ON INTEGERS n RELATIVELY PRIME TO $[\alpha n]$

G. L. WATSON

1. Introduction. The object of this paper is to consider a problem suggested by Dr. K. F. Roth, on the distribution of integers n that are relatively prime to the integral part of αn, α being a fixed real number. He conjectured that the number of positive integers up to x with this property is asymptotic to $6x/\pi^2$ (or in other words that they have the density $6/\pi^2$), for irrational α. I prove this and rather more in the following

THEOREM. *For every real number α the positive integers n such that*

$$(n, [\alpha n]) = 1 \tag{1}$$

have a density $\delta(\alpha)$. For every irrational α, $\delta(\alpha) = 6/\pi^2$. For rational $\alpha = a/q$, with $(a, q) = 1$ and $q > 0$, $\delta(\alpha)$ depends only on q and has the value

$$q^{-1}\sum_{u=1}^{q-1} u^{-1}\,\phi(u),$$

which tends to the limit $6/\pi^2$ as $q \to \infty$.

Notation. Throughout the paper, Greek letters denote real numbers, ϵ being positive and arbitrarily small. Latin letters denote rational integers, n, q, q', x, d, R being positive, and a, q coprime. $\phi(x)$ and $\mu(x)$ are the functions of Euler and Möbius, $d(x)$ is the number of divisors of x, and (y, z) is the highest common factor of y and z (not both zero). $[\alpha]$ is the greatest integer not exceeding α. The constants implied by the O-notation are absolute, except in formulae containing ϵ, in which they depend on ϵ only.

We define

$$f(x, \alpha) = \sum_{n \leqslant x,\,(1)} 1,$$

where (1) refers to equation (1) above, and

$$\psi(q, r) = \sum_{-r \leqslant u < q-r} |u|^{-1}\,\phi(|u|), \qquad u \neq 0$$

where an empty sum is to be interpreted as zero. Thus $\delta(\alpha)$ is the limit (to be proved to exist) of $x^{-1}f(x, \alpha)$ as $x \to \infty$, and we have to show that $\delta(a/q) = \psi(q, 0)$.

2. Preliminary. The first of the following lemmas is a known result due to Vinogradov [**1**, chap. II, Ex. 19b], but the proof is reproduced as it is short.

Received March 4, 1953.

https://doi.org/10.4153/CJM-1953-050-9 Published online by Cambridge University Press

LEMMA 1. *For any l, u, x with $u \neq 0$ we have*

$$\sum_{n \leqslant x,\, (n+l,u)=1} 1 = x|u|^{-1}\,\phi(|u|) + O(d(|u|)).$$

Proof. We may suppose $u > 0$, and $0 \leqslant l < u$. Then the result for positive l follows from that for $l = 0$ by putting first $x + l$ and then l for x, and subtracting; so we may suppose $l = 0$. With these preliminaries the sum to be estimated is equal to

$$\begin{aligned}\sum_{n=1}^{x} \sum_{d|u,\, d|n} \mu(d) &= \sum_{d|u} \sum_{n \leqslant x,\, d|n} \mu(d)\\ &= \sum_{d|u} \left[\frac{x}{d}\right] \mu(d)\\ &= \sum_{d|u} \frac{x}{d}\,\mu(d) + O(d(u))\\ &= xu^{-1}\,\phi(u) + O(d(u)).\end{aligned}$$

LEMMA 2.

$$\psi(q, r) = \frac{6q}{\pi^2} + O(q^{\epsilon}) + O(|r|^{\epsilon}).$$

Proof. It is sufficient to consider the case $r = 0$, which follows by partial summation from the known result [**2**, p. 266, Theorem 330]

$$\sum_{n=1}^{x} \phi(n) = \frac{3x^2}{\pi^2} + O(x \log x).$$

LEMMA 3. *For $R \geqslant q^2$,*

$$R^{-1} \sum_{r=0}^{R-1} \psi(q, r) = \frac{6q}{\pi^2} + O(1). \tag{2}$$

and

$$R^{-1} \sum_{r=-R}^{-1} \psi(q, r) = \frac{6q}{\pi^2} + O(1). \tag{3}$$

Proof. We count the number of times a summand with given u occurs in the double sum obtained by substituting the sum defining $\psi(q, r)$ in that on the left of (3); with unimportant exceptions it is precisely q. Thus

