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AV-Courant Algebroids and Generalized CR Structures

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Abstract. We construct a generalization of Courant algebroids that are classified by the third cohomology group $H^3(A,V)$, where A is a Lie Algebroid, and V is an A-module. We see that both Courant algebroids and $\mathcal{E}^1(M)$ structures are examples of them. Finally we introduce generalized CR structures on a manifold, which are a generalization of generalized complex structures, and show that every CR structure and contact structure is an example of a generalized CR structure.

1 Introduction

Courant algebroids and the Dirac structures associated with them were first introduced by Courant and Weinstein (see [6, 7]) to provide a unifying framework for studying such objects as Poisson and symplectic manifolds. Aïssa Wade later introduced the related $\mathcal{E}^1(M)$-Dirac structures in [27] to describe Jacobi structures.

In [13], Hitchin defined generalized complex structures that are further described by Gualtieri [12]. Generalized complex structures unify both symplectic and complex structures, interpolating between the two, and have appeared in the context of string theory [17]. In [14] Iglesias and Wade describe generalized contact structures, an odd-dimensional analog to generalized complex structures, using the language of $\mathcal{E}^1(M)$-Dirac structures.

In this paper, we shall define AV-Courant Algebroids, a generalization of Courant algebroids that also allows one to describe $\mathcal{E}^1(M)$-Dirac structures. We will show that these have a classification similar to Ševera's classification of exact Courant algebroids in [24].

To be more explicit, let M be a smooth manifold, $A \to M$ be a Lie algebroid with anchor map $a\colon A \to TM$, and $V \to M$ a vector bundle that is an A-module. If we endow V with the structure of a trivial Lie algebroid (that is, trivial bracket and anchor), then it is well known that the extensions of A by V are a geometric realization of $H^2(A,V)$ (see [18]). In this paper, we introduce AV-Courant algebroids and describe how they are a geometric realization of $H^3(A,V)$.

We then go on to show how to simplify the structure of certain AV-Courant algebroids by pulling them back to certain principal bundles. Indeed, in the most interesting cases, the pullbacks will simply be exact Courant algebroids.

We then introduce AV-Dirac structures, a special class of subbundles of an AV-Courant algebroid which generalize Dirac structures. Finally, we will introduce a special class of AV-Dirac structures, called generalized CR structures, which allow

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us to describe any complex, symplectic, CR or contact structure on a manifold, as well as many interpolations of those structures. We show that associated with every generalized CR structure is a Jacobi bundle, introduced by Charles-Michel Marle [21] and Kirillov [16].

It is important to note that there are other constructions related to AV-Courant algebroids. For instance, recently Z. Chen, Z. Liu, and Y.-H. Sheng introduced the notion of E-Courant algebroids [5] in order to unify the concepts of omni-Lie algebroids (introduced in [3], see also [4]) and generalized Courant algebroids or Courant–Jacobi algebroids (introduced in [23] and [10] respectively; they are equivalent concepts; see [23]). The key property that both E-Courant algebroids and AV-Courant algebroids share is that they replace the $\mathbb{R}$-valued bilinear form of Courant algebroids with one taking values in an arbitrary vector bundle (E or V respectively). Nevertheless, while there is some overlap between E-Courant algebroids and AV-Courant algebroids in terms of examples, these constructions are not equivalent; indeed, AV-Courant algebroids are classified by $H^3(A,V)$, while there is no simple classification of E-Courant algebroids. Moreover, this paper is distinguished from [5] by having the definition of generalized CR manifolds as one of its main goals.

Meanwhile, generalized CRF structures, introduced and studied in great detail by Izu Vaisman in [26], and generalized CR structures describe similar objects. To summarize, a complex structure on a manifold M is a subbundle $H \subset TM \otimes \mathbb{C}$ such that

$$H \oplus \bar{H} = TM \otimes \mathbb{C} \tag{1.1}$$

and $[H,H] \subset H$. The definition of a CR structure simply relaxes (1.1) to $H \cap \bar{H} = 0$. On the other hand, the definition of a generalized complex structure replaces TM with the standard Courant algebroid $\mathbb{T}M = T^*M \oplus TM$ in the definition of a complex structure, and in addition, requires $H \subset \mathbb{T}M \otimes \mathbb{C}$ to be isotropic.

The definition of a generalized CRF structure parallels the definition of a generalized complex structure, but relaxes the requirement that $H \oplus \bar{H} = \mathbb{T}M \otimes \mathbb{C}$ to $H \cap \bar{H} = 0$. Among numerous interesting examples of generalized CRF structures are normal contact structures and normalized CR structures (namely those CR structures $H \subset TM \otimes \mathbb{C}$ for which there is a splitting $TM \otimes \mathbb{C} = H \oplus \bar{H} \oplus Q_c$ and $[H, Q_c] \subset H \oplus Q_c$).

Generalized CR structures differ from generalized CRF structures in multiple ways. In particular, they replace the standard Courant algebroid with an AV-Courant algebroid $\mathbb{A}$, and furthermore, they take a different approach to describe contact and CR structures, using only maximal isotropic subbundles but allowing $H \cap \bar{H}$ to contain "infinitesimal" elements.

2 *AV*-Courant Algebroids

Let M be a smooth manifold, $A \to M$ a Lie algebroid, and $V \to M$ a vector bundle that is an A-module, that is, there is a $C^\infty(M)$-linear Lie algebra homomorphism

$$\mathcal{L}_\cdot : \Gamma(A) \to \mathrm{End}(\Gamma(V)) \tag{2.1}$$

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satisfying the Leibniz rule. (See [18] for more details.)

For any A-module V, the sections of $V \otimes \wedge^* A^*$ have the structure of a graded right $\wedge^*\Gamma(A^*)$-module, and there are several important derivations of its module structure that we shall use throughout this paper. The first is the interior product with a section $X \in \Gamma(A)$,

$$\iota_X \colon \Gamma(V \otimes \wedge^i A^*) \to \Gamma(V \otimes \wedge^{i-1} A^*),$$

a derivation of degree -1.

The second is the Lie derivative, a derivation of degree 0, defined to be the unique derivation of $V \otimes \wedge^* A^*$ whose restriction to V is given by (2.1), and such that the graded commutator with $\iota_\cdot$ satisfies $[\mathcal{L}_X, \iota_Y] = \iota_{[X,Y]}$. Finally, the differential d, a derivation of degree 1, is defined inductively by the graded commutator $\mathcal{L}_X = [d, \iota_X]$ (for all $X \in \Gamma(A)$).

It is easy to check that $d^2 = 0$, and the cohomology groups of the complex $(\Gamma(V \otimes \wedge^\bullet A^*), d)$ are denoted $H^\bullet(A, V)$.

2.1 Definition of AV-Courant Algebroids

Let A be a Lie algebroid and V an A-module.

Definition 2.1 (AV-Courant Algebroid) Let $\mathbb{A}$ be a vector bundle over M, with a V-valued symmetric bilinear form $\langle \cdot, \cdot \rangle$ on the fibres of $\mathbb{A}$, and a bracket $[\![\cdot, \cdot]\!]$ on sections of $\mathbb{A}$. Suppose further that there is a short exact sequence of bundle maps

$$0 \to V \otimes A^* \xrightarrow{j} \mathbb{A} \xrightarrow{\pi} A \to 0 \tag{2.2}$$

such that for any $e \in \Gamma(\mathbb{A})$ and $\xi \in \Gamma(V \otimes A^*)$,

$$\langle e, j(\xi) \rangle = \iota_{\pi(e)} \xi. \tag{2.3}$$

The bundle $\mathbb{A}$ with these structures is called an AV-Courant algebroid if, for $f \in C^\infty(M)$ and $e, e_i \in \Gamma(\mathbb{A})$, the following axioms are satisfied:

(AV-1) $[\![e_1, [\![e_2, e_3]\!]]\!] = [\![[\![e_1, e_2]\!], e_3]\!] + [\![e_2, [\![e_1, e_3]\!]]\!]$

(AV-2) $\pi([\![e_1, e_2]\!]) = [\pi(e_1), \pi(e_2)]$

(AV-3) $[\![e, e]\!] = \frac{1}{2} D\langle e, e \rangle$, where $D = j \circ d$

(AV-4) $\mathcal{L}_{\pi(e_1)} \langle e_2, e_3 \rangle = \langle [\![e_1, e_2]\!], e_3 \rangle + \langle e_2, [\![e_1, e_3]\!] \rangle$

we will often refer to $[\![\cdot, \cdot]\!]$ as the Courant bracket.

Remark 2.2 Axioms (AV-1) and (AV-4) state that $[\![e, \cdot]\!]$ is a derivation of both the Courant bracket and the bilinear form, while Axiom (AV-2) describes the relation of the Courant bracket to the Lie algebroid bracket of A. One should interpret Axiom (AV-3) as saying that the failure of $[\![\cdot, \cdot]\!]$ to be skew symmetric is only an "infinitesimal" $D(\cdot)$.

