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HOMOTOPY PULL-BACKS AND APPLICATIONS TO DUALITY

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Introduction. The topic of homotopy pull-backs and push-outs has recently been discussed by a number of authors; Boardman and Vogt [**5**], Bousfield and Kan [**6**], Fantham [**7**], Mather [**11**], and Vogt [**16**]. Mather develops the theory with an eye to applications and of particular interest is his cube theorem which appears in this paper as Theorem (1.10); the significance of this theorem to applications is shown in [**11**]. As often occurs in homotopy theory the dual is not true. The purpose of this paper is to examine approximations to the dual in order to obtain new information concerning classical problems of duality.

Given an arbitrary number of fibrations with the same base, Svarc ([**15**, Chapter II, Section 1]) describes a fibration whose fibre is the join of the fibres. In the case of two fibrations Svarc's result was rediscovered by Hall [**9**] and called the *Whitney Sum*. Nomura ([**12**; **13**]) extended the result to the situation of arbitrary maps and calls his construction the *Whitney Join*. Independent of Svarc and Hall, Ganea [**8**] described the Whitney Sum of two fibrations $F \to E \to B$ and $\Omega B \to * \to B$. It is also recognized (Ganea [**8**] and Nomura [**12**]) that the results on the Whitney Join are in a certain sense dual to the results of Blakers and Massey ([**2**; **3**; **4**]) on the homotopy groups of a triad. In the language of homotopy pull-backs and push-outs this duality has a succinct formulation; see Theorems (1.12) and (1.13). The problem of determining to what extent the dual of (1.13) is valid has been the subject of much research; see [**1**] and [**12**]. In Section 4, Theorem (4.2) gives new information concerning this problem.

Also, in Section 3, Theorem (3.2) provides an approximation to a dual of a theorem of Sugawara [**14**] on a necessary condition when a space is an H-space.

Section 1 overlaps somewhat with [**5**] and [**16**] and especially with [**11**]. It was decided for the purposes of exposition to avoid the more complicated formulation of theory as in [**5**] and [**16**]. Also as the topic is unfamiliar to many, reformulation of certain aspects of [**11**] was deemed appropriate.

1. Preliminaries. All spaces will be furnished with a base point $*$, and all maps and homotopies will be considered as base-point preserving.

If $G, H: X \times I \to Y$ are two homotopies such that $G(., 0) = H(., 0)$ and $H(., 1) = G(., 1)$, G and H are said to be *equivalent* if there is a map Φ: $X \times I \times I \to Y$ such that

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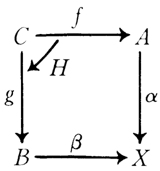
(i) $\Phi(X, s, 0) = G(x, s)$
(ii) $\Phi(x, s, 1) = H(x, s)$
(iii) $\Phi(x, 1, t) = \Phi(x, 1, t')$ and $\Phi(x, 0, t) = \Phi(x, 0, t')$ for $t, t' \in I$.

Given a homotopy $G : X \times I \to Y$, the homotopy $-G : X \times I \to Y$ is defined by $-G(x, t) = G(x, 1 - t)$; given a homotopy $H : X \times I \to Y$ such that $G(., 1) = H(., 0)$, the homotopy $H + G : X \times I \to Y$ is defined by:

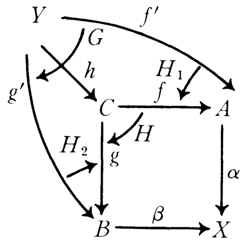
$$H + G(x, t) = \begin{cases} G(x, 2t), & 0 \leqq t \leqq \frac{1}{2} \\ H(x, 2t - 1), & \frac{1}{2} \leqq t \leqq 1. \end{cases}$$

A diagram:



of spaces and maps together with a homotopy $H : C \times I \to X$ such that $H(., 0) = \alpha f$ and $H(., 1) = \beta g$ is called a *homotopy commutative* square.

A diagram of spaces, maps and homotopies of the form



where the homotopies $H : C \times I \to X$, $H_1 : Y \times I \to A$, $H_2 : Y \times I \to B$, and $G : Y \times I \to A$, are defined so that:

(i) $H(., 0) = \alpha f$ and $H(., 1) = \beta g$
(ii) $H_1(., 0) = f'$ and $H_1(., 1) = fh$
(iii) $H_2(., 0) = g'$ and $H_2(., 1) = gh$
(iv) $G(., 0) = \alpha f'$ and $G(., 1) = \beta g'$

is *called homotopy commutative* if the homotopies G and $\beta(-H_2) + H(h \times 1) + \alpha H_1$ are equivalent.

(1.1) For each subset J of $\{1, 2, 3\}$, let X_J be a topological space and for $i \in J$, let $f_J{}^{J-\{i\}} : X_J \to X_{J-\{i\}}$ be a map.

