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ON MINIMAL SUBGROUPS OF FINITE GROUPS

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Abstract. Let G be a finite group. A minimal subgroup of G is a subgroup of prime order. A subgroup of G is called S-quasinormal in G if it permutes with each Sylow subgroup of G. A group G is called an MS-group if each minimal subgroup of G is S-quasinormal in G. In this paper, we investigate the structure of minimal non-MS-groups (non-MS-groups all of whose proper subgroups are MS-groups).

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1. Introduction. Throughout this paper G will denote a finite group. We write $\sigma(G)$ for the set of prime divisors of the order of G and $|\sigma(G)|$ for their number. A minimal subgroup of G is a subgroup of prime order. A group G is called a PN-group if each minimal subgroup of G is normal in G. Two subgroups H and K of a group G are said to permute if $HK = KH$. It is easily seen that H and K permute if and only if the set HK is a subgroup of G. We say, following Kegel [**7**], that a subgroup of G is S-quasinormal in G if it permutes with each Sylow subgroup of G. A group G is called an MS-group if each minimal subgroup of G is S-quasinormal in G.

Schmidt and others [**9**] determined the structure of minimal non-nilpotent groups (non-nilpotent groups all of whose proper subgroups are nilpotent), and Doerk [**2**] also determined the structure of minimal non-supersolvable groups (non-supersolvable groups all of whose proper subgroups are supersolvable). In [**8**], Sastry investigated the structure of minimal non-PN-groups (non-PN-groups all of whose proper subgroups are PN-groups) and proved that if G is not of prime power order and G is a minimal non-PN-group, then one of the following two statements is true:

(1) $G = PQ$, where P is a normal Sylow p- subgroup of G, $P =< x >^G$; P is elementary abelian; and Q is a non-normal cyclic Sylow q-subgroup of G.

(2) $G = P < x >$, where P is a normal ultraspecial 2-subgroup of G of order 2^{3s} (a p-group P is called ultraspecial if $P' = \Omega_1(P) = \Phi(P) = Z(P)$) and $|x|$ is a prime dividing $2^s + 1$.

In this paper, we investigate the structure of minimal non-MS-groups (non-MS-groups all of whose proper subgroups are MS-groups). In Section 2, we prove that if G is not of prime power order and is a minimal non-PN-group, then it is a minimal non-MS-group. However, the converse statement is not true, as the following example shows:

Example. Let P be an extraspecial group of order 3^7 and exponent 3 (a p-group P is called extraspecial if $P' = \Phi(P) = Z(P)$ and $|Z(P)| = p$). Then, by [**3**, Lemma 20.13, p. 83], Aut(P) contains an element α of order 7 which acts irreducibly on $P/\Phi(P)$. Let G be the semi-direct product of P with $< \alpha >$. Then it follows easily that

(i) P contains a non-normal minimal subgroup $< x >$;
(ii) G is a minimal non-nilpotent group; and
(iii) G is a minimal non-MS-group.

We prove the following theorem:

THEOREM . *If G is a minimal non-MS-group, then $G = PQ$, where P is a normal Sylow p-subgroup of G and Q is a non-normal cyclic Sylow q-subgroup of G ($q \neq p$), and one of the following statements is true:*

(a) G is supersolvable.
(b) $P =< x >^G$, where $| < x > | = p$ and $< x >$ is not S-quasinormal in G.
(c) P is a non-abelian 2-group, $\Omega_1(P) \leq Z(G)$ and $|Q| = q$.

2. Preliminaries. In this section we collect some of the results used later.
(2.1) If each minimal subgroup of G is normal in G, then G is solvable.

Proof. This is [**5**, Satz 5.7, p. 436]. □

(2.2) (a) If $H \leq K \leq G$ and H is S-quasinormal in G, then H is S-quasinormal in K.
(b) If H is S-quasinormal in G, then H is subnormal in G.
(c) Let H be a p-subgroup for some prime p. If H is S-quasinormal in G, then $O^p(G) \leq N_G(H)$, where

$$O^p(G) =< Q|Q \text{ is a Sylow } q\text{-subgroup of G, where } q \neq p > .$$

(d) If H and K are S-quasinormal in G, then $< H, K >$ is S-quasinormal in G.