Remark 2.3 The bracket is also derivation of $\mathbb{A}$ as a $C^\infty(M)$-module in the sense that

$$[\![e_1, fe_2]\!] = f[\![e_1, e_2]\!] + a \circ \pi(e_1)(f) \cdot e_2$$

for any $e_1, e_2 \in \Gamma(\mathbb{A})$ and $f \in C^\infty(M)$. In fact if $e_3 \in \Gamma(\mathbb{A})$,

$$\begin{aligned}
&\langle a \circ \pi(e_1)(f) \cdot e_2 + f[\![e_1, e_2]\!] - [\![e_1, fe_2]\!], e_3\rangle \\
&\quad \text{(by (AV-4))} \;= \langle a \circ \pi(e_1)(f) \cdot e_2 + f[\![e_1, e_2]\!], e_3\rangle - \pi(e_1)\langle fe_2, e_3\rangle \\
&\qquad\qquad\qquad + \langle fe_2, [\![e_1, e_3]\!]\rangle \\
&\qquad\qquad\quad\;= a \circ \pi(e_1)(f)\langle e_2, e_3\rangle - \pi(e_1)\langle fe_2, e_3\rangle \\
&\qquad\qquad\qquad + f(\langle [\![e_1, e_2]\!], e_3\rangle + \langle e_2, [\![e_1, e_3]\!]\rangle) \\
&\quad \text{(by (AV-4))} \;= a \circ \pi(e_1)(f)\langle e_2, e_3\rangle - \pi(e_1)\langle fe_2, e_3\rangle + f\pi(e_1)\langle e_2, e_3\rangle \\
&\qquad\qquad\quad\;= 0,
\end{aligned}$$

where the last equality follows from the fact that V is an A module. Since this holds for all $e_3 \in \Gamma(\mathbb{A})$, and $\langle \cdot\,, \cdot \rangle$ is non-degenerate, the statement follows.

Remark 2.4 One notices that (2.3) and exactness of (2.2) implies that the map $\mathbb{A} \to V \otimes \mathbb{A}^*$, given by $e \to \langle e, \cdot \rangle$, is an injection. Consequently, if V is a line bundle (as in all the known interesting examples), it follows that $\mathbb{A} \simeq V \otimes \mathbb{A}^*$, and j must be the composition

$$j\colon V \otimes A^* \xrightarrow{\mathrm{id}\otimes\pi^*} V \otimes \mathbb{A}^* \simeq \mathbb{A}.$$

Remark 2.5 Any $\mathfrak{D}E$ E-Courant algebroid (an AV-Courant algebroid with $V = E$ and $A = \mathfrak{D}E$, the gauge Lie algebroid of E) is an E-Courant algebroid. However, not every E-Courant algebroid is a $\mathfrak{D}EE$-Courant algebroid, since there is no requirement in the definition of E-Courant algebroids for the sequence (2.2) in Definition 2.1 of AV-Courant algebroids to be exact, and the map (2.2) $j\colon E \otimes (\mathfrak{D}E)^* \to \mathbb{A}$ is only defined on the first jet bundle $\mathfrak{J}^1E \subset E \otimes (\mathfrak{D}E)^*$.

One could imagine some generalization of both AV-Courant algebroids and E-Courant algebroids that ignores the requirement that (2.2) be exact in the above definition (and perhaps allows j to be defined on a smaller domain).

Conversely, if $\mathbb{A}$ is an AV-Courant algebroid, then there is a natural Lie algebroid morphism $\phi\colon A \to \mathfrak{D}V$ resulting from the fact that V is an A-module. Consequently, $(\mathbb{A}, \langle \cdot\,, \cdot \rangle, [\![\cdot\,, \cdot]\!], \phi \circ \pi)$ is an E-Courant algebroid (with $E = V$). So an AV-Courant algebroid can be thought of as an E-Courant algebroid with some additional structure, such as a exact sequence (2.2) and a factorization of the anchor map through a Lie algebroid A. This additional structure allows for a more comprehensive understanding of AV-Courant algebroids, including a simple classification of AV-Courant algebroids by $H^3(A, V)$, and when A is a transitive Lie algebroid, a means of understanding both AV-Courant algebroids and AV-Dirac structures by relating them to standard Courant algebroids and Dirac structures on principal bundles.

2.2 Splitting

We call $\phi\colon A \to \mathbb{A}$ an isotropic splitting, if it splits the exact sequence (2.2), and $\phi(A)$ is an isotropic subspace of $\mathbb{A}$ with respect to the inner product.

Remark 2.6 Such splittings exist. In fact we may choose a splitting $\lambda\colon A \to \mathbb{A}$, which is not necessarily isotropic.

Then we have a map $\gamma\colon A \to V \otimes A^*$ given by the composition

$$\gamma\colon A \xrightarrow{\lambda} \mathbb{A} \xrightarrow{e \to \langle e,\cdot\rangle} V \otimes \mathbb{A}^* \xrightarrow{\mathrm{id}\otimes\lambda^*} V \otimes A^*.$$

We let $\phi = \lambda - \frac{1}{2} j \circ \gamma$. It is easy to check that ϕ is an isotropic splitting.

If $\phi\colon A \to \mathbb{A}$ is an isotropic splitting, then we have an isomorphism $\phi \oplus j\colon A \oplus (V \otimes A^*) \to \mathbb{A}$.

Proposition 2.7 *Let $\phi\colon A \to \mathbb{A}$ be an isotropic splitting. Then under the above isomorphism, the bracket on $A \oplus (V \otimes A^*)$ is given by*

$$(2.4)\qquad [\![X + \xi, Y + \eta]\!]_\phi = [X, Y] + \mathcal{L}_X \eta - \iota_Y d\xi + \iota_X \iota_Y H_\phi,$$

where $X, Y \in \Gamma(A)$, $\xi, \eta \in \Gamma(V \otimes A^*)$ and $H_\phi \in \Gamma(V \otimes \wedge^3 A^*)$, with $dH_\phi = 0$.

Furthermore, if $\psi\colon A \to \mathbb{A}$ is a different choice of isotropic splitting, then $\psi(X) = \phi(X) + j(\iota_X \beta)$ and $H_\psi = H_\phi - d\beta$, where $\beta \in \Gamma(V \otimes \wedge^2 A^*)$.

The proof is relegated to the appendix, since it is parallel to the proof for ordinary Courant algebroids (see [2, 24]).

Theorem 2.8 *Let A be a Lie algebroid, and let V be an A-module. Then the isomorphism classes of AV-Courant algebroids are in bijective correspondence with $H^3(A, V)$.*

Proof If $H \in \Gamma(V \otimes \wedge^3 A^*)$, and $dH = 0$, then let $\mathbb{A} = A \oplus (V \otimes A^*)$. We define $\langle\cdot\,,\cdot\rangle$ by

$$(2.5)\qquad \langle X + \xi, Y + \eta\rangle = \iota_X \eta + \iota_Y \xi,$$

where $\xi, \eta \in \Gamma(V \otimes A^*)$ and $X, Y \in \Gamma(A)$. We define the bracket to be given by equation (2.4). It is not difficult to check that this satisfies the axioms of an AV-Courant algebroid.

Conversely, by the above proposition, every AV-Courant algebroid defines a unique element of $H^3(A, V)$. ∎

3 Examples

Example 3.1 Let M be a point, then a Lie algebroid A is simply a Lie algebra, and an A-module V is a finite dimensional representation of A as a Lie algebra. $H^i(A, V)$ is simply the V-valued Lie algebra cohomology, and $H^3(A, V)$ classifies the AV-Courant algebroids over a point. Note that an AV-Courant algebroid over a point is a Lie algebra if and only if V is a trivial A-representation.

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Example 3.2 (Exact Courant Algebroids) If we let $A \simeq TM$ and $V = M \times \mathbb{R}$ be the trivial line bundle over M with a trivial TM-module structure, then we may identify T^*M with $V \otimes T^*M$ by the map $\alpha \to 1 \otimes \alpha$. It follows that the class of $TM\,\mathbb{R}$-Courant algebroids over M corresponds to the class of exact Courant algebroids (see [6,7]) on M,

$$0 \longrightarrow T^*M \xrightarrow{\pi^*} \mathbb{A} \xrightarrow{\pi} TM \longrightarrow 0.$$

Theorem 2.8 then corresponds to Ševera's classification of exact Courant algebroids.

Example 3.3 ($\mathcal{E}^1(M)$ Structures) The bundle $\mathcal{E}^1(M)$ was introduced by A. Wade in [27] and is uniquely associated with a given manifold M. Within the context of this paper, it is easiest to define $\mathcal{E}^1(M)$ by using the language of AV-Courant algebroids.

We let $A = TM \oplus L$, where $L \simeq \mathbb{R}$ is spanned by the abstract symbol $\frac{\partial}{\partial t}$. The bracket is given by

$$\Big[X \oplus f\tfrac{\partial}{\partial t}, Y \oplus g\tfrac{\partial}{\partial t}\Big]_A = [X,Y]_{TM} \oplus (X(g) - Y(f))\tfrac{\partial}{\partial t},$$

where $X, Y \in \mathfrak{X}(M)$ and $f, g \in C^\infty(M)$.

