WHICH GRAPHS HAVE ONLY SELF-CONVERSE ORIENTATIONS?

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An orientation of a graph G is an assignment of a unique direction to each line of G. The result is called an oriented graph. Two orientations of a graph are regarded as equivalent if the resulting oriented graphs are isomorphic as directed graphs. For example, the graph C_3 consisting of a cycle of length 3 (a triangle) shown in Figure 1(a), has exactly two orientations D_1 and D_2; see Figure 1(b) and (c). Both of these orientations are self-converse, i.e., the reversal of every direction results in the same directed graph.

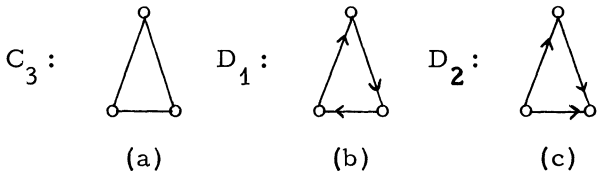

Figure 1. The two orientations of a triangle.

Not quite every graph has this property. For instance the graph P_3 consisting of a path having 3 points, shown in Figure 2(a), has the 3 orientations given in Figure 1(b), (c), (d). Only the first of these is self-converse, the other two being converses of each other.

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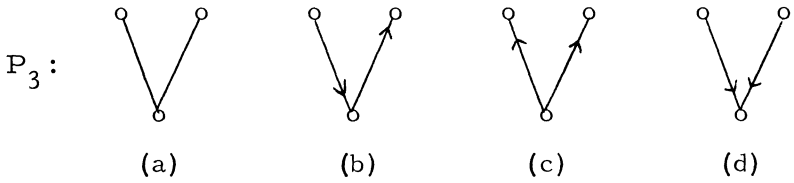

Figure 2. The three orientations of a three-point path.

A graph is called entirely self-converse if each of its orientations is self-converse. In this note, we determine all such graphs and find that there are just five of them. We also handle the same problem for multigraphs (in which more than one line may join a pair of points), where we see that there are an infinite family with this property, all having the same underlying graph. Finally, we observe conditions for general graphs (in which loops can occur as well as multiple lines) to be self-converse.

Each of the cycles on four and five points, denoted by C_4 and C_5 respectively, has exactly four orientations (see [3]). One quickly verifies that both of these graphs are entirely self-converse. For convenience, the complete graphs on one and two points are denoted by C_1 and C_2 respectively. We refer to [1, 2, 5] for graph theoretic notation and definitions not given here. In [4] we enumerate self-converse digraphs with a given number of points and lines.

THEOREM. The only connected graphs for which every orientation is self-converse are the "small cycles" C_1, C_2, C_3, C_4 and C_5.

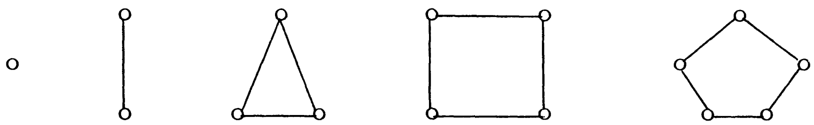

Figure 3. The five entirely self-converse graphs.

Proof. We show how to impose an orientation which is not self-converse on any graph G that is not one of the small cycles.

CASE 1. G is not regular.[1]

For each line $x = uv$ of G, either the degrees of u and v are equal or they are not. If they are equal, orient line uv arbitrarily; if not, orient this line from the point of higher degree to the other point. The orientation of G thus obtained is obviously not self-converse.

CASE 2. G is regular of degree $2r \geq 4$.

Since G is connected and has even degree, we can let G have the orientation induced by a directed eulerian walk; see [2]. Then each point has both indegree and outdegree equal to $r \geq 2$. Now choose any point v of G and reverse all of the arrows on lines coming from points adjacent to v. In the resulting orientation, G has a transmitter but no receivers, and hence this orientation is not self-converse.

CASE 3. G is regular of degree $2r - 1 \geq 5$.

First we form the join $C_1 + G$ by adding a new point to G and new lines joining it with every point of G. Since $C_1 + G$ is eulerian, there is an orientation of $C_1 + G$ as in Case 2, such that each point has both indegree and outdegree equal to $r \geq 3$. In the orientation thus induced on G itself, each point has both indegree and outdegree at least 2. Again by reversing the direction of all lines to a particular point, an orientation with a transmitter but no receiver is obtained.

CASE 4. G is cubic (regular of degree 3).

As in Case 3, we can construct an orientation on G such that each point has indegree 2 and outdegree 1 or vice versa. Let U be the collection of points with outdegree 2 and let W be the rest. Suppose u_1 and u_2 are points of U and u_1u_2 is a directed line. If the direction of this line is reversed, u_2 is a transmitter and there are no receivers in the resulting orientation of G. Similar considerations show that we may also assume

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[1] A graph is <u>regular</u> if all points have the same degree. The <u>degree</u> of a point is the number of lines incident with it.

that no two of the points in W are adjacent. Now choose any point v in W and suppose u_1v and u_2v are directed lines. By reversing the directions of these two lines, an orientation of G is again obtained which has a transmitter, namely v , but no receivers.

CASE 5. G is a cycle with more than 5 points.

We exhibit in Figure 4 an orientation of C_6 not isomorphic with its converse. The larger cycles are handled similarly.

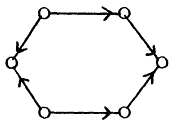

Figure 4. An orientation of C_6 which is not self-converse.

The corresponding result for multigraphs follows quickly from the theorem.

COROLLARY. The only connected multigraphs which are entirely self-converse are the graphs C_1, C_3, C_4 and C_5 and those multigraphs whose underlying graph is C_2 .

If a connected multigraph is entirely self-converse, its underlying graph must be C_1, C_2, C_3, C_4 or C_5 . All multiples of C_2 clearly have this property. The method in the theorem is easily used to construct an orientation which is not self-converse whenever there are multiple lines and the underlying graph is C_3, C_4 or C_5 . For example, a multigraph whose underlying graph is C_3 is shown in Figure 5 together with an orientation which is not self-converse.

Figure 5. An orientation of a multigraph which is not self-converse.

Entirely self-converse general graphs are all obtained from

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multigraphs with this property by simply adding exactly the same number of loops at each point. Two such general graphs are shown in Figure 6.

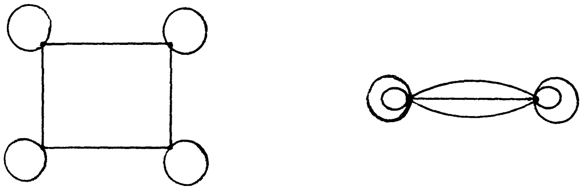

Figure 6. Two self-converse general graphs.

REFERENCES

1. F. Harary, A seminar on graph theory. New York, Holt Rinehart and Winston, 1967, pp. 1-41.

2. F. Harary, R. Norman and D. Cartwright, Structural models: an introduction to the theory of directed graphs. New York, Wiley, 1965.

3. F. Harary and E. Palmer, On the number of orientations of a given graph. Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. 14 (1966), pp. 125-128.

4. F. Harary and E. Palmer, Enumeration of self-converse digraphs. Mathematika 13 (1966), pp. 151-157.

5. D. Kőnig, Theorie der endlichen und unendlichen Graphen. Leipzig, 1936; reprinted New York, Chelsea, 1950.

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