For simplicity subsets of $\{1, 2, 3\}$ are shown without set brackets or commas. The resulting diagram is said to be a *homotopy commutative cube* if it is fitted with homotopies as follows:

(1) for each subset J of $\{1, 2, 3\}$ and each subset $\{i, j\}$ of J, there is a homotopy $H_J{}^{J-\{i,j\}} : (X_J, *) \times I \to (X_{J-\{i,j\}}, *)$ connecting the maps $f_J{}^{J-\{i,j\}} \circ f_J{}^{J-\{i\}}$ and $f_{J-\{i\}}{}^{J-\{i,j\}} \circ f_J{}^{J-\{j\}}$ which is directed as shown below.

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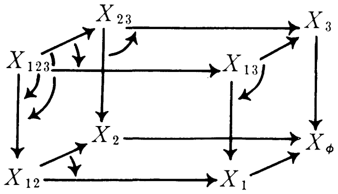
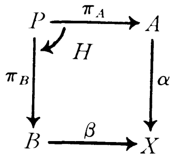
(2) the homotopies $H_{13}^{\phi}(f_{123}^{13} \times 1) + f_3^{\phi}H_{123}^{3} + H_{23}(f_{123}^{23} \times 1)$ and $f_1^{\phi}(-H_{123}^{1}) + H_{12}^{\phi}(f_{123}^{12} \times 1) + f_2^{\phi}H_{123}^{2}$ are equivalent.

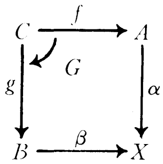


Definition (1.2). A *homotopy pull-back* of a diagram $A \xrightarrow{\alpha} X \xleftarrow{\beta} B$ is a homotopy commutative square

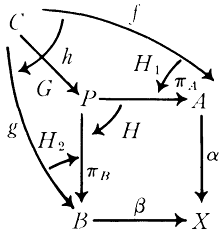


which satisfies the conditions:

(1) If

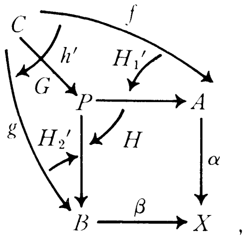


is a homotopy commutative square, then there is an induced map $h : C \to P$ and appropriate homotopies making the diagram



homotopy commutative.

(2) If there is another homotopy commutative diagram



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then

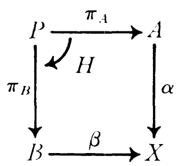
(i) there is a homotopy $F: C \times I \rightarrow P$ such that $F(.,0) = h$ and $F(.,) = h'$

(ii) H_1' is equivalent to $\pi_A F + H_1$ and H_2' is equivalent to $\pi_B F + H_2$.

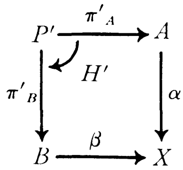
The notion of the *homotopy push-out* of a diagram $B \overset{g}{\leftarrow} C \overset{f}{\rightarrow} A$ is defined dually.

The proofs that homotopy push-outs exist and are unique appear in [**5**; **7**; **11**; **16**]; accordingly we summarize as follows.

THEOREM (1.3). *The homotopy pull-back of a diagram $A \overset{\alpha}{\rightarrow} X \overset{\beta}{\leftarrow} B$ exists and is unique in the sense that if*

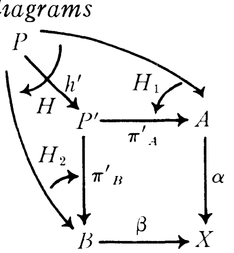


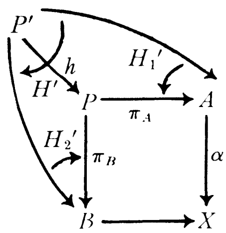
and



are two homotopy pull-backs then:

(i) *the two diagrams*

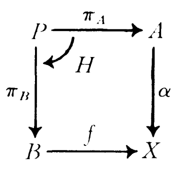




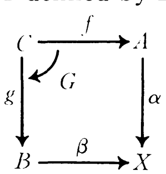
are homotopy commutative; the homotopies H_1, H_2', and H_2' and the maps h and h' are induced according to Definition (1.2).

(ii) *P is homotopy equivalent to P'.*

Remarks (1.5). 1) From now on we shall use the expression *standard homotopy pull-back* to denote the homotopy pull-back

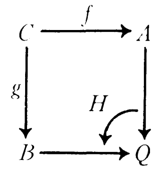


with $P = \{(a, b, \gamma) \in A \times B \rightarrow X^I : \gamma(0) = a$ and $\gamma(1) = b\}$, π_A and π_B projections, and $H : P \times I \rightarrow X$ defined by $H(a, b, \gamma), t) = \gamma(t)$. Also if



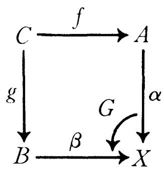
is homotopy commutative the words *standard induced* map shall refer to the induced map $h : C \to P$ defined by $h(c) = (f(c), g(c), G(c, .))$.

