq H_n \subseteq \cdots \subseteq H_1 = H$$

such that H_{i+1} is a normal Hopf subalgebra of H_i, for all i, and all quotients $H_i/H_i H_{i+1}^{+}$ are trivial. A Hopf subalgebra $A \subseteq H$ is called normal if $h_1 A S(h_2) \subseteq A$, for all $h \in H$. Dually, H is called upper semisolvable if there exists a chain of quotient Hopf algebras

$$H_{(0)} = H \xrightarrow{\pi_1} H_{(1)} \xrightarrow{\pi_2} \cdots \xrightarrow{\pi_n} H_{(n)} = k$$

such that $H_{(i-1)}^{\mathrm{co}\,\pi_i}$ is a normal Hopf subalgebra of $H_{(i-1)}$, and all $H_{(i-1)}^{\mathrm{co}\,\pi_i}$ are trivial.

By [8, Corollary 3.3], we have that H is upper semisolvable if and only if H^* is lower semisolvable. We call H semisolvable if it is upper or lower semisolvable.

Remark 2.3 Let K be a proper normal Hopf subalgebra of H. Then

$$K \hookrightarrow H \to \overline{H} := H/HK^{+}$$

is an exact sequence of Hopf algebras. If K is lower semisolvable and $\overline{H}$ is trivial, then H is lower semisolvable. On the other hand, if K is trivial and $\overline{H}$ is upper semisolvable, then H is upper semisolvable.

Proposition 2.4 *Let H be a semisimple Hopf algebra of dimension pq^n, where p, q are prime numbers and n is a nonnegative integer. Suppose that H has a nontrivial central group-like element g. Then H is upper semisolvable.*

Proof Our proof follows from an induction on n. First, the semisimple Hopf algebras of dimension pq are upper semisolvable by the classification of such Hopf algebras [2]. Then we assume that all semisimple Hopf algebras of dimension pq^i $(1 \le i \le n-1)$ are upper semisolvable.

Let $\langle g \rangle$ denote the subgroup of $G(H)$ generated by g. Since g is central in H, $k\langle g \rangle$ is a normal Hopf subalgebra of H. Hence, $\overline{H} := H/Hk\langle g \rangle^+$ is a Hopf algebra, and we have an exact sequence of Hopf algebras:

$$k\langle g \rangle \hookrightarrow H \to \overline{H}.$$

By [12], the order of g divides $\dim H = pq^n$. Suppose that the order of g is of the form pq^i $(0 \le i \le n)$. In this case, $\dim \overline{H} = q^{n-i}$. By [8, Theorem 3.5], $\overline{H}$ is upper semisolvable. So H is upper semisolvable by Remark 2.3.

Suppose that the order of g is of the form q^i $(1 \le i \le n)$. In this case, $\dim \overline{H} = pq^{n-i}$. By induction, $\overline{H}$ is upper semisolvable. So H is upper semisolvable, also by Remark 2.3. ■

3 Main Result

Since the trivial Hopf algebras are automatically upper and lower semisolvable, we assume in this section that H is a nontrivial semisimple Hopf algebra of dimension pq^n, where p, q are prime numbers with $p^2 < q$ and n is a nonnegative integer.

By [4, Proposition 9.9], H has a nontrivial 1-dimensional simple module. Equivalently, the order of $G(H^*)$ is greater than 1. Moreover, the order of $G(H^*)$ divides the dimension of H by [12].

Theorem 3.1 *Either*

(i) *H or H^* contains a nontrivial central group-like element, or*
(ii) *$|G(H)|$ and $|G(H^*)|$ have the same prime factor p.*

Proof We consider the projection $\pi\colon H \to (kG(H^*))^*$ obtained by transposing the inclusion $kG(H^*) \hookrightarrow H^*$.

By the discussion in Section 2, we know that the coinvariant subspace $H^{\mathrm{co}\,\pi}$ is a Yetter–Drinfeld H-submodule of H and $\dim H^{\mathrm{co}\,\pi} \dim(kG(H^*))^* = \dim H$.