Let V be the trivial line bundle spanned by the abstract symbol e^t, so that $\Gamma(V) = \{e^t h | h \in C^\infty(M)\}$. V has an A-module structure (as suggested by the choice of symbols) given by

$$\big(X \oplus f\tfrac{\partial}{\partial t}\big)(e^t h) = e^t(X(h) + fh).$$

We let $\mathbb{A} := (TM \oplus L) \oplus (V \otimes (T^*M \oplus L^*))$, and define a bracket on sections by equation (2.4). It is clear that this data defines an AV-Courant algebroid on M. If we set $H = 0$ in equation (2.4), then the pair $(\mathbb{A}, [\![\cdot,\cdot]\!])$ associated with M is the $\mathcal{E}^1(M)$-Structure, as introduced by Wade in [27].

Example 3.4 (Equivariant AV-Courant Algebroids on Principal Bundles) Let $\nu\colon P \to M$ be a G-principal bundle. Suppose that A is a Lie algebroid over P and V is an A-module, and that there is an AV-Courant algebroid on P,

$$0 \to V \otimes A^* \to \mathbb{A} \to A \to 0.$$

If the action of G on P lifts to an action by bundle maps on V, A and $\mathbb{A}$, such that all the structures involved are G-equivariant, then the quotient,

$$0 \to (V \otimes A^*)/G \to \mathbb{A}/G \to A/G \to 0,$$

is an $A/G\ V/G$-Courant algebroid.

Example 3.5 Let $\nu\colon P \to M$ be a G-principal bundle, and let W be a k-dimensional vector space possessing a linear action of G. We regard W as a trivial bundle over P, and we consider the bundle $\mathbb{T} := TP \oplus (W \otimes T^*P)$, endowed with a W-valued symmetric bilinear form given by equation (2.5). We also define a bracket on sections of $\mathbb{T}$ by equation (2.4), where $H \in \Omega^3(P, W)^G$ is closed, then

$$0 \to W \otimes T^*P \xrightarrow{j} \mathbb{T} \xrightarrow{\pi} TP \to 0$$

is an equivariant $TP\,W$-Courant algebroid on P (where j and π are the obvious inclusion and projection). Thus (as in Example 3.4), we have an AV-Courant algebroid on P/G, where $A = TP/G$ is the Atiyah algebroid, and $V = P \times_G W$.

Note, if W is 1-dimensional, then the $TP\,W$-Courant algebroid given above is simply an exact Courant algebroid.

As it turns out, this is quite a general example. Indeed, if A is a transitive Lie algebroid, then locally all AV-Courant algebroids result from such a construction (see Section 5).

Remark 3.6 In the above example, one could replace $P \times W$ with any flat bundle.

Example 3.7 As a special case of Example 3.5, if we take $G = \mathbb{R}$, then $P = M \times \mathbb{R}$ is a $\mathbb{R}$-principal bundle where the action is translation. We let $W = P \times \mathbb{R}$ be the trivial line bundle over P and let $\lambda \in \mathbb{R}$ act on W by scaling by $e^{-\lambda}$.

To describe the G-action explicitly,

$$\lambda \cdot ((x, s), t) = ((x, s + \lambda), e^{-\lambda} t),$$

where $\lambda \in \mathbb{R}$, $(x, s) \in M \times \mathbb{R} = P$, and $((x, s), t) \in P \times \mathbb{R} = W$.

The quotient of the $TP\,W$-Courant algebroid on P with $H = 0$ under this action is precisely the $\mathcal{E}^1(M)$-Structure on $M = P/\mathbb{R}$.

Example 3.8 If A is a Lie algebroid over M, V is an A-module, and $\mathbb{A}$ is an AV-Courant algebroid on the manifold M, and if $F \subset M$ is a leaf of the singular foliation defined by $a(A)$, then $i^*\mathbb{A}$ is an $i^*A\, i^*V$-Courant algebroid on F, where $i \colon F \to M$ is the inclusion.

Remark 3.9 At this point, in the most interesting examples of AV-Courant algebroids, V is a line bundle. Nevertheless, as mentioned in Theorem 2.8, for any Lie algebroid $A \to M$, any A-module V over M, and any element $\gamma \in H^3(A, V)$, there is an AV-Courant algebroid (unique up to isomorphism) classified by γ. It is not yet known if these examples are of any importance.

4 AV-Dirac Structures

Definition 4.1 (AV-Dirac Structure) Let M be a manifold, let $A \to M$ be a Lie algebroid over M, let $V \to M$ be an A-module, and let $\mathbb{A}$ be an AV-Courant algebroid. Suppose that $L \subset \mathbb{A}$ is a subbundle, since $\mathbb{A}$ has a non-degenerate inner product, we can define $L^\perp = \{v \in \mathbb{A} \mid \langle v, u\rangle = 0\ \forall u \in L\}$.

We call L an *almost AV-Dirac structure* if $L^\perp = L$. An *AV-Dirac* structure is an almost AV-Dirac structure, $L \subset \mathbb{A}$ that is involutive with respect to the bracket $[\![\cdot\,,\,\cdot]\!]$.

Remark 4.2 If $L \subset \mathbb{A}$ is an AV-Dirac structure, then $[\![e, e]\!] = \frac{1}{2}D\langle e, e\rangle = 0$ for any section $e \in \Gamma(L)$, so $[\![\cdot\,,\,\cdot]\!]$ is skew-symmetric when restricted to L, and then by the other properties of the bracket, it follows that $a \circ \pi \colon L \to TM$ is a Lie algebroid, and $\pi \colon L \to A$ is a Lie algebroid morphism.

Example 4.3 (Invariant Dirac Structure on a Principal Bundle) Using the notation of Example 3.4, suppose that the A/G V/G-Courant algebroid $\mathbb{A}/G$ on M is the quotient of a AV-Courant algebroid $\mathbb{A}$ on P. If $L \subset \mathbb{A}$ is an AV-Dirac structure which is G invariant, then it is clear that $L/G \subset \mathbb{A}/G$ is an A/G V/G-Dirac structure (see Example 3.4).

Example 4.4 ($\mathcal{E}^1(M)$-Dirac Structures) Using Example 3.3, we can describe $\mathcal{E}^1(M)$, the bundle introduced by Wade in [27], as an AV-Courant algebroid. In this context, the $\mathcal{E}^1(M)$-Dirac structures (also introduced by Wade in [27]) correspond directly to the AV-Dirac structures.

5 Transitive Lie Algebroids

5.1 Simplifying *AV*-Courant Algebroids

Suppose that A is a Lie algebroid, V is an A-module, and $\mathbb{A}$ is an AV-Courant algebroid over M (where we use the notation given in the definition of AV-Courant algebroids). We will assume for the duration of this section that M is connected, and we require that A be a transitive Lie algebroid, namely the anchor map $a\colon A \to TM$ is surjective (see [18] for more details).

Since $\mathbb{A}$ may be quite complicated, we wish to examine whether this AV-Courant algebroid is the quotient of a much simpler $A'V'$-Courant algebroid on a principal bundle over M, where A' is a very simple Lie algebroid and V' is a very simple A'-module. To be more explicit, we wish to examine whether $\mathbb{A}$ results from the construction in Example 3.5. For this to be true, it is clearly necessary that A be the Atiyah algebroid of that principal bundle; namely, if P is the principal bundle, then $A = TP/G$. The existence of such a principal bundle is equivalent to the integrability of A as a Lie algebroid:

Proposition 5.1 *Suppose that $A \to M$ is an integrable transitive Lie algebroid, that is to say, there exists a source-simply connected Lie groupoid*

$$\Gamma \underset{t}{\overset{s}{\rightrightarrows}} M$$

with Lie algebroid A (see [18] for more details). Then A is the Atiyah algebroid of a principal bundle.

Conversely, if A is the Atiyah algebroid of a principal bundle, then A is an integrable Lie algebroid.

Proof Suppose first that A is integrable, then using the notation in the statement of the proposition, where $s\colon \Gamma \to M$ is the source map and $t\colon \Gamma \to M$ is the target map, let $x \in M$, let $P = \Gamma_x := s^{-1}(x)$, and let $G = \Gamma_x^x := s^{-1}(x) \cap t^{-1}(x)$.

Since A is transitive, $t\colon P \to M$ is a surjective submersion. For clarity, we define $p := t|_P$. Furthermore, if $y \in M$, and $g \in \Gamma_x^y$, then $g\colon p^{-1}(x) \to p^{-1}(y)$ is a diffeomorphism, so $p\colon P \to M$ is a fibre bundle, with its fibre diffeomorphic to G. In addition, G has a right action on P, given by right multiplication in the Lie groupoid. If $p^{-1}(y) = \Gamma_x^y$ is a fibre, and $g \in \Gamma_x^y$, then the diffeomorphism $g\colon p^{-1}(x) \to p^{-1}(y)$ is given by left groupoid multiplication while the action of G on P is given by right groupoid multiplication, so it is clear that the two operations commute, from which it follows that G preserves the fibres of P, acting transitively and freely on them. Thus P is a principal G bundle.

Since A is the Lie algebroid of Γ, it can be identified with the right invariant vector-fields on Γ tangent to the source fibres. However, since A is transitive, any two source fibres are diffeomorphic by right multiplication by some element. Thus A can be identified with the G invariant vector fields on P.