Proof. (a), (b): see Kegel [**7**]. □

(c) Let Q be any Sylow q-subgroup of G, where $q \neq p$. Since H is S-quasinormal in G, it follows that HQ is a subgroup of G. By (a) and (b), H is subnormal in HQ, and since H is a p-subgroup of G, it follows that H is normal in HQ for each Sylow q-subgroup Q of G, where $q \neq p$. Hence $O^p(G) \leq N_G(H)$. □

(d) By the hypothesis, $HP = PH$ and $KP = PK$ for all Sylow subgroups P of G. Now it follows easily that $P< H, K >=< H, K >P$, and so $< H, K >$ is S-quasinormal in G. □

We can now prove the following:
(2.3) If G is not of prime power order and G is a minimal non-PN-group, then G is a minimal non-MS-group.

Proof. Suppose that G is an MS-group. By the hypothesis, there exists a minimal subgroup H of order, say, p such that H is not normal in G. Then H is S-quasinormal in G. Hence $O^p(G) \leq N_G(H) < G$ by (2.2)(c). Let P be a Sylow p-subgroup of G such that $H \leq P$. By the hypothesis, G is not of prime power order and is a minimal non-PN-group, so H is normal in P. Since H is normal in P and $O^p(G) \leq N_G(H)$, we have that H is normal in G, a contradiction. □

(2.4) Let P be a Sylow p-subgroup of G for some odd prime p. If $\Omega_1(P) \leq Z(G)$, then G is p-nilpotent.

Proof. This is [**5**, Satz 5.5(a), p. 435]. □

(2.5) If A is a p'-group of automorphisms of the abelian p-group P which acts trivially on $\Omega_1(P)$, then $A = 1$.

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Proof. This is [**4**, Theorem 4.2, p. 178]. □

(2.6) Any non-abelian simple group, all of whose subgroups are solvable, is isomorphic to one of the following simple groups:
(1) $PSL(2, p)$, where p is a prime with $p > 3$ and $5 \nmid p^2 - 1$;
(2) $PSL(2, 2^q)$, where q is a prime;
(3) The Suzuki group $Sz(2^q)$, where q is an odd prime;
(4) $PSL(2, 3^q)$, where q is an odd prime;
(5) $PSL(3, 3)$.

Proof. Thompson [**10**]; see also [**5**, Bemerkung 7.5, p. 190]. □

(2.7) If G is any one of the simple groups mentioned in (2.6) other than $PSL(3, 3)$, then G is a Zassenhaus group of degree $n + 1$, where $n = r$ or r^2 according to $G = PSL(2, r)$ or $G = Sz(r)$; and the stabilizer N of a point is a maximal subgroup of G. Further, N is a Frobenius group with kernel K of order n and a cyclic complement H. If $G = PSL(2, r)$, then $|H| = (r - 1)/d$, where $d = (r - 1, 2)$; and if $G = Sz(r)$, then $|H| = r - 1$. Also, $N' = K$ and H contains a Sylow ℓ-subgroup of G for any odd prime divisor ℓ of $|H|$, with $(\ell, r) = 1$.

Proof. See [**4**, Theorem 8.2, p. 41]; see also [**6**, pp. 182–189]. □

(2.8) Let H be a proper subgroup of G and suppose that H is subnormal in K whenever $H \leq K < G$ but is not subnormal in G. Then H is contained in a unique maximal subgroup of G.

Proof. This is [**3**, Lemma 14.9, p. 49]. □

(2.9) Let G be a solvable group. If each subgroup of $F(G)$ of prime order or order 4 is S-quasinormal in G, then G is supersolvable.