$$\begin{aligned}\sum_{r=-R}^{-1} \psi(q, r) &= \sum_{r=-R}^{-1} \sum_{-r \leqslant u < q-r,} |u|^{-1}\,\phi(|u|) \qquad u \neq 0\\ &= q \sum_{1 \leqslant u < R} u^{-1}\phi(u) - \sum_{u=1}^{q-1} (q-u)\, u^{-1}\phi(u)\\ &\qquad + \sum_{v=1}^{q} (q - v + 1)(R + v - 1)^{-1}\phi(R + v - 1)\\ &= q\psi(R, 0) + O(q^2).\end{aligned}$$

Hence, using $R \geqslant q^2$, the left member of (3) is

$$qR^{-1}\psi(R, 0) + O(1),$$

and (3) follows on putting R, 0, $\frac{1}{2}$ for q, r, ϵ in Lemma 2.

The proof of (2) is similar.

3. The case of rational α. In this section we take $\alpha = a/q$, and prove a lemma which is a slight generalization of the latter part of the Theorem.

LEMMA 4. *We consider n satisfying (for fixed a, q, r)*

$$\left(n, \left[\frac{an+r}{q}\right]\right) = 1, \tag{4}$$

and define

$$F(x; a, q, r) = \sum_{n \leqslant x, (4)} 1.$$

Then (interpreting 0 log 0 *as* 0) *we have*

$$F(x; a, q, r) = xq^{-1}\psi(q, r) + O(q \log q) + O(|r| \log |r|) + O(1).$$

Proof We write

$$n = qm + l, \qquad 0 \leqslant l < q, \tag{5}$$

$$\left[\frac{an+r}{q}\right] = am + l'. \tag{6}$$

We can choose a', q' so that $a'q - aq' = \pm 1$, whence

$$(y, z) = (ay - qz, a'y - q'z),$$

and so

$$\left(n, \left[\frac{an+r}{q}\right]\right) = (u, m + l'')$$

where l'' is independent of m and

$$u = al - ql'. \tag{7}$$

It is clear from (5) to (7) that $-r \leqslant u < q - r$ and that u runs with l through a complete set of residues modulo q. Hence for $l = 0, 1, \ldots, q-1$, u takes the values $-r, -r+1, \ldots, q-r-1$, in some order, each just once.

Now we break up the sum $F(x; a, q, r)$ into a double sum over l, m, or equivalently, over u, m. For $u = 0$, the inner sum over m is $O(1)$, since (4) can hold only if $m = -l'' \pm 1$. For other u, we have to sum over $m = 0, 1, \ldots, [x/q]$, and possibly $[x/q] + 1$. But with error $O(1)$ we can omit the values 0, $[x/q] + 1$. Then using Lemmas 1 and 2 we find (for $u \neq 0$)

$$F(x; a, q, r) = \sum_{-r \leqslant u < q-r} \sum_{\substack{1 \leqslant m \leqslant [x/q], \\ (u, m+l'')=1}} 1 + O(q)$$

https://doi.org/10.4153/CJM-1953-050-9 Published online by Cambridge University Press

$$= \sum_{-r \leqslant u < q-r,} \left[\frac{x}{q}\right] |u|^{-1} \phi(|u|) + O\Big(\sum_{-r \leqslant u < q-r,} d(|u|)\Big) + O(q)$$

$$= xq^{-1}\,\psi(q, r) + O\Big(\sum_{-r \leqslant u < q-r} d(|u|)\Big) + O(q).$$

The Lemma now follows from

$$d(1) + d(2) + \ldots + d(x) = O(x \log x);$$

more precise results implying this are well known [**2**, p. 262, Theorems 318 to 320].