Conversely, if A is the Atiyah algebroid of some principal bundle, it obviously integrates to the gauge groupoid associated with that principal bundle (see [9] or Remark 5.5), and we may take Γ to be the source-simply connected cover of the gauge groupoid. ■

We now examine whether V is an associated vector bundle.

Proposition 5.2 *Suppose that A is an integrable transitive Lie algebroid, and $V \to M$ is an A-module. Then there exists a (possibly disconnected) Lie group G and a simply connected principal G-bundle $P \to M$ such that V is the quotient bundle of $P \times \mathbb{R}^k$, for some G action on $\mathbb{R}^k$. In this setting, the standard action of $\mathfrak{X}(P)$ on $C^\infty(P, \mathbb{R}^k)$ induces the module structure on V.*

Proof Using the notation and the Lie groupoid described in the previous proposition, we consider $\Gamma_x \times V_x$, where V_x is the fibre of V at x. We may assume that Γ is source-simply connected, and, consequently, since V is an A-module, by Lie's second theorem there exists a Lie groupoid morphism $\Gamma \to \mathbf{GL}(V)$.[1] Thus Γ acts on V, and we have a map $\tilde{p}\colon \Gamma_x \times V_x \to V$ given by $(g, v) \to gv$. This is clearly a surjective submersion.[2] Furthermore,

$$\tilde{p}(g, v) = \tilde{p}(g', v') \Leftrightarrow g^{-1}g' \in \Gamma_x^x \quad \text{and} \quad v = (g^{-1}g')v'.$$

Thus, letting $G = \Gamma_x^x$ and $P = \Gamma_x$, we have $V \simeq (\Gamma_x \times V_x)/G \simeq (P \times V_x)/G$.

Furthermore, identifying V_x with $\mathbb{R}^k$, if $X \in \mathfrak{X}(P) \simeq \mathfrak{X}(\Gamma_x)$, and $\sigma \in C^\infty(P, \mathbb{R}^k)$, then the standard action of X on σ is given by $X(\sigma)_z = \frac{\partial}{\partial t}|_{t=0}\sigma(e^{tX}z)$ for any $z \in P \simeq \Gamma_x$. If we suppose that X and σ are G invariant, then

$$\tilde{p}\Big(\tfrac{\partial}{\partial t}|_{t=0}\sigma\big(e^{tX}(z)\big)\Big) = \tfrac{\partial}{\partial t}|_{t=0}\big(e^{-tX}\tilde{p}(\sigma)\big)_{p(z)} = (\mathcal{L}_X\tilde{p}(\sigma))_{p(z)},$$

since we defined the action of Γ on V in terms of the A-module structure of V. ■

Proposition 5.3 *Suppose that A is an integrable Lie algebroid, and $V \to M$ is an A-module. Then $\mathbb{A}$ results from the construction given in Example* 3.5*. Namely, there*

[1] See, for instance, [8, 19, 22] for more details. Here $\mathbf{GL}(V)$ is the Lie groupoid of linear isomorphisms of the fibres of V, namely $\mathbf{GL}(V)_x^y = \mathrm{Hom}(V_x, V_y)$.

[2] Since A is transitive and M is connected, $t\colon \Gamma_x \to M$ is a surjective submersion. Let $y \in M$, and let $\sigma\colon U \to \Gamma_x$ be a section (so that $t \circ \sigma = \mathrm{id}$). Then $(z, v) \to \sigma(z)(v)\colon U \times V_x \to V_U$ is a diffeomorphism.

exist a Lie group G and a principal G-bundle $P \to M$ such that $\mathbb{A}$ is the quotient of a $TP\,\mathbb{R}^k$-Courant algebroid Furthermore, if $L \subset \mathbb{A}$ is an AV-Dirac structure, then it is also the quotient of a corresponding $TP\,\mathbb{R}^k$-Dirac structure on P.

Consequently, if V is a line-bundle, then $\mathbb{A}$ is simply the quotient of an exact Courant algebroid on P.

Proof We choose some isotropic splitting of $\mathbb{A}$, so that $\mathbb{A} \simeq A \oplus (V \otimes A^*)$. The bracket is given by equation (2.4), and the symmetric bilinear form by equation (2.5). Then we can use the previous propositions to lift the right-hand side to a principal bundle.

By the above propositions, there exist a (possibly disconnected) Lie group G and a simply connected G-principal bundle, $\nu \colon P \to M$, such that $A \simeq TP/G$. In addition to this there is a G-action on $W := \mathbb{R}^{\dim(V)}$, say $\lambda \colon G \to \mathbf{GL}(W)$, such that $V = P \times_G W$. In this setting, $\Gamma(V \otimes \wedge^i A^*) \simeq \Omega^i(P, W)^G$, and $d \colon \Gamma(V \otimes \wedge^i A^*) \to \Gamma(V \otimes \wedge^{i+1} A^*)$ is the restriction of the exterior derivative d to $\Omega^*(P, W)^G$.

Thus since $H \in \Gamma(V \otimes \wedge^3 A^*) \simeq \Omega^3(P, W)^G$, it is clear that we may view H as a G-invariant element of $\Omega^3(P, W)$ and define the $TP\,W$-Courant algebroid $W \otimes T^*P \to \mathbb{T} \to TP$ in terms of it: Namely, $\mathbb{T} \simeq TP \oplus (W \otimes T^*P)$ endowed with a W-valued symmetric bilinear form given by equation (2.5), and the bracket given by equation (2.4). (See Example 3.5 for more details on this construction.)

It is clear that $\mathbb{A}$ is the quotient of this $TP\,W$-Courant algebroid.

Equivalently, it is easy to see that $TP = \nu^* A$, $W = \nu^* V$, and $\mathbb{T} = \nu^* \mathbb{A}$. The W-valued symmetric bilinear form on $\mathbb{T}$ is simply the pullback of the V-valued symmetric bilinear form on $\mathbb{A}$, and if $e_1, e_2 \in \Gamma(\mathbb{A})$, then $[\![\nu^* e_1, \nu^* e_2]\!] = \nu^* [\![e_1, e_2]\!]$, and the bracket on $\mathbb{T}$ is then extended to arbitrary sections of $\mathbb{T}$ by Axiom (AV-3) and Remark 2.2.

Next, let $\tilde{L} = \nu^*(L) \subset \mathbb{T}$. It is obvious that $L^\perp = L \Rightarrow \tilde{L}^\perp = \tilde{L}$, and, similarly, since L is involutive, so is $\tilde{L}$.

Thus $\tilde{L} \subset \mathbb{T}$ is a $TP\,W$-Dirac structure, and $\tilde{L}/G = L$. ■

Example 5.4 If $A = TM$ and V is a flat vector bundle over M, then following the proof of Proposition 5.3 we see that $G = \pi_1(M)$ is the fundamental group, and $P = \tilde{M}$ is the simply connected covering space of M over which the pullback of V is a trivial vector bundle.

Remark 5.5 The above propositions construct the principal bundle P and the Lie group G. Suppose however, that we already have a Lie group G' and a connected G'-principal bundle $\nu' \colon P' \to M$ such that $A \simeq TP'/G'$. It will not be difficult to see that $\mathbb{A}$ is the quotient of a AV-Courant algebroid on P'.

Let $\mathcal{G} = (P' \times P')/G'$, where we take the quotient by the diagonal action. Then $\mathcal{G} \underset{t}{\overset{s}{\rightrightarrows}} M$ is a Lie groupoid with Lie algebroid A, where the source map is $s \colon [u, v] \to \nu'(v)$, the target map is $t \colon [u, v] \to \nu'(u)$, and the multiplication is $[u, v] \cdot [v, w] = [u, w]$.[3] Hence by Lie's second theorem (see [8, 19, 22] for more details), since Γ, the Lie groupoid used in the proof of Proposition 5.3, is source-simply connected, there

[3] An element of $\mathcal{G}$ is an equivalence class, which we may view as a subset of $\nu'^{-1}(y) \times \nu'^{-1}(z)$ that is G invariant. As such, we may view it as the graph of an equivariant diffeomorphism $\nu'^{-1}(y) \to \nu'^{-1}(z)$. The multiplication in $\mathcal{G}$ is simply the composition of these diffeomorphisms. See [9] for details.

is a unique Lie groupoid morphism $\Phi\colon \Gamma \to \mathcal{G}$ that restricts to the identity map on the Lie algebroid A.

It follows that $\Phi|_P\colon P \to P'$ is a covering map,[4] and $\Phi|_G\colon G \to G'$ is a covering morphism of Lie groups.[5] It is easy to see that $H = \ker(\Phi|_G) \simeq \pi(P')$ and $P' = P/H$.

Thus, we may take the quotient of the $\mathbb{T}P\,W$-Courant algebroid on P (constructed in Proposition 5.3) by H, to form a $\mathbb{T}P'\ W/H$-Courant algebroid on P' whose quotient by G' is $\mathbb{A}$. It is important to note that while W is a trivial vector bundle, W/H is a flat vector bundle.

Remark 5.6 Proposition 5.3 was observed for $\mathcal{E}^1(M)$ structures in [15].