2) Dually, *standard homotopy push-out* refers to the homotopy push-out



with Q being the space obtained from $C \times I/* \times I$ by attaching A and B according to the maps $(c, 0) \to f(c)$ and $(c, 1) \to g(c)$. Points of Q may be represented as $[c, t]$, $[a, 0]$, and $[b, 1]$ with the understanding that $[c, 0] = [f(c), 0]$, $[c, 1] = [g(c), 1]$, and $[*, t_1] = [*, t_2]$ for $0 \leqq t_1, t_2 \leqq 1$. The maps $A \to Q$ and $B \to Q$ are the inclusions $a \to [a, 0]$ and $b \to [b, 1]$ and the homotopy $H : C \times I \to Q$ is defined by $H(c, t) = [c, t]$.

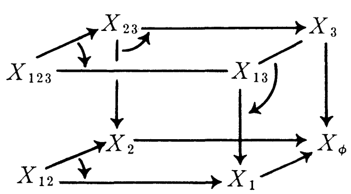
Also if



is homotopy commutative then the *standard induced map* $h : Q \to X$ is defined by

$$\begin{aligned} h &: [c, t] \mapsto H(c, t) \\ &: [a, 0] \mapsto \alpha(a) \\ &: [b, 0] \mapsto \beta(b). \end{aligned}$$

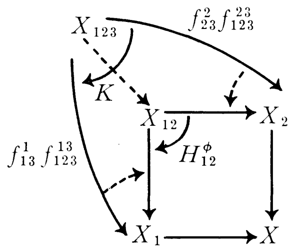
Consider



as a portion of the diagram of (1.1).

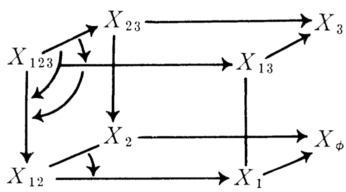
LEMMA (1.6). *If in the above diagram the bottom square is a homotopy pull-back there is an induced map* $X_{123} \to X_{12}$ *and homotopies on the front and left faces so that the resulting cube is homotopy commutative.*

Proof. Form the diagram



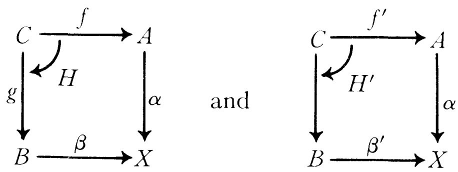
where $K = H_{13}{}^{\phi}(f_{123}{}^{13} \times 1) + f_3{}^{\phi}H_{123}{}^{3} + H_{123}{}^{\phi}(f_{123}{}^{23} \times 1)$. Then apply Definition (1.2).

Dually, consider



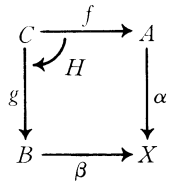
LEMMA (1.7). *If the top face is a homotopy push-out, there is an induced map* $X_3 \to X_\phi$ *and homotopies on the right and back faces so that the resulting cube is homotopy commutative.*

Given homotopy commutative squares



and homotopies $G_1 : C \times I \to A$ and $G_2 : B \times I \to X$ so that $G_1(., 0) = f'$, $G_1(., 1) = f$, $G_2(., 0) = \beta$ and $G_2(., 1) = \beta'$, according to Lemma 6 of [**11**].

LEMMA (1.8). *If* H' *is equivalent to* $G_2(g \times 1) + H + \alpha G_1$,



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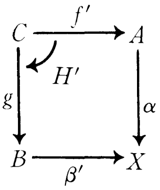
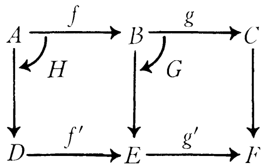
is a homotopy pull-back (push-out) if and only if

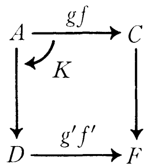


is a homotopy pull-back (push-out).

Consider next the diagram



consisting of two adjacent homotopy commutative squares. Letting $K = H + G(f \times 1)$ we have the third homotopy commutative square



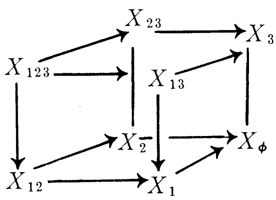
which is called the *composition*. According to [**11**] we state the following result.

THEOREM (1.9). *In the first diagram*

(i) *if two of the three squares are homotopy pull-backs, then so is the third;*
(ii) *if the left and right squares are homotopy push-outs, so is the large square;*
(iii) *if the left and large squares are homotopy push-outs, so is the right square.*

The following theorem due to Mather [**11**] is fundamental to applications.

THEOREM (1.10). *If in the homotopy commutative diagram*



the front and left faces are homotopy pull-backs, the top and bottom homotopy push-outs, then the right and back faces are homotopy pull-backs.

COROLLARY (1.11). *In the homotopy commutative cube of Theorem* (1.10) *sup-*

pose all vertical faces are homotopy pull-backs; if one of the top or bottom faces is a homotopy push-out, then so is the other.

Remark. As shown in a preliminary version of [**11**] neither the dual of Theorem (1.10) of Corollary (1.11) hold in general.

From now on when speaking of homotopy pull-backs or push-outs, maps and spaces will often be omitted; in these cases it is understood that the reader should consider the standard construction of (1.5). Similarly explicit reference is omitted in the case of maps *from* spaces obtained by taking homotopy pull-backs or maps *to* spaces obtained by taking homotopy push-outs; again it is assumed that we refer to the standard maps as described in (1.5).