Proof. This is [**1**, Corollary 2]. □

3. Proofs.

We first prove the following lemmas:

LEMMA 3.1. *If G is an MS-group, then G is solvable.*

Proof. We proceed by induction on the order of G. If each minimal subgroup of G is normal in G, then G is solvable by (2.1). Therefore, we may assume that some minimal subgroup H, with $|H| = p$ say, is not normal in G. By the hypothesis, H is S-quasinormal in G. Then $O^p(G) \leq N_G(H) < G$ by (2.2)(c). By (2.2)(a), $O^p(G)$ is an MS-group, so $O^p(G)$ is solvable by induction on the order of G, and since $G/O^p(G)$ is a p-group, it follows that G is solvable. □

LEMMA 3.2. *Let G be a non-solvable minimal non-MS-group. Then $\Phi(G) \neq 1$, and each minimal subgroup of $\Phi(G)$ is normal in G.*

Proof. By the hypothesis, each proper subgroup of G is an MS-group. Then, by Lemma 3.1, each proper subgroup of G is solvable, and since G is non-solvable, it follows that each maximal subgroup of G is non-normal in G. Let M be a maximal subgroup of G. If $M \cap M^x = 1$ for each $x \in G\backslash M$, then G is a Frobenius group [**4**, Theorems 7.7 and 7.6(i), pp. 38–39], and so G is solvable, a contradiction. Therefore, $M \cap M^x \neq 1$ for some x$\in G\backslash M$. Let H be a minimal subgroup of $M \cap M^x$ of order, say, p. By the hypothesis, H is S-quasinormal in K whenever $H \leq K < G$. Then, by (2.2)(b), H is subnormal in K whenever $H \leq K < G$. If H is not subnormal in G, then H is contained in a unique

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maximal subgroup of G by (2.8), a contradiction. Therefore, H is subnormal in G, and so $H \leq O_p(G)$ by [**3**, Lemma 8.6 (a)]. If $O_p(G)$ is not contained in $\Phi(G)$, then there exists a maximal subgroup M_1 of G such that $G = O_p(G)M_1$. Then G is solvable, a contradiction. Hence $H \leq O_p(G) \leq \Phi(G)$, and so $\Phi(G) \neq 1$.

Now we argue that each minimal subgroup of $\Phi(G)$ is normal in G. If not, then $\Phi(G)$ contains a minimal subgroup H with order, say, p such that H is not normal in G. Since $\Phi(G)$ is nilpotent, it follows that H is subnormal in G, and so $H \leq O_p(G)$. Clearly, $O_p(G)Q < G$ for each Sylow subgroup Q of G with $(p, |Q|) = 1$. By the hypothesis, H is S-quasinormal in $O_p(G)Q$, and so $HQ \leq O_p(G)Q < G$. By (2.2) (a), H is S-quasinormal in HQ, so H is subnormal in HQ by (2.2) (b). Since H is subnormal Hall in HQ, it follows that H is normal in HQ. Then $O^p(G) \leq N_G(H) < G$. Since $O^p(G)$ is solvable and $G/O^p(G)$ is a p-group, it follows that G is solvable, a contradiction. Therefore, each minimal subgroup of $\Phi(G)$ is normal in G. □

LEMMA 3.3. *Let G be a non-solvable minimal non-MS-group. Let M be a subgroup of G and Q be a Sylow q-subgroup of M, where q is an odd prime. If $\Omega_1(Q) \leq \Phi(G) \leq M$, then M is q-nilpotent.*

Proof. Since G is non-solvable and each proper subgroup of G is solvable by Lemma 3.1, it follows that $G = G'$. Since $\Omega_1(Q) \leq \Phi(G)$, it follows that each minimal subgroup H of Q is normal in G by Lemma 3.2, and so $G/C_G(H)$ is abelian. Then $G = G' \leq C_G(H)$, and so $\Omega_1(Q) \leq Z(G)$; in particular $\Omega_1(Q) \leq Z(M)$. Hence M is q-nilpotent by (2.4). □

LEMMA 3.4. *Let G be a non-solvable minimal non-MS-group. Then $G/\Phi(G)$ contains no subgroup isomorphic to S_4.*