Corollary 5.7 *Suppose that V is an A-module, and M is contractible, then $\mathbb{A}$ is the quotient of a $\mathbb{T}P$ $\mathbb{R}^k$-Courant algebroid $\mathbb{R}^k \otimes T^*P \to \mathbb{T} \to TP$ on some principal G-bundle, P. (See Example* 3.4*). Furthermore, if $L \subset \mathbb{A}$ is an AV-Dirac structure, then it is also the quotient of a $\mathbb{T}P$ $\mathbb{R}^k$-Dirac structure $\tilde{L} \subset \mathbb{T}$.*

Proof Every transitive Lie algebroid is integrable over a contractible space; see [18] for details. ■

5.2 Contact Manifolds

Iglesias and Wade show how to describe contact manifolds as $\mathcal{E}^1(M)$-Dirac structures in [14]. Thus in light of Example 4.4, we can describe them as AV-Dirac structures. We will now describe this same construction from a more geometric perspective, similar to their description in [15].

To simplify things, we assume that (M, ξ) is a co-oriented contact manifold, namely $\xi \subset TM$ can be given as the kernel of a nowhere vanishing 1-form $\alpha \in \Omega^1(M)$, and we use the fact that there is a one-to-one correspondence between co-oriented contact manifolds and symplectic cones (see [1]). Recall, as in [1], that a symplectic manifold (N, ω_N) is a symplectic cone if

- N is a principal $\mathbb{R}$ bundle over some manifold B, called the base of the cone, and
- the action of $\mathbb{R}$ expands the symplectic form exponentially, namely $\rho_\lambda^*\omega_N = e^\lambda \omega_N$, where ρ_λ denotes the diffeomorphism defined by $\lambda \in \mathbb{R}$.

In particular, let

$$N = \{q \in T^*M \mid q \in T_x^*M \text{ and } q = e^\tau \cdot \alpha_x \text{ for some } x \in M \text{ and } \tau \in \mathbb{R}\} \subset T^*M,$$

then $\mathbb{R}$ acts on N by $\rho_\lambda(q) = e^\lambda \cdot q$ (for any $\lambda \in \mathbb{R}$, and $q \in N \subset T^*M$). Furthermore let $\omega_N \in \Omega^2(N)$ be the restriction to N of the canonical symplectic form on T^*M, then (N, ω_N) is a symplectic cone over the base M if and only if (M, ξ) is a co-oriented contact manifold.

Since ω_N is expanded exponentially by the $\mathbb{R}$-action, we can simplify things by instead considering the $\mathbb{R}$-invariant section $1 \otimes \omega_n$ of $\Omega^2(N, W)$, where $W = N \times \mathbb{R}$

[4] Here we use the identifications $P = \Gamma_x$ and $P' = \mathcal{G}_x$. It is a covering map, since the right invariant vector fields, which are identified with the sections of A, span the tangent space of the source fibres.

[5] Here we use the identifications $G = \Gamma_x^x$ and $G' = \mathcal{G}_x^x$.

is the trivial line-bundle over N on which $\lambda \in \mathbb{R}$ acts by scaling by $e^{-\lambda}$ (see Example 3.7).

Now $X \to 1 \otimes \iota_X \omega_N$ defines an isomorphism $TN \hookrightarrow\!\!\!\!\rightarrow W \otimes T^*N$. We let $L \subset TN \oplus (W \otimes T^*N)$ be the graph of this morphism. It is easy to check that L is a maximal isotropic subbundle of $TN \oplus (W \otimes T^*N)$, and since ω_N is closed, L is a $TN\,W$-Dirac subbundle of the $TN\,W$-Courant algebroid on N defined in Example 3.7. We note that (M, ξ) is a contact manifold if and only if L is the graph of an isomorphism, or simply $L \cap W \otimes T^*N = 0$ and $L \cap TN = 0$.

As described in Example 3.7, the quotient of the $TN\,W$-Courant algebroid on N by the $\mathbb{R}$ action yields an $\mathcal{E}^1(M)$ bundle on M or an AV-Courant algebroid, where $A = TN/\mathbb{R}$, and V is the trivial line bundle on M.

Since $1 \otimes \omega_N$ is $\mathbb{R}$-invariant, it follows that its graph, L, is $\mathbb{R}$-invariant; consequently, L defines an $\mathcal{E}^1(M)$-Dirac structure that we denote by $\tilde{L}_\xi$. It is perhaps important to note that $\tilde{L}_\xi$ is defined intrinsically. We may conclude the following(as shown in [14]).

Proposition 5.8 *(M, ξ) is a contact manifold if and only if $\tilde{L}_\xi \cap V \otimes A^* = 0$ and $\tilde{L}_\xi \cap A = 0$ (under the canonical splitting).*

6 CR-structures and Courant Algebroids

Suppose M is a smooth manifold; let $H \subset TM$ be a subbundle, and suppose $J \in \Gamma(\mathrm{Hom}(H, H))$ is such that $J^2 = -\mathrm{id}$. Then (H, J) is called an almost CR structure. We let $H_{1,0} \subset \mathbb{C} \otimes H \subset \mathbb{C} \otimes TM$ denote the $+i$-eigenbundle of J. If $H_{1,0}$ is involutive, then it is called a CR-structure. It is possible to describe this as a Courant algebroid.

We consider the bundle $H^* \oplus H \simeq T^*M \oplus H/\operatorname{Ann}(H)$ and the bundle map $\mathbb{J} := -J^* \oplus J \in \Gamma(\mathrm{Hom}(H^* \oplus H, H^* \oplus H^*))$. It is clear that $\mathbb{J}^2 = -\mathrm{id}$. Let $L = \ker(\mathbb{J} - i) \oplus \operatorname{Ann}(H) \subset \mathbb{C} \otimes (TM \oplus T^*M)$.

Proposition 6.1 *L is involutive under the standard Courant bracket if and only if J defines a CR structure.*

Proof We notice that $L = H_{1,0} \oplus \operatorname{Ann}(H_{1,0})$. Therefore, L is involutive under the Courant bracket only if $\pi(L) = H_{1,0}$ is involutive, where $\pi\colon TM \oplus T^*M \to TM$ is the projection. Thus J defines a CR structure.

Conversely, suppose that $H_{1,0}$ is involutive. Then if I is the ideal generated by $\operatorname{Ann}(H_{1,0})$ in $\Gamma(\mathbb{C} \otimes \wedge T^*M)$, then I is closed under the differential: $dI \subset I$.

In particular, if we restrict our attention to a local neighborhood on M, and α_i is a local basis for $\operatorname{Ann}(H_{1,0})$ and $\xi \in \Gamma(\operatorname{Ann}(H_{1,0}))$, then $d\xi = \sum_i \beta_i \wedge \alpha_i$ for some $\beta_i \in \Omega^1(M, \mathbb{C})$. Thus, for any $X \in \Gamma(H_{1,0})$, we have,

$$\iota_X d\xi = \sum_i \beta_i(X)\alpha_i \in \Gamma(\operatorname{Ann}(H_{1,0})),$$

and

$$\mathcal{L}_X \xi = d\iota_X \xi + \iota_X d\xi = \iota_X d\xi \in \Gamma(\operatorname{Ann}(H_{1,0})).$$

It follows that L is involutive under the standard Courant bracket. ∎

In the next section we shall generalize this construction.

7 Generalized CR structures

Suppose that M is a manifold; A is a Lie algebroid over M; V is an A-module of rank one over M, and $\mathbb{A}$ is an AV-Courant algebroid over M. Suppose further that A has some distinguished subbundle $H \subset A$, and consider the bundle given by

$$\mathbb{H} = q(\pi^{-1}(H)), \text{ where } q\colon \pi^{-1}(H) \to \pi^{-1}(H)/j(V \otimes \operatorname{Ann}(H)).$$

Then the pairing on $\mathbb{A}$ restricts non-degenerately to $\mathbb{H}$, and we have an exact sequence

$$0 \to V \otimes H^* \xrightarrow{j} \mathbb{H} \xrightarrow{\pi} H \to 0.$$

Definition 7.1 $\mathbb{J} \in \Gamma(\operatorname{Hom}(\mathbb{H}, \mathbb{H}))$ is called a generalized CR structure if:

(i) $\mathbb{J}$ is orthogonal (preserves the pairing on $\mathbb{H}$);
(ii) $\mathbb{J}^2 = -1$;
(iii) $L := q^{-1}(\ker(\mathbb{J} - i)) \subset \mathbb{C} \otimes \mathbb{A}$ is involutive.

Remark 7.2 We have that $L := q^{-1}(\ker(\mathbb{J} - i)) \subset \mathbb{C} \otimes \mathbb{A}$ is a maximal isotropic subspace of $\mathbb{A}$, since $\ker(\mathbb{J} - i)$ is a maximal isotropic subspace of $\mathbb{H}$. In particular, since we assume that L is involutive, it is an AV-Dirac structure.

Remark 7.3 Here we have relaxed the requirement $L \cap \bar{L} = 0$ in the definition of a generalized complex structure. While we have allowed $L \cap \bar{L}$ to be non-trivial, it must lie in $j(V \otimes \operatorname{Ann}(H)) \subset V \otimes A^*$. As pointed out in Remark 2.2, this can be interpreted as saying that $L \cap \bar{L}$ only fails to be trivial up to an "infinitesimal". On the other hand, we still require that L be an AV-Dirac structure.