In the list of examples below, examples (ii), (iii), (v) and (vi) may be considered as definitions of the spaces $X \flat Y$, $X \hat{*} Y$, $X * Y$, and $X \# Y$, and as such correspond to the usual definitions.

Examples. (1) The homotopy pull-back of the diagram $* \rightarrow X \leftarrow *$ is of the form

$$\begin{array}{ccc} \Omega X & \longrightarrow & * \\ \Big| & & \Big\downarrow \\ * & \longrightarrow & X \end{array}$$

(2) If $i : X \vee Y \rightarrow X \times Y$ is the inclusion map, the homotopy pull-back of $* \rightarrow X \times Y \overset{i}{\leftarrow} X \vee Y$ is of the form

$$\begin{array}{ccc} X \flat Y & \longrightarrow & X \vee Y \\ \Big\downarrow & & \Big\downarrow i \\ * & \longrightarrow & X \times Y \end{array}$$

where $X \flat Y$ is the *flat product* of X and Y.

(3) If $i_X : X \rightarrow X \vee Y$ and $i_Y : Y \rightarrow X \vee Y$ are the inclusion maps, the homotopy pull-back of the diagram $X \overset{i_X}{\rightarrow} X \vee Y \overset{i_Y}{\leftarrow} Y$ is of the form

$$\begin{array}{ccc} X \hat{*} Y & \longrightarrow & X \\ \Big\downarrow & & \Big\downarrow i_X \\ Y & \longrightarrow & X \vee Y \end{array}$$

where $X \hat{*} Y$ is the *co-join* of X and Y.

(4) Homotopy push-outs of the diagram $X \leftarrow * \rightarrow Y$ and $* \leftarrow X \rightarrow *$ are of

the form

$$\begin{array}{ccc} * & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \vee Y \end{array} \quad \text{and} \quad \begin{array}{ccc} X & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & \Sigma X \end{array}$$

(5) If $\mathrm{pr}_1 : X \times Y \to X$ and $\mathrm{pr}_2 : X \times Y \to Y$ are the projections, the homotopy push-out of the diagram $X \overset{\mathrm{pr}_1}{\longleftarrow} X \times Y \overset{\mathrm{pr}_2}{\longrightarrow} Y$ is of the form

$$\begin{array}{ccc} X \times Y & \xrightarrow{\mathrm{pr}_2} & Y. \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow \\ X & \longrightarrow & X * Y \end{array}$$

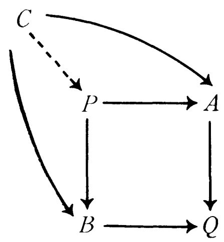
(6) The homotopy push-out of the diagram $X \times Y \overset{i}{\leftarrow} X \vee Y \to *$ is of the form

$$\begin{array}{ccc} X \vee Y & \longrightarrow & * \\ \downarrow & & \downarrow \\ X \times Y & \longrightarrow & X \# Y \end{array}$$

where $X \# Y$ is the smash product of X and Y.

By using the ideas developed so far it is possible to improve exposition and simplify proofs in a number of areas. In particular the Blakers-Massey Theorem ([**3**, Theorem I] and [**4**, Theorem I]) has the following statement.

THEOREM (1.12). (Blakers-Massey). *If in a homotopy commutative diagram*

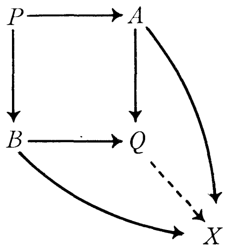


the outside square is a homotopy push-out and the inside a homotopy pull-back, and if the maps $C \to A$ *and* $C \to B$ *are respectively* p *and* q *connected with* $\min(p, q) > 1$, *then the induced map* $C \to P$ *of Definition* (1.2) *is* $p + q - 1$ *connected and* $\pi_{p+q}(C \to P) \approx \pi_p(C \to A) \otimes \pi_q(C \to B)$.

The Svarc-Ganea-Nomura Theorem ([**15**, Chapter II, Section 1]; [**8**, Theorem (1,1)]; **12**; **13**]) as expressed below may be considered dual to the Blakers-

Massey Theorem above. Its proof is an application of previous techniques; see [**11**, Theorem 47].

THEOREM (1.13) (Svarc-Ganea-Nomura). *If in a homotopy commutative diagram*

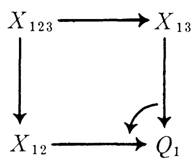


the outside square is a homotopy pull-back and the inside square is a homotopy push-out and if F and G are the fibres of $A \to X$ and $B \to X$ respectively, then the fibre of the induced map $Q \to X$ of Definition (1.2) *is F^*G. Consequently if $A \to X$ and $B \to X$ are respectively p and q connected, then $Q \to X$ is $p + q + 1$ connected.*

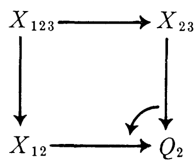
2. Dual cube theorems. In this section we establish approximations to the duals of Theorem (1.10) and Corollary (1.11)

In the homotopy commutative cube of (1.1), suppose the top and bottom faces are homotopy pull-backs and the right and back faces homotopy push-outs.