We assume that the decomposition of $H^{\mathrm{co}\,\pi}$ into simple Yetter–Drinfeld submodules of H is

$$H^{\mathrm{co}\,\pi} = k\mathbf{1} \oplus \sum_{i \in I} V_i,$$

where $k\mathbf{1}$ is the trivial Yetter–Drinfeld H-submodule and V_i's are the nontrivial ones.

Suppose that the order of $G(H^*)$ is q^n. Then $\dim H^{\mathrm{co}\,\pi} = p$. Since $p^2 < q$, the dimension of any V_i $(i \in I)$ cannot be of the form qm for some $m \in \mathbb{N}$. Therefore, the dimension of every V_i is 1. We should notice that the dimension of every simple Yetter–Drinfeld H-module divides $\dim H = pq^n$, as mentioned in Section 2. It follows that H has a nontrivial central group-like element by Lemma 2.1.

Suppose that the order of $G(H^*)$ is not q^n. That is, it is of the form

$$p^i q^j, \quad 0 \le i \le 1,\ 0 \le j \le n-1,\ i + j \ne 0.$$

Then we may write $\dim H^{\mathrm{co}\,\pi} = qm$ for some $m \in \mathbb{N}$. In this case, if there exists $i \in I$ such that $\dim V_i = 1$, then we are done. However, if the dimension of every V_i is greater than 1, then there must exist $i \in I$ such that $\dim V_i = p$. In fact, if it is not this case, we will encounter a contradiction $qm = 1 + qs$ for some $s \in \mathbb{N}$. Let V be a

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simple Yetter–Drinfeld H-submodule with $\dim V = p$. We decompose $V \otimes V^*$ into simple Yetter–Drinfeld H-modules as follows:

$$V \otimes V^* = k\mathbf{1} \oplus \sum_{i \in I} U_i.$$

Since $p^2 < q$, the dimension of U_i is 1 or p. Obviously, not all $\dim U_i$'s are p, otherwise we will encounter a contradiction $p^2 = 1 + pm$ for some $m \in \mathbb{N}$. Therefore, there exists $i \in I$ such that $\dim U_i = 1$.

By [11, Theorem 10], the 1-dimensional simple Yetter–Drinfeld H-modules (equivalently, the 1-dimensional simple $D(H)$-modules) entering the decomposition of $V \otimes V^*$ form a subgroup G of $G(D(H)^*)$. Moreover, the order of G divides p^2 [1, Lemma 2.1]. By the discussion in the paragraph above, we know that G is not trivial. In other words, $G(D(H)^*)$ contains an element of order p.

By [14, Propositions 10], every group-like element of $D(H)^*$ is of the form $g \otimes \eta$, where $g \in G(H)$ and $\eta \in G(H^*)$. Let $g \otimes \eta \in G(D(H)^*)$ be the group-like element of order p. If the orders of g and η are different, then the proof of [1, Corollary 2.7] shows that either H or H^* contains a nontrivial central group-like element. If g and η have the same orders, then both of them are equal to p. Hence, $|G(H)|$ and $|G(H^*)|$ have the same prime factor p. This completes the proof. ■

Corollary 3.2 *H possesses one of the following structures:*

(i) *H is upper or lower semisolvable;*
(ii) *H is a Radford biproduct $R\#kG$, where kG is the group algebra of group G of order p, and R is a semisimple Yetter–Drinfeld Hopf algebra in ${}^{kG}_{kG}\mathcal{YD}$ of dimension q^n.*

Proof If H (or H^*) contains a nontrivial central group-like element, then H (or H^*) is upper semisolvable by Proposition 2.4. In addition, if H^* is upper semisolvable, then H is lower semisolvable (as recalled in Section 2). So the first part follows.

If $|G(H)|$ and $|G(H^*)|$ have the same prime factor p, then the second part follows from [10, Lemma 4.1.9]. ■

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