Proof. Suppose that $G/\Phi(G)$ contains a subgroup $M/\Phi(G)$ such that $M/\Phi(G) \cong S_4$. If $M = G$, then $G/\Phi(G) \cong S_4$, and since $\Phi(G)$ is nilpotent, it follows that G is solvable, a contradiction. Then we may assume that M is a proper subgroup of G. Let Q be a Sylow 3-subgroup of M. If $\Omega_1(Q) \leq \Phi(G) \leq M$, then M is 3-nilpotent by Lemma 3.3, a contradiction. Therefore, Q contains some minimal subgroup H such that H is not contained in $\Phi(G)$. By the hypothesis, H is S-quasinormal in M, and so $H\Phi(G)$ is S-quasinormal in M. Then $H\Phi(G)/\Phi(G)$ is S-quasinormal in $M/\Phi(G)$. By (2.2) (b) $H\Phi(G)/\Phi(G)$ is subnormal in $M/\Phi(G)$, and since $H\Phi(G)/\Phi(G)$ is a Sylow 3-subgroup of $M/\Phi(G)$, it follows that $H\Phi(G)/\Phi(G)$ is normal in $M/\Phi(G) \cong S_4$, a contradiction. □

LEMMA 3.5. *Let G be a non-solvable minimal non-MS-group. Then $G/\Phi(G)$ is not isomorphic to $A_5 \cong PSL(2, 5) \cong PSL(2, 4)$.*

Proof. Suppose that $G/\Phi(G) \cong A_5$. Then $G/\Phi(G)$ contains a subgroup $M/\Phi(G)$ isomorphic to A_4. Let Q be a Sylow 3-subgroup of M. Clearly, $Q \ldots$, is a Sylow 3-subgroup of G, too. If $\Omega_1(Q) \leq \Phi(G)$, then G is 3-nilpotent by Lemma 3.3, and so G is solvable by Lemma 3.1, a contradiction. Therefore, Q contains some minimal subgroup H such that H is not contained in $\Phi(G)$. By the hypothesis, H is S-quasinormal in M, and so $H\Phi(G)/\Phi(G)$ is S-quasinormal in $M/\Phi(G)$. By (2.2) (b), $H\Phi(G)/\Phi(G)$ is subnormal in $M/\Phi(G)$, and since $H\Phi(G)/\Phi(G)$ is a Sylow 3-subgroup of $M/\Phi(G)$, it follows that $H\Phi(G)/\Phi(G)$ is normal in $M/\Phi(G) \cong A_4$, a contradiction. □

LEMMA 3.6. *Let G be a non-supersolvable minimal non-MS-group. Suppose that $G = PQ$, where P is a normal Sylow p- subgroup of G and Q is a non-normal Sylow q-subgroup* *of G. Then $P = H^G$ for some non-S-quasinormal minimal subgroup H of G or $\Omega_1(P) \leq Z(G)$; P is a non-abelian 2-group and $|Q| = q$.*

Proof. Suppose that the result is not true. We treat with the following three cases:

Case 1. Each minimal subgroup of P is S-quasinormal in G and $p > 2$. Then each subgroup of $F(G)$ of prime order or order 4 is S-quasinormal in G. Hence G is supersolvable by (2.9), a contradiction.