Let



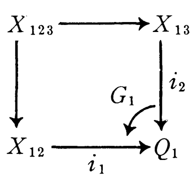
and



be homotopy push-outs and let $Q_1 \to X_1$ and $Q_2 \to X_2$ be induced according to Definition (1.2). In this context we have the following theorem which we consider as an approximation to the dual of Theorem (1.10).

THEOREM (2.1). *If the maps $X_{123} \to X_{12}$, $X_{123} \to X_{13}$ and $X_{123} \to X_{23}$ are respectively p, q and r connected with* min $(p, q, r) > 1$, *then the induced maps $Q_1 \to X_1$ and $Q_2 \to X_2$ are $p + q + r$ connected.*

Proof. It suffices to prove the result for $Q_1 \to X_1$. With no loss of generality let



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be the homotopy push-out and $Q_1 \to X_1$ the induced map constructed in (1.5). Construct homotopy pull-backs

$$\begin{array}{ccc} P_1 & \longrightarrow & X_{23} \\ {\scriptstyle \pi_1}\downarrow & & \downarrow \\ X_{12} & \longrightarrow & X_2 \end{array} \qquad\qquad \begin{array}{ccc} P_2 & \longrightarrow & X_3 \\ \downarrow & & \downarrow \\ X_1 & \longrightarrow & X_\phi \end{array}$$

and induced maps $X_{123} \to P_1$ and $X_{13} \to P_2$ as in (1.5). Write

$$P_1 = \{ (x_{12}, x_{23}, \gamma) \in X_{12} \times X_{23} \times X_2{}^I : \gamma(0) = f_{12}{}^2(x_{12}) \quad \text{and} \quad \gamma(1) = f_{23}{}^2(x_{23})\}$$

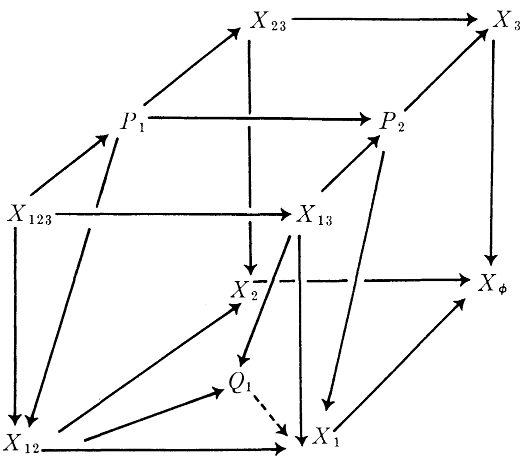
and

$$P_2 = \{x_1, x_3, \gamma) \in X_1 \times X_3 \times X_\phi{}^I : \gamma(0) = f_1{}^\phi(x_1) \quad \text{and} \quad \gamma(1) = f_3{}^\phi(x_3)\}.$$

By Lemma (1.6) there is an induced map $P_1 \to P_2$ defined by

$$(x_{12}, x_{23}, \gamma) \mapsto (f_{12}{}^1(x_{12}), f_{23}{}^3(x_{23}), H_{23}{}^\phi(x_{23}, .) + f_2{}^\phi(\gamma) - H_{12}{}^\phi(x_{12}, .)).$$

In the diagram below this map makes: (a) the inner cube homotopy commutative, (2) the front square of the top face homotopy commutative, and (3) the back square of the top face and the front face of the inner cube strictly commutative



Applying Lemma (1.8) and Theorem (1.9) to the inner cube, it follows that the back square of the top face is a homotopy pull-back. By Corollary (1.11) the square

$$\begin{array}{ccc} P_1 & \longrightarrow & P_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow \\ X_{12} & \longrightarrow & X_1 \end{array}$$

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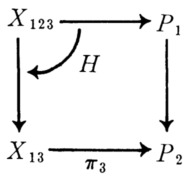
is a homotopy push-out. Also writing the homotopy, $H : X_{123} \times I \to P_2$, on the front square of the top face as:

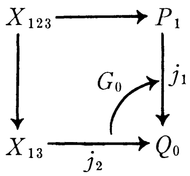
$$H(x_{123}, t) = (-H_{123}{}^1(x_{123}, t), H_{123}{}^3(x_{123}, t), \phi),$$

where ϕ is a path in $(X_\phi)^I$ derived from the equivalence making the outside cube homotopy commutative, it follows that $\pi_3 H = H_{123}{}^3$. By Theorem (1.9)

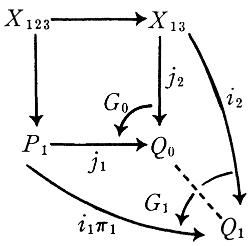


is a homotopy pull-back.