4. Proof of the Theorem. For rational α, we have only to take $r = 0$ in Lemmas 2 and 4.

Now let α be irrational, and let a/q, a'/q' be two successive convergents to its infinite continued fraction expansion. (In the case of negative α, which we could of course avoid, the convergents are those of the continued fraction for $|\alpha|$, with the signs of the numerators changed.) For large x, we choose q to satisfy

$$q \leqslant x(\log x)^{-2} < q'. \tag{8}$$

Clearly q tends to infinity with x, and the theorem follows if we prove

$$f(x, \alpha) = \frac{6x}{\pi^2} + O(x/\log x) + O(xq^{-\frac{1}{2}}). \tag{9}$$

We define $r = r(n) = r(n, \alpha, q)$ by

$$r = [n(q\alpha - a)], \tag{10}$$

whence

$$[\alpha n] = \left[\frac{an + r}{q}\right]. \tag{11}$$

As n takes the values $1, 2, \ldots, x$, we note that r takes the values $0, 1, \ldots, R - 1$ or $-1, -2, \ldots, -R$, according to the sign of $\alpha - a/q$, where, by (8) and since $|\alpha - a/q| < 1/qq'$, we have

$$R < 1 + x/q' < 1 + \log^2 x. \tag{12}$$

If $q > \log^3 x$, (11) and (12) show that $[\alpha n] = [an/q]$ except possibly for n in $O(R) = O(q/\log x)$ residue classes (mod q). Now by (8) there are, up to x, only $O(x/\log x)$ such n, so (9) follows from

$$\begin{aligned} f(x, \alpha) &= f(x, a/q) + O(x/\log x) \\ &= F(x; a, q, 0) + O(x/\log x) \\ &= xq^{-1}\,\psi(q, 0) + O(q \log q) + O(x/\log x) \\ &= xq^{-1}\,\psi(q, 0) + O(x/\log x), \end{aligned}$$

using Lemmas 2 and 4 (with $r = 0$) and (8).

https://doi.org/10.4153/CJM-1953-050-9 Published online by Cambridge University Press

We may therefore assume

$$q \leqslant \log^3 x. \tag{13}$$

We write

$$f(x, \alpha) = \sum_r \sum_{X_r < n \leqslant Y_r, (1)} 1,$$

where $X_r + 1, \ldots, Y_r$ $(= X_{r\pm 1}$ unless $r = R - 1$ or $-R)$ are the consecutive values of n for which r takes a given value. The outer sum is over $0 \leqslant r < R$ or $0 > r \geqslant -R$ as the case may be. By (11) and Lemma 4, the inner sum is

$$F(Y_r; a, q, r) - F(X_r; a, q, r) = xq^{-1}\psi(q, r) + O(q \log q) + O(R \log R).$$

Hence using (12) and (13) we find

$$f(x, \alpha) = \sum_r (Y_r - X_r)\, q^{-1}\, \psi(q, r) + O(\log^6 x). \tag{14}$$

Now we may assume

$$R \geqslant q^2. \tag{15}$$

For otherwise (9) follows immediately from (14) and Lemma 2.

We next note that, except for $r = R - 1$ or $-R$, $Y_r - X_r$ can take only two different values; these are consecutive integers, the smaller of which is $[|q\alpha - a|^{-1}]$. It easily follows that

$$Y_r - X_r = \begin{cases} xR^{-1} + O(xR^{-1}), & \text{if } r = R - 1 \text{ or } -R, \\ xR^{-1} + O(xR^{-2}), & \text{otherwise.} \end{cases}$$

Substituting in (14) we find

$$f(x, \alpha) = xq^{-1} R^{-1} \sum_r \psi(q, r) + O(xR^{-1}) + O(\log^6 x).$$

Now (9) follows from (15) and Lemma 3, and so the proof is complete.

5. Conclusion. It would not be difficult to prove a similar result (with $6/\pi^2 k^2$ in place of $6/\pi^2$) for n satisfying $(n, [\alpha n]) = k$ in place of (1).

I am indebted to Professor Davenport and Dr. Roth for reading earlier drafts of this paper, and pointing out some obscurities and errors.

REFERENCES

1. I. M. Vinogradov, *Osnovy teorii čisel* (Moscow and Leningrad, 1949).
2. G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers* (Oxford, 1938).

University College, London