Case 2. Each minimal subgroup of P is S-quasinormal in G and $p = 2$. Let H be any minimal subgroup of P. Then $H \neq P$, because Q is a non-normal Sylow q-subgroup of G. Since H is S-quasinormal in G, we have that HQ is a subgroup of G. By (2.2)(a), H is S-quasinormal in HQ, and so H is subnormal in HQ by (2.2) (b). Since H is a subnormal Sylow 2-subgroup of HQ, we have that H is normal in HQ. But Q is normal in HQ, because $|H| = 2$. Then $HQ = H \times Q$ for each minimal subgroup H of P. If $\Omega_1(P) = P$, then $G = PQ = P \times Q$, a contradiction. Thus $\Omega_1(P) < P$, and $\Omega_1(P)Q$ is a proper subgroup of G. Since G is a minimal non-MS-group and since each minimal subgroup of P is S-quasinormal in G, it follows that there exists a minimal subgroup L of Q such that L is not S-quasinormal in G, and so $G = PQ = PL$. By the hypothesis, $Q = L$ is S-quasinormal in $\Omega_1(P)Q$, so Q is a subnormal Sylow q-subgroup of $\Omega_1(P)Q$. Then Q is normal in $\Omega_1(P)Q$, and since $\Omega_1(P)$ is normal in $\Omega_1(P)Q$, we have $Q \leq C_G(\Omega_1(P))$. If $C_G(\Omega_1(P)) < G$, then Q is S-quasinormal in $C_G(\Omega_1(P))$ by the hypothesis. By (2.2)(b), Q is subnormal in $C_G(\Omega_1(P))$, and since $C_G(\Omega_1(P))$ is normal in G, we have that Q is subnormal in G, and so Q is normal in G, a contradiction. Therefore, $\Omega_1(P) \leq Z(G)$, and since Q is not normal in G, it follows that P is a non-abelian 2-group by (2.5), a contradiction

Case 3. There exists some non-S-quasinormal minimal subgroup H of G with $H \leq P$.

Suppose $H^G \neq P$. Then $H^G Q$ is a proper subgroup of G, and so H is S-quasinormal in $H^G Q$. In particular, H permutes with Q. We can repeat this argument with any Sylow q-subgroup, and H permutes with P, so H is S-quasinormal in G, a contradiction. □

We can now prove the main theorem.

Proof. For the sake of clarity, we break the proof into five parts.

(1) G is solvable.

Suppose that G is non-solvable. Then:

(i) $G = G'$ and $G/\Phi(G)$ is a non-abelian simple group, because every proper subgroup of G is solvable by Lemma 3.1.

(ii) $G/\Phi(G)$ is not isomorphic to $PSL(2, 4) \cong PSL(2, 5) \cong A_5$ by Lemma 3.5.

(iii) $G/\Phi(G)$ is not isomorphic to $PSL(2, r)$ or $Sz(r)$, where $r = 2^q$ and q is an odd prime.

Suppose that $G/\Phi(G) \cong PSL(2, r)$ or $Sz(r)$, where $r = 2^q$ and q is an odd prime. Let ℓ be an odd prime dividing $r - 1$. By (2.7), $\overline{G} = G/\Phi(G)$ contains a proper Frobenius subgroup $\overline{M} = M/\Phi(G)$ with kernel $\overline{K} = K/\Phi(G)$ and cyclic complement $\overline{H} = H/\Phi(G)$ of order $r - 1$, and $\overline{H}$ contains a Sylow ℓ-subgroup $L\Phi(G)/\Phi(G)$ of $\overline{G}$, where L is a sylow ℓ-subgroup of G. If $\Omega_1(L) \leq \Phi(G)$, then G is 3-nilpotent by Lemma 3.3, and so G is solvable by Lemma 3.1, a contradiction. Therefore, we may assume that there exists a minimal subgroup A of L such that A is not contained in $\Phi(G)$. By the hypothesis, A is S-quasinormal in M, and so $\overline{A} = A\,\Phi(G)/\Phi(G)$ is S-quasinormal in $\overline{M}$. Then $\overline{A}$ is normal in $\overline{K}\,\overline{A}$, and this is a contradiction because $\overline{M}$ is a Frobenius group.

(iv) $G/\Phi(G)$ is not isomorphic to $PSL(2, p)$ or $PSL(2, 3^q)$, where p is a prime with $p > 3$ and q is an odd prime.
The assertion in (iii) implies that there is no odd prime dividing $p - 1$ or $3^q - 1$. Then $p - 1 = 2^n$ for some natural number n. By (ii), $n \geq 4$. Also, $3^q - 1 = 2^m$ for some natural number $m \geq 4$. Since $p^2 - 1 \equiv 0\,(16)$ and $3^{2q} - 1 \equiv 0\,(16)$, it follows that $G/\Phi(G)$ contains a subgroup isomorphic to S_4, contradicting Lemma 3.4.