Let

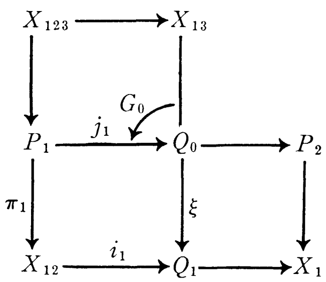


be a homotopy push-out and $Q_0 \to P_2$ the induced map. Letting $\xi : Q_0 \to Q_1$ be the induced map in the diagram

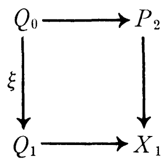


it follows that $\xi j_2 = i_2$, $\xi j_1 = i_1\pi_1$, and $\xi G_0 = G_1$.

Consider the diagram

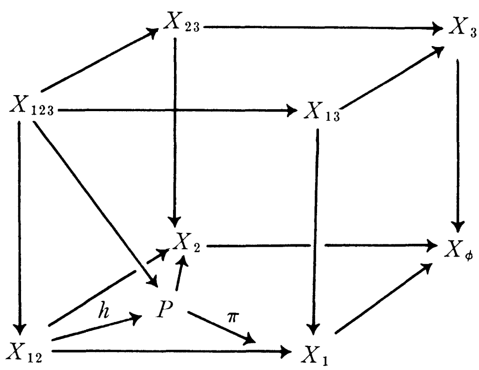


in which each of the three bottom squares is strictly commutative. Since $\xi G_0 = G_1$ it follows from Theorem (1.9), that the bottom left square is a homotopy push-out. Again by Theorem (1.9)



is a homotopy push-out. Since by Theorem (1.13) $Q_0 \to P_2$ is $p + q + r$ connected, so also is $Q_1 \to X_1$. This completes the proof.

Again in the homotopy commutative cube (1.1) suppose the top face is a homotopy pull-back and the vertical faces are homotopy push-outs. If P is the homotopy pull-back of the diagram $X_1 \to X_\phi \leftarrow X_2$, according to Lemma (1.6) and Definition (1.2) there are induced maps $X_{123} \to P$ and $h : X_{12} \to P$ so that there is a diagram



In this context the following theorem is an approximation to the dual of Corollary (1.11)

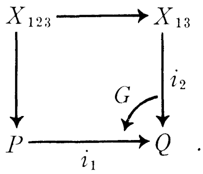
THEOREM (2.2). *If the maps $X_{123} \to X_{12}$, $X_{123} \to X_{13}$, and $X_{123} \to X_{23}$ are respectively p, q, and r connected, then the induced map $h : X_{12} \to P$ is $p + q + r - 1$ connected.*

Proof. Without loss of generality, let P be the homotopy pull-back and $h : X_{12} \to P$ the induced map constructed in (1.5). Write $P = \{(x_2, x_1, \gamma) \in X_2 \times X_1 \times X_\phi{}^I : \gamma(0) = x_2, \gamma(1) = x_1\}$. The map $X_{123} \to P$ is then defined by

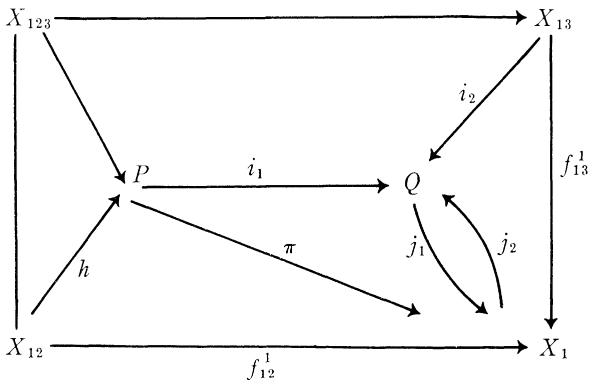
$$x_{123} \mapsto (f_{23}{}^2 f_{123}{}^{23}(x_{123}), f_{13}{}^1 f_{123}{}^{13}(x_{123}), H_{13}{}^\phi(f_{123}{}^{13} \times 1) + f_3{}^\phi H_{123}{}^3 + H_{23}{}^\phi(f_{123}{}^{23} \times 1))$$

Suppose the front face is the standard homotopy push-out described in (1.5);

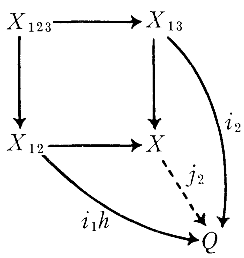
construct the standard homotopy push-out

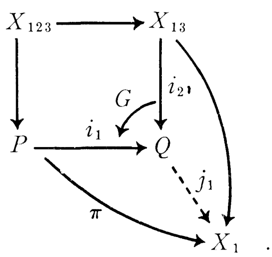


We have the diagram



The maps $j_1 : Q \to X_1$ and $j_2 : X_1 \to Q$ are the standard induced maps of (1.5) in the diagrams

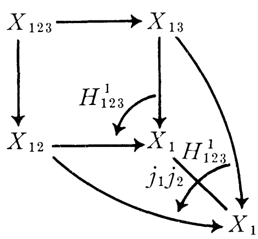




It follows that:

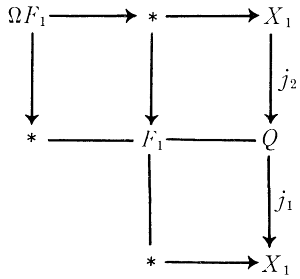
$$j_2 f_{12}^1 = i_1 h \qquad \text{and} \qquad f_{13}^1 = j_1 i_2$$
$$j_2 f_{13}^1 = i_2 \qquad\qquad\qquad \pi = j_1 i_1.$$

Therefore $j_1 j_2 f_{12}^1 = j_1 i_1 h = \pi h = f_{12}^1$ and $j_1 j_2 f_{13}^1 = j_1 i_2 = f_{13}^1$. Consequently the diagram



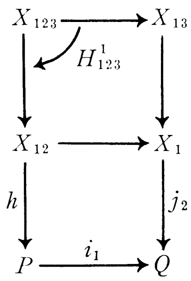
is homotopy commutative so that $j_1 j_2 \simeq 1_{X_1}$.

Let F_1 = fibre (j_1) and F_2 = fibre (j_2). Using Theorem (1.9) each square of



is a homotopy pull-back so that F_2 is homotopy equivalent to ΩF_1. By Theorem (1.13) $j_1 : Q \to X_1$ is $p + q + r$ connected; it follows that $j_2 : X_1 \to Q$ is $p + q + r - 1$ connected.

In the diagram

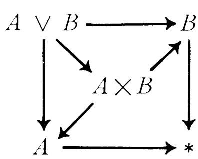


it may be shown that the homotopy $j_2 H_{123}{}^1$ is equivalent to G, so by Theorem (1.9) the bottom square is a homotopy push-out. Therefore the connectivity of h equals the connectivity of j_2.

3. Applications. Theorems (1.12) and (1.13) are not precise duals of one another; although (1.13) gives precise information concerning the fibre of $Q \to X_1$, (1.12) provides little information concerning $C \to P$. The task of approximating $C \to P$ has been the subject of much research; see [**1**] and [**12**].

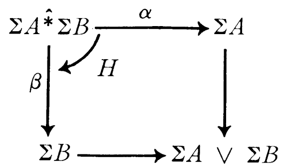
If the exact dual of Theorem (1.13) were true, then in the diagram of Theorem (1.12) the cofibre of $C \to P$ would be the cojoin of the cofibres of the maps $C \to A$ and $C \to B$. In general this is not true as shown below.

Example (3.1). Given spaces A and B, if $A \vee B \to A$, $A \times B \to A$ and $A \vee B \to B$, $A \times B \to B$ are projections onto the first and second factors, there is the commutative diagram

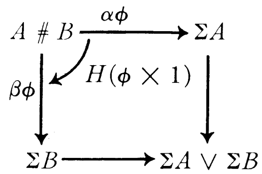


in which the inner square is a homotopy pull-back and the outer a homotopy push-out. The induced map $A \vee B \to A \times B$ of (1.5) becomes the inclusion map. Observe that ΣA is the cofibre of $A \vee B \to B$, ΣB is the cofibre of $A \vee B \to A$, and $A \# B$ is the cofibre of $A \vee B \to A \times B$.

Let



be a homotopy pull-back. If the dual of Theorem (1.13) were true there would be a homotopy equivalence $\phi : A \# B \to \Sigma A \hat{*} \Sigma B$ so that

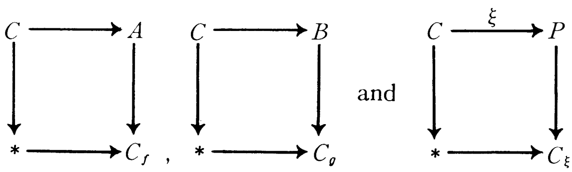


is also a homotopy pull-back. This however is not necessarily true as seen by a spectral sequence argument with $A = S^2$ and $B = S^7$.

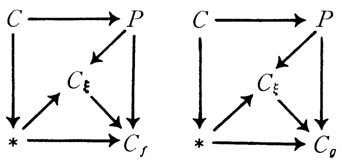
The result below represents an approximation to the dual of Theorem (1.13).

THEOREM (3.2). *In the diagram of Theorem* (1.12), if C *is* r *connected and the maps* $f : C \to A$ *and* $g : C \to B$ *are respectively* p *and* q *connected with* min $(p, q, r + 1) > 1$, *there is a map from the cofibre of the map* $\xi : C \to P$ *to the cojoin of the cofibres of* $C \to A$ *and* $C \to B$ *that is* $p + q + r$ *connected.*

Proof. Let

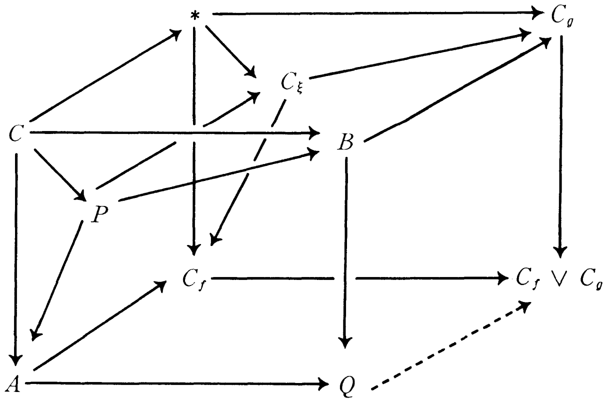


be cofibre squares. Let $C_\xi \to C_f$ and $C_\xi \to C_g$ be the maps induced according to Definition (1.2) in the diagrams