This is in contrast to the approach taken by generalized CRF structures, introduced by Izu Vaisman in [26], which requires $L \cap \bar{L} = 0$, but does not require L to be a Dirac structure.

It is well known that one can canonically associate a Poisson structure with every generalized complex structure. The analogue for generalized CR structures is to endow $V \otimes A^*$ with a non-trivial Lie algebroid structure, which we shall do in a canonical fashion following the corresponding argument given for generalized complex structures in [12].

We have an inclusion $i\colon H \to A$, and consequently, a map $\mathbb{J} \circ j \circ (\mathrm{id} \otimes i^*)\colon V \otimes A^* \to \mathbb{H}$, which (abusing notation), we shall simply call $\mathbb{J}$. We consider the family of subspaces of $\mathbb{A}$ given by

$$D_t := e^{t\mathbb{J}}(V \otimes A^*) + V \otimes \operatorname{Ann}(H) = q^{-1}(e^{t\mathbb{J}}(V \otimes H^*)).$$

Since $e^{t\mathbb{J}} = \cos(t) + \sin(t)\mathbb{J}\colon \mathbb{H} \to \mathbb{H}$ is orthogonal, and $j(V \otimes H^*)$ is a lagrangian subspace of $\mathbb{H}$, it follows that D_t is lagrangian for each t.

The following proposition is a slight generalization of a result of Gualtieri [12].

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Proposition 7.4 (Gualtieri) *The family D_t of almost AV-Dirac structures is integrable for all t.*

Proof Let $\xi_1, \xi_2 \in \Gamma(V \otimes A^*)$, then since $V \otimes A^* \subset L \oplus \bar{L}$, we may choose $X_j \in \Gamma(L)$ and $Y_j \in \Gamma(\bar{L})$, such that $\xi_j = X_j + Y_j$. It follows that $\mathbb{J}\xi_j = iX_j - iY_j + V \otimes \mathrm{Ann}(H)$. In fact, since $L \cap \bar{L} = V \otimes \mathrm{Ann}(H)$, by choosing X_j and Y_j appropriately, we may suppose that $iX_j - iY_j$ is any given representative of $\mathbb{J} \circ i^*(\xi_j)$ in $\pi^{-1}(H)$. Abusing notation, we will use the term $\mathbb{J}(\xi_j)$ and our particular choice of representative $iX_j - iY_j$ interchangeably. Then,

$$\begin{aligned}[\![\mathbb{J}\xi_1, \mathbb{J}\xi_2]\!] - [\![\xi_1, \xi_2]\!] &= [\![iX_1 - iY_1, iX_2 - iY_2]\!] - [\![X_1 + Y_1, X_2 + Y_2]\!] \\ &= -2[\![X_1, X_2]\!] - 2[\![Y_1, Y_2]\!]\end{aligned}$$

and

$$\begin{aligned}[\![\mathbb{J}\xi_1, \xi_2]\!] - [\![\xi_1, \mathbb{J}\xi_2]\!] &= [\![iX_1 - iY_1, X_2 + Y_2]\!] - [\![X_1 + Y_1, iX_2 - iY_2]\!] \\ &= 2i[\![X_1, X_2]\!] - 2i[\![Y_1, Y_2]\!].\end{aligned}$$

Thus, since L and hence $\bar{L}$ are involutive, we have $[\![\mathbb{J}\xi_1, \mathbb{J}\xi_2]\!] - [\![\xi_1, \xi_2]\!] + V \otimes \mathrm{Ann}(H) = \mathbb{J}([\![\mathbb{J}\xi_1, \xi_2]\!] - [\![\xi_1, \mathbb{J}\xi_2]\!]) + V \otimes \mathrm{Ann}(H)$.

We let $a = \cos(t)$ and $b = \sin(t)$, and we have,

$$\begin{aligned}&[\![(a + b\mathbb{J})\xi_1, (a + b\mathbb{J})\xi_2]\!] \\ &\quad = ab([\![\xi_1, \mathbb{J}\xi_2]\!] + [\![\mathbb{J}\xi_1, \xi_2]\!]) + b^2[\![\mathbb{J}\xi_1, \mathbb{J}\xi_2]\!] \\ &\quad = ab([\![\xi_1, \mathbb{J}\xi_2]\!] + [\![\mathbb{J}\xi_1, \xi_2]\!]) + b^2([\![\mathbb{J}\xi_1, \mathbb{J}\xi_2]\!] - [\![\xi_1, \xi_2]\!]).\end{aligned}$$

So modulo $V \otimes \mathrm{Ann}(H)$, we see that

$$[\![(a+b\mathbb{J})\xi_1, (a+b\mathbb{J})\xi_2]\!] + V \otimes \mathrm{Ann}(H) = b(a+b\mathbb{J})([\![\xi_1, \mathbb{J}\xi_2]\!] + [\![\mathbb{J}\xi_1, \xi_2]\!]) + V \otimes \mathrm{Ann}(H)$$

Since $[\![\xi_1, \mathbb{J}\xi_2]\!] + [\![\mathbb{J}\xi_1, \xi_2]\!] \in V \otimes A^*$, it follows that $(\cos(t) + \sin(t)\mathbb{J})(V \otimes A^*) + V \otimes \mathrm{Ann}(H)$ is involutive. ■

We next consider the map $P\colon V \otimes A^* \to H \xrightarrow{i} A$, which for $\xi, \eta \in V \otimes A^*$, is given by

$$\langle P(\xi), \eta\rangle = \Big\langle \tfrac{\partial}{\partial t}\big|_{t=0} e^{t\mathbb{J}}(\xi), \eta\Big\rangle = \langle \mathbb{J}\xi, \eta\rangle \qquad (= \langle i \circ \pi \circ \mathbb{J} \circ j \circ i^*(\xi), \eta\rangle).$$

Clearly, since $\mathbb{J}$ is an orthogonal almost complex structure on $\mathbb{H}$, P will be given by an element of $\Gamma(V^* \otimes \wedge^2 A)$, which we will also denote by P. Adapting a proposition given in [12], we have the following.

Proposition 7.5 (Gualtieri) *The bivector field $P = i \circ \pi \circ \mathbb{J} \circ j \circ i^*\colon V \otimes A^* \to A$ defines a Lie algebroid structure on $V \otimes A^*$. The bracket is given by*

$$[\xi, \eta] = \iota_{P(\cdot, \xi)} d\eta - \iota_{P(\cdot, \eta)} d\xi + d(P(\xi, \eta))),$$

where $\xi, \eta \in V \otimes A^*$, and the anchor map is given by $\xi \to a \circ P(\xi, \cdot) V \otimes A^* \to TM$, where $a\colon A \to TM$ is the anchor map of A. Furthermore, the map $\xi \to a \circ P(\xi, \cdot)\colon V \otimes A^* \to A$ is a Lie-algebroid morphism.

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The proof is an adaptation of one found in [12].

Proof We choose a splitting of the AV-Courant algebroid and use the isomorphism and notation described in Proposition 2.7. Then if we choose t sufficiently small, the AV-Dirac structures D_t can be described as the graphs of $\beta_t \in \Gamma(V^* \otimes \wedge^2 A)$.

In [24], it was shown that the integrability condition of a twisted Poisson structure β over a 3-form background γ is $[\beta, \beta] = \wedge^3 \tilde{\beta}(\gamma)$, where $\tilde{\beta}\colon T^*M \to TM$ is given by $\tilde{\beta}(\xi)(\eta) = \beta(\xi, \eta)$. We would like to derive a similar equation for β_t, but we have not defined a bracket for sections of $V^* \otimes \wedge^2 A$. In order to define such a bracket, we first define a sheaf of rings over M.

We let $\mathcal{F} := (S(V) \otimes S(V^*))/I$, where $S(V)$ denotes the symmetric algebra generated by V, and I is the ideal generated by $u \otimes f - f(u)$ for $f \in \Gamma(V^*)$ and $u \in \Gamma(V)$. Since V is one dimensional, if $t \in \Gamma(V)$ is a local basis, then $\mathcal{F}$ is locally isomorphic to $C^\infty(M)[t, t^{-1}]$ as a ring. It is clear that it has a well-defined $\mathbb{Z}$ grading, which for a homogeneous $v \in \mathcal{F}$, we denote by $\tilde{v}$.

$\Gamma(S(V) \otimes S(V^*))$ is a $\Gamma(A)$ module, where sections of $\Gamma(A)$ act as derivations, and it is easy to check that $\Gamma(I)$ is a sub-module. Thus it is clear that $\Gamma(A)$ acts on $\Gamma(\mathcal{F})$ by derivations satisfying the Leibniz rule with respect to the ring structure on $\mathcal{F}$.

We define a bracket on $\mathcal{F} \otimes \wedge^* A$, as follows (for $v, w \in \Gamma(\mathcal{F})$ and $P, Q \in \Gamma(\wedge^* A)$):

- $[X, v] = Xv$ for any $X \in \Gamma(A)$, and $[v, w] = 0$;
- $[P \wedge Q, v] = P \wedge [Q, v] + (-1)^{|Q|}[P, v] \wedge Q$;
- $[P, Q]$ is given by the Schouten–Nijenhuis bracket;
- $[vP, wQ] = (v[P, w])Q - (-1)^{(|P|-1)(|Q|-1)}(w[Q, v])P + vw[P, Q]$.