(v) $G/\Phi(G)$ is not isomorphic to $PSL(3, 3)$.
Suppose that $G/\Phi(G) \cong PSL(3, 3)$. Take $Y = PSL(3, 3)$. Let x be an involution in the centre of a Sylow 2-subgroup of Y. Then $C_Y(x) \cong GL(2, 3)$ by [**6**, Lemma 5.1, p. 341]. Let $M/\Phi(G)$ be a subgroup of $G/\Phi(G)$ such that $M/\Phi(G) \cong GL(2, 3)$. Let Q be a Sylow 3-subgroup of M. If $\Omega_1(Q) \leq \Phi(G) \leq M$, then M is 3-nilpotent by Lemma 3.3, a contradiction. Therefore, we may assume that there exists a minimal subgroup L of Q such that L is not contained in $\Phi(G)$. By the hypothesis, L is S-quasinormal in M, and so $L\Phi(G)/\Phi(G) = Q\Phi(G)/\Phi(G)$ is S-quasinormal in $M/\Phi(G)$. Hence $L\Phi(G)/\Phi(G)$ is normal in $M/\Phi(G) \cong GL(2, 3)$, a contradiction.

Since none of the simple groups mentioned in (2.6) can be isomorphic to $G/\Phi(G)$, it follows that G is solvable.

(2) $|\sigma(G)| = 2$.
Clearly, G is not of prime power order, because nilpotent groups are MS-groups. Suppose that $|\sigma(G)| \geq 3$. By the hypothesis, there exists a minimal subgroup H of order, say, p such that H is not S-quasinormal in G. We argue that H is subnormal in G. By the hypothesis, H is S-quasinormal in K whenever $H \leq K < G$. Then H is subnormal in K whenever $H \leq K < G$. If H is not subnormal in G, then H is contained in a unique maximal subgroup of G by (2.8). Since $|\sigma(G)| \geq 3$ and G is solvable by (1), it follows that H is contained in more than one maximal subgroup of G, a contradiction. Therefore, H is subnormal in G, so $H \leq O_p(G)$. Clearly, $HP = PH = P$ for each Sylow p-subgroup P of G. Since $|\sigma(G)| \geq 3$, it follows that $O_p(G)Q < G$ for each Sylow subgroup Q of G with $(|Q|, p) = 1$. By the hypothesis, H is S-quasinormal in $O_p(G)Q$, so HQ is a subgroup of G. Then H is S-quasinormal in G, a contradiction. Therefore, $|\sigma(G)| = 2$.

(3) G has a normal Sylow subgroup.
Suppose that G has no normal Sylow subgroup. By (1), G is solvable, so G has a normal subgroup M of prime index, say, q. Let P be a Sylow p-subgroup of G, and let Q be a Sylow q-subgroup of G, where p and q are distinct primes. Clearly, $P \leq M$. By the hypothesis, M is an MS-group. Then by (2.2)(d), $\Omega_1(P)$ is S-quasinormal in M, and hence $\Omega_1(P)$ is subnormal in M, and since M is normal in G, it follows that $\Omega_1(P)$ is subnormal in G. If $\Omega_1(P) = P$, then P is normal in G, a contradiction. Therefore, we may assume that $\Omega_1(P) < P$. Since $\Omega_1(P)$ is subnormal in G, we have $\Omega_1(P) \leq O_p(G) < P$. Then $\Omega_1(P) = \Omega_1(O_p(G))$ is normal in G, and so each minimal subgroup of G of order p is S-quasinormal in G and $C_G(\Omega_1(P))$ is normal in G. Hence, by the hypothesis, there exists a subgroup H of order q such that H is not S-quasinormal in G. Since $H\Omega_1(P) < G$ is an MS-group, it follows that $H\Omega_1(P)= H\times \Omega_1(P)$, so $H \leq C_G(\Omega_1(P))$. We treat with the following two cases:

Case 1. $C_G(\Omega_1(P)) < G$.
Then H is S-quasinormal in $C_G(\Omega_1(P))$, and hence H is subnormal in $C_G(\Omega_1(P))$, and since $C_G(\Omega_1(P))$ is normal in G, it follows that H is subnormal in G, so $H \leq O_q(G) < Q$. Now it follows easily that H is S-quasinormal in G, a contradiction.