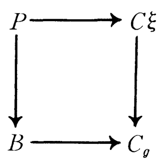


where the maps $P \to C_f$ and $P \to C_g$ are the compositions $P \to A$ with $A \to C_f$

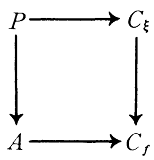
and $P \to B$ with $B \to C_g$ respectively. We have the following diagram



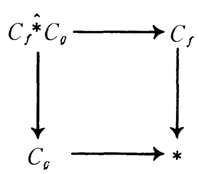
where the map $Q \to C_f \vee C_g$ is induced according to Lemma (1.7) making the outer cube homotopy commutative. Applying Lemma (1.8) and Theorem (1.9), each face of the outer cube is a homotopy push-out. The inner cube may be shown to be homotopy commutative and using Theorem (1.9) the squares



and

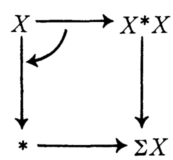


may be shown to be homotopy push-outs. Since

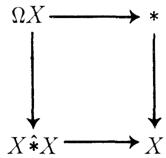


is a homotopy pull-back, the result follows as an application of Theorem (2.2).

If X is an H-space with multiplication $m : X \times X \to X$ the Sugawara Theorem [**14**] says there is a homotopy pull-back

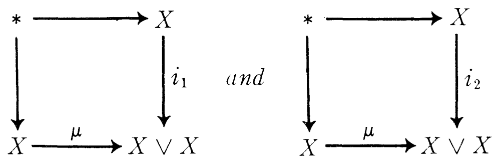


where the map $X \to X * X$ is inessential. As shown by Hilton ([**10**, p. 215]) the dual of this result is not necessarily true, i.e. if X is a co-H space it does not necessarily follow that there is a homotopy push-out



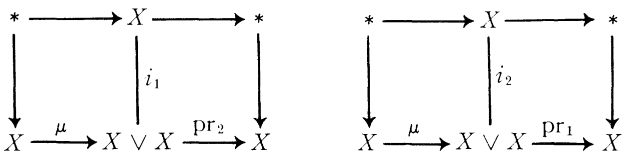
with the map $X \hat{*} X \to X$ inessential. It is reasonable to ask if the dual is true in some approximate sense.

LEMMA (3.3). *Given a simply connected space* X, *a map* $\mu : X \to X \vee X$ *makes* X *a co-H space if and only if the commutative diagrams*



are homotopy push-outs.

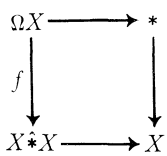
Proof. Consider the commutative diagrams



in which pr_1 and pr_2 are the projections onto the first and second factors respectively. Observe that the right hand squares are homotopy push-outs. Since the large squares are homotopy push-outs if and only if $\mathrm{pr}_2\mu$ and $\mathrm{pr}_1\mu$ are homotopic to 1_X the result follows according to the remarks following Theorem (1.9).

Using this result we have the following approximation to a dual Sugawara Theorem.

THEOREM (3.4). *If* X *is an* n *connected co-H space with* $n > 1$, *there is a diagram,*



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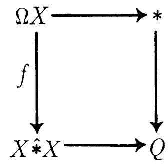
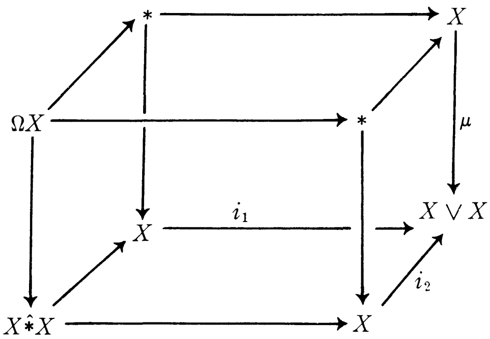
with $X \mathbin{\hat{*}} X\, X$ *inessential, which is a homotopy push-out to dimension* $3n$ (i.e., *if*

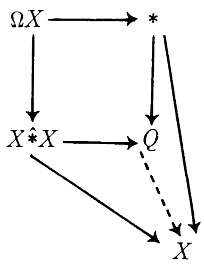


is a homotopy push-out the induced map $Q \to X$ *of Definition* (1.2) *is* $3n$ *connected*).

Proof. Consider the diagram



in which i_1 and i_2 are inclusions in, respectively, the first and second factors, the top and bottom faces are homotopy pull-backs, the back and right faces are homotopy push-outs, and the map $\Omega X \to X \mathbin{\hat{*}} X$ is induced according to Lemma (1.6) making the cube homotopy commutative. Putting in the homotopy push-out of the front face we have the diagram



in which by Theorem (2.1) the induced map $Q \to X$ is $3n$ connected. According to [**12**, Lemma (2.1)], $X * X \to X$ is inessential.

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