If we write $|vP| = i$ for $P \in \wedge^i A$, and $\deg(vP) = (\tilde{v}, |vP|)$, then it is clear that our bracket satisfies the following identities (for homogeneous $a, b, c \in \Gamma(\mathcal{F} \otimes \wedge^* A)$):

- $\deg(ab) = \deg(a) + \deg(b)$ and $\deg([a, b]) = \deg(a) + \deg(b) - (0, 1)$;
- $(ab)c = a(bc)$ and $ab = (-1)^{|a||b|} ba$;
- $[a, bc] = [a, b]c + (-1)^{(|a|-1)|b|} b[a, c]$;
- $[a, b] = -(-1)^{(|a|-1)(|b|-1)}[b, a]$;
- $[a, [b, c]] = [[a, b], c] + (-1)^{(|a|-1)(|b|-1)}[b, [a, c]]$.

We next extend d to a map $d\colon \mathcal{F} \otimes \wedge^i A^* \to \mathcal{F} \otimes \wedge^{i+1} A^*$ in the obvious way. We also have a natural $\mathcal{F}$-bilinear pairing on $\Gamma(\mathcal{F} \otimes \wedge^* A^*) \times \Gamma(\mathcal{F} \otimes \wedge^* A)$, which for $v_i, w_j \in \mathcal{F}$, $\alpha_i \in \Gamma(A^*)$, and $X_j \in \Gamma(A)$, is given by

$$\langle (v_1 \otimes \alpha_1) \cdots (v_p \otimes \alpha_p), (w_1 \otimes X_1) \cdots (w_q \otimes X_q) \rangle = \begin{cases} 0 & \text{if } p \neq q, \\ \det(v_i w_j \otimes \alpha_i(X_j)) & \text{if } p = q. \end{cases}$$

We define a morphism $\iota\colon \mathcal{F} \otimes \wedge^* A \to \operatorname{End}(\mathcal{F} \otimes \wedge^* A^*)$ by $\langle \xi, PQ \rangle = \langle \iota_P \xi, Q \rangle$. For $P \in \mathcal{F} \otimes A$, ι_P is a derivation.

We also define a morphism $\breve{\iota}\colon \mathcal{F} \otimes \wedge^* A^* \to \operatorname{End}(\mathcal{F} \otimes \wedge^* A)$ by $\langle \xi\eta, P \rangle = \langle \xi, \breve{\iota}(\eta)P \rangle$. For $\alpha \in \mathcal{F} \otimes A^*$, $\breve{\iota}(\alpha)$ is a derivation on the right. Namely, $\breve{\iota}(\alpha)(PQ) = P\breve{\iota}(\alpha)Q + (-1)^{|Q|}(\breve{\iota}(\alpha)P)Q$ (where $P, Q \in \mathcal{F} \otimes \wedge^* A$ are homogeneous).

Next, we notice that $\iota_{[P,Q]} = -[[\iota_Q, d], \iota_P]$. This is easy to check, following the argument given in [20]. Also following an argument in [20] one can verify that, for

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$\eta \in \Gamma(\mathcal{F} \otimes A^*)$,

$$\breve{\iota}(\eta)[P, Q] - [P, \breve{\iota}(\eta)Q] - (-1)^{|Q|-1}[\breve{\iota}(\eta)P, Q] = \\ (-1)^{|Q|-2}(\breve{\iota}(d\eta)(PQ) - P\breve{\iota}(d\eta)Q - (\breve{\iota}(d\eta)P)Q).$$

From this, we calculate, for any $\beta \in \Gamma(\mathcal{F} \otimes \wedge^2 A)$ and $\xi, \eta \in \Gamma(\mathcal{F} \otimes A^*)$,

$$[\breve{\iota}(\xi)\beta, \breve{\iota}(\eta)\beta] = \tfrac{1}{2}\breve{\iota}(\xi\eta)[\beta, \beta] + [\beta, \langle \eta\xi, \beta\rangle] + \tfrac{1}{2}(\breve{\iota}(\eta d\xi)\beta^2 - \breve{\iota}(\xi d\eta)\beta^2) \\ - \langle d\xi, \beta\rangle\breve{\iota}(\eta)\beta + \langle d\eta, \beta\rangle\breve{\iota}(\xi)\beta.$$

Furthermore, it is not difficult to verify that $[\beta, \langle \eta\xi, \beta\rangle] = \breve{\iota}(d\beta(\eta, \xi))\beta$, while

$$\tfrac{1}{2}(\breve{\iota}(\eta d\xi)\beta^2 - \breve{\iota}(\xi d\eta)\beta^2) - \langle d\xi, \beta\rangle\breve{\iota}(\eta)\beta + \langle d\eta, \beta\rangle\breve{\iota}(\xi)\beta = \breve{\iota}(\iota_{\breve{\iota}(\xi)\beta}d\eta - \iota_{\breve{\iota}(\eta)\beta}d\xi)\beta.$$

Thus, we have, for $\beta \in \Gamma(V^* \otimes \wedge^2 A)$,

$$\begin{aligned} &[\![-\breve{\iota}(\xi)\beta + \xi, -\breve{\iota}(\eta)\beta + \eta]\!]_\phi \\ &\quad= [\breve{\iota}(\xi)\beta, \breve{\iota}(\eta)\beta] - \iota_{\breve{\iota}(\xi)\beta}d\eta + \iota_{\breve{\iota}(\eta)\beta}d\xi + d(\beta(\xi, \eta)) + \iota_{\breve{\iota}(\xi)\beta}\iota_{\breve{\iota}(\eta)\beta}H \\ &\quad= \breve{\iota}(\iota_{\breve{\iota}(\xi)\beta}d\eta - \iota_{\breve{\iota}(\eta)\beta}d\xi - d(\beta(\xi, \eta)))\beta - \iota_{\breve{\iota}(\xi)\beta}d\eta + \iota_{\breve{\iota}(\eta)\beta}d\xi + d(\beta(\xi, \eta)) \\ &\qquad+ \tfrac{1}{2}\breve{\iota}(\xi\eta)[\beta, \beta] + \iota_{\breve{\iota}(\xi)\beta}\iota_{\breve{\iota}(\eta)\beta}H. \end{aligned}$$

It follows that β_t defines an *AV*-Dirac structure under our chosen splitting if and only if $\frac{1}{2}\breve{\iota}(\eta\xi)[\beta_t, \beta_t] = \breve{\iota}(\iota_{\breve{\iota}(\xi)\beta_t}\iota_{\breve{\iota}(\eta)\beta_t}H)\beta_t$. To rewrite this, we let $\tilde{\beta}\colon \mathcal{F} \otimes A^* \to \mathcal{F} \otimes A$ be the map $\alpha \to -\breve{\iota}(\alpha)\beta$. The condition is then $[\beta_t, \beta_t] = 2 \wedge^3 \tilde{\beta}_t(H)$. We differentiate both sides by t and evaluate at 0. Since we have $P = \frac{\partial}{\partial t}\big|_0 \beta_t$ and $\beta_0 = 0$, the cubic term vanishes, and we see that the condition is $[P, P] = 0$. The result follows immediately from this. ∎

We also have a bracket $\{\,\cdot\,,\,\cdot\,\}$ on $\Gamma(V)$, which for $v, w \in \Gamma(V)$ is given by

$$\{v, w\} = P(dv, dw). \tag{7.1}$$

It satisfies the following properties (for $f \in C^\infty(M)$):

- $\{\,\cdot\,,\,\cdot\,\}$ is bilinear;
- $\{v, w\} = -\{w, v\}$;
- $\{v, fw\} = f\{v, w\} + (a \circ P(dv)(f))w$;
- $\{u, \{v, w\}\} = \{\{u, v\}, w\} + \{v, \{u, w\}\}$ (for any $u, v, w \in \Gamma(V)$).

Since V is a line-bundle, this is quite similar to a Poisson structure. In particular, if $U \subset M$ is an open set on which $\sigma \in \Gamma(V|_U)$ is a local basis such that $P(\sigma) = 0$, then we have a morphism

$$\rho\colon C^\infty(U) \xrightarrow{f \to f\sigma} \Gamma(V|_U),$$

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which allows us to define a Poisson structure on U, by

$$\{f, g\} = \rho^{-1}\{\rho(f), \rho(g)\}.$$

In particular, if in some neighborhood $U \subset M$, V admits a non-zero A-parallel section $\sigma \in \Gamma(V|_U)$, then $P(\sigma) = 0$, and thus U is endowed with a Poisson structure. In fact, the Poisson structure associated with U in this way is unique up to a constant multiple. Furthermore, if it exists at one point on a leaf of A, then it exists for any neighborhood of any point in that leaf.