Case 2. $\Omega_1(P) \leq Z(G)$.

Since Q is not normal in G, it follows that $p = 2$ by (2.4). Since P is a non-normal Sylow 2-subgroup of G and $\Omega_1(P) \leq Z(G)$, we have $\Omega_1(P) \leq O_2(G) < P$. By the hypothesis, $O_2(G)H$ is an MS-group, and so H is S-quasinormal in $O_2(G)H$. By (2.2)(b), H is subnormal in $O_2(G)H$, and since H is a Sylow q-subgroup of $O_2(G)H$, we have $H \leq C_G(O_2(G))$. If $C_G(O_2(G)) < G$, then H is subnormal in G. Hence $H \leq O_q(G) < Q$ for each Sylow q-subgroup Q of G. Clearly, $PO_q(G)$ is a proper subgroup of G for each Sylow p-subgroup P of G. By the hypothesis, H is S-quasinormal in $PO_q(G)$, so H permutes with P, and since H permutes with each q-subgroup Q of G we have that H is S-quasinormal in G, a contradiction. Therefore, $O_2(G) \leq Z(G)$, and so

$$\overline{1} < F(G/O_2(G)) = F(G)/O_2(G) = O_q(G)O_2(G)/O_2(G) \cong O_q(G).$$

Clearly, each minimal subgroup of $O_q(G)$ is S-quasinormal in G. Then each minimal subgroup of $F(G/O_2(G))$ is S-quasinormal in $G/O_2(G)$, and since $q > 2$, it follows that $G/O_2(G)$ is supersolvable by (2.9). Hence $G/O_2(G)$ possesses a Sylow tower of supersolvable type, and so $O_2(G)Q/O_2(G)$ is normal in $G/O_2(G)$. Then $O_2(G)Q$ is normal in G, and since $O_2(G) \leq Z(G)$, we have $O_2(G)Q = O_2(G) \times Q$. Hence Q is a characteristic in $O_2(G)Q$, and since $O_2(G)Q$ is normal in G, we have that Q is normal in G, a contradiction.

(4) G has a non-normal cyclic Sylow subgroup.

By (3), G has a normal Sylow subgroup P for some prime, say p. Let Q be any Sylow q-subgroup of G, where p and q are distinct primes. By (2), $G = PQ$. If Q is normal in G, then G is nilpotent, a contradiction. Therefore, Q is not normal in G. Suppose that Q is non-cyclic. Let H be any minimal subgroup of Q. Then $PH < G$, and so $PH = P \times H$. If $\Omega_1(Q) = Q$, then $Q \leq C_G(P)$, and so G is nilpotent, a contradiction. Therefore, we may assume that $\Omega_1(Q) < Q$. By (2.2)(d), $\Omega_1(Q)$ is S-quasinormal in $\Omega_1(Q)P$, so $\Omega_1(Q)P = \Omega_1(Q) \times P$. Then $\Omega_1(Q)$ is normal in G, and so each minimal subgroup of G of order q is S-quasinormal in G. Then, by the hypothesis, there exists a minimal subgroup L of P such that L is not S-quasinormal in G. Since Q is non-cyclic, it follows that Q contains two distinct maximal subgroups, say Q_1 and Q_2. Then $PQ_i < G$ and L is normal in LQ_i, where $i = 1, 2$. Hence LQ is a subgroup of G for each Sylow q-subgroup Q of G, and since $LP = PL = P$, it follows that L is S-quasinormal in G, a contradiction.

(5) Finishing the proof.

By (2), (3) and (4), $G = PQ$, where P is a normal Sylow p-subgroup of G and Q is a cyclic Sylow q-subgroup of G. Hence by Lemma 3.6 one of the statements (a),(b) and (c) of the theorem holds. □

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