Remark 7.6 (Poisson Structure on a Leaf of A) Suppose that $F \subset M$ is a connected leaf of the foliation given by A, then $a\colon A|_F \to TF$ is a Lie algebroid, and we have an exact sequence of Lie algebroids given by $0 \to L = \ker(a) \to A|_F \to TF \to 0$, where L is actually a bundle of Lie algebras. The following are equivalent:

- V admits an $A|_F$-parallel section for any neighborhood $U \subset F$;
- L acts trivially on $V|_F$;
- L_x acts trivially on V_x, for some point $x \in F$.[6]

Note that, up to a constant multiple, there is a unique A-parallel section of $V|_F$. Thus, if $\sigma \in \Gamma(V|_F)$ is a non-zero A-parallel section, we can associate a Poisson structure with F, unique up to a constant multiple.

Remark 7.7 (Jacobi Bundle) A Jacobi bundle, introduced by Marle in [21] and Kirillov in [16], is a line bundle $P \to M$ over a manifold M, together with a bilinear map $\{\cdot\,,\cdot\}\colon \Gamma(P)\times\Gamma(P) \to \Gamma(P)$ on the sections of P and a map $\Gamma(P) \xrightarrow{s\to X_s} \Gamma(TM)$ such that

- $\{\cdot\,,\cdot\}$ is bilinear;
- $\{v, w\} = -\{w, v\}$ (for any $v, w \in \Gamma(P)$);
- $\{v, fw\} = f\{v, w\} + (X_v(f))w$ (for any $f \in C^\infty(M)$ and $v, w \in \Gamma(P)$);
- $\{u, \{v, w\}\} = \{\{u, v\}, w\} + \{v, \{u, w\}\}$ (for any $u, v, w \in \Gamma(P)$).

It follows that V together with the bracket (7.1) is a Jacobi bundle canonically associated with the generalized CR structure.

Suppose for some $U \subset M$ there is a choice of a local basis $\sigma \in \Gamma(V|_U)$. We may consider the isomorphism

$$\rho\colon C^\infty(U) \xrightarrow{f\to f\sigma} \Gamma(V|_U),$$

which allows us to define a bracket on $C^\infty(U)$ by $[f, g]_\sigma = \rho^{-1}\{\rho(f), \rho(g)\}$. One notices that this bracket endows $C^\infty(U)$ with a Lie algebra structure that is local in the sense that the linear operator

$$D_f\colon C^\infty(U) \xrightarrow{g\to[f,g]_\sigma} C^\infty(U)$$

[6]This follows from the fact that for any $x, y \in F$ there is a Lie algebroid morphism of A covering a diffeomorphism of M that takes x to y. In addition these morphisms can be assumed to come from flowing along a section of A, and hence extend to V.

is local for all $f \in C^\infty(U)$. It is an important result (see [11, 16, 25]) that for any local Lie algebra structure, there exists unique $\Lambda \in \Gamma(\wedge^2 TM)$, and $E \in \Gamma(TM)$ with $[\Lambda, \Lambda] = -2\Lambda \wedge E$ and $[\Lambda, E] = 0$ such that

$$[f, g]_\sigma = \{f, g\}_\Lambda + f\mathcal{L}_X g - g\mathcal{L}_X f,$$

where $\{f, g\}_\Lambda = \check{\iota}_{df}\check{\iota}_{dg}\Lambda$.

The triple (U, Λ, E) is then called a Jacobi structure. Note however the dependence of Λ and E on σ; this is unlike the local Poisson structure that (if it exists) is unique up to a constant multiple.

Example 7.8 (CR Structures) As described in Section 6, a CR-structure on a manifold M can be described by a generalized CR structure. In this case, V can be taken to be the trivial bundle, and A can be taken to be TM. It follows from the above discussion that there is a Poisson structure $P \in \Gamma(\wedge^2 TM)$ associated with the CR structure.

If $L \subset \mathbb{C} \otimes TM$ is the CR-structure, and $H = \mathbf{Re}(L \oplus \bar{L}) \subset TM$, then $P(T^*M) \subset H$. So the symplectic foliation associated with P is everywhere tangent to H.

Example 7.9 (Quotients of Generalized Complex Structures) If the procedures described in Examples 4.3 and 3.4 are applied to a generalized complex structure, then one obtains a generalized CR structure.

Example 7.10 (Contact Structures and Generalized Contact Structures) Suppose that M is a contact manifold, then there is a canonical way to associate a generalized CR structure with M. In particular, if $N = M \times \mathbb{R}$ is its symplectization, then N admits a generalized complex structure corresponding to its symplectic structure. $\mathbb{R}$ acts on N, and the quotient is a generalized CR structure on M (in the sense of Examples 3.7 and 4.4).

This procedure is also described in [14, 15], where they describe it as a generalized contact structure. In fact any generalized contact structure results from the quotient of generalized complex structure, and as such can also be described as a generalized CR structure.

Since the Lie algebroid A and the vector bundle V describe an $\mathcal{E}^1(M)$ structure, as given in Example 3.3, it can be checked that V does not admit parallel sections, and thus, in general, $P \in \Gamma(V^* \otimes \wedge^2 A)$ does not describe a Poisson structure, but rather a Jacobi structure. When the generalized contact structure is simply a contact structure, then P corresponds to a Jacobi structure describing the contact structure.

To be more explicit, we let M be a contact manifold with contact distribution $\xi \subset TM$, and $N = M \times \mathbb{R}$ its symplectization, where we let $t\colon M \times \mathbb{R} \to \mathbb{R}$ be the projection to the second factor, and $\omega \in \Omega^2(N)$ denote the corresponding symplectic form. (That is, $\omega = e^t(d\eta + dt \wedge \eta)$, where $\eta \in \mathrm{Ann}(\xi)$ is nowhere vanishing.) We note that $\mathcal{L}_{\frac{\partial}{\partial t}}\omega = \omega$.

Since N is a symplectic manifold, we can associate a canonical generalized complex structure $\mathcal{J}\colon TN \oplus T^*N \to TN \oplus T^*N$ with it on the standard Courant algebroid

$$0 \to T^*N \to TN \oplus T^*N \to TN \to 0$$

(see [12] for details).

The Poisson bivector $\pi \in \Gamma(\wedge^2 TN)$ associated with this generalized complex structure has the property that $\mathcal{L}_{\partial/\partial t}\pi = -\pi$ (since it is the Poisson bivector corresponding to ω). It follows that we can write $\pi = e^{-t}(\Lambda + \partial/\partial t \wedge E)$ for $E \in \Gamma(M)$, and $\Lambda \in \Gamma(\wedge^2 M)$. Then $[\pi, \pi] = 0$ implies that

$$\begin{aligned} 0 = [\pi, \pi] &= \Big[e^{-t}\big(\Lambda + \tfrac{\partial}{\partial t} \wedge E\big), e^{-t}\big(\Lambda + \tfrac{\partial}{\partial t} \wedge E\big) \Big] \\ &= e^{-2t}[\Lambda, \Lambda] - 2e^{-2t}\Lambda \wedge E + 2e^{-2t}\tfrac{\partial}{\partial t} \wedge [\Lambda, E]. \end{aligned}$$

From this it follows that $[\Lambda, \Lambda] = -2\Lambda \wedge E$ and $[\Lambda, E] = 0$, which are the defining conditions for a Jacobi structure (Λ, E) on M.

Now, we consider the $TM \oplus \mathbb{R} - \mathbb{R}$ Courant algebroid structure on M, given by taking the quotient by the $G = \mathbb{R}$ action on $N = M \times \mathbb{R}$,

$$0 \to T^*N/G \to (TN \oplus T^*N)/G \to TN/G \to 0,$$

and the generalized CR structure on M given by quotient homomorphism

$$\mathbb{J} := \mathcal{J}/G \colon (TN \oplus T^*N)/G \to (TN \oplus T^*N)/G.$$

They define an AV-Courant algebroid, where $A = TN/G$, and the bundle $V \to M$ is trivial, with $\Gamma(V) \simeq C^\infty(N)^G$ (this is in fact an $\mathcal{E}^1(M)$ structure; see [14]). Abusing notation, we denote by $e^t \in \Gamma(V)$ the section associated with the G-invariant function $e^t \in C^\infty(N)$.

Then the bivector $P \in \Gamma(V^* \otimes \wedge^2 A)$ associated with the generalized CR structure on M is simply $e^{-t}(\Lambda + \frac{\partial}{\partial t} \wedge E)$, and it defines a Jacobi structure on M, with bivector field Λ and vector field E. Since $\Lambda^n \wedge E \neq 0$ (where $\dim(M) = 2n + 1$), this Jacobi structure corresponds to a contact structure. In fact, the contact distribution is given by $\operatorname{span}\{\breve{\iota}_\alpha \Lambda \mid \alpha \in T * M\}$, and if $\theta \in \Omega^1(M)$ satisfies $\breve{\iota}_\theta \Lambda = 0$ and $\breve{\iota}_\theta E = 1$, then θ is a contact form. It is not difficult to see that this is the original contact structure, ξ, defined on M. (In fact, if $\omega = e^t(d\eta + dt \wedge \eta)$ is the symplectic form on N (where $\eta \in \operatorname{Ann}(\xi)$ is nowhere vanishing), then E is a reeb vector field for η and $\theta = \eta$.)

We must note that, if instead of trivializing V by the section $e^t \in \Gamma(V)$, we made the transformation $e^t \to fe^t$, for some nowhere vanishing $f \in C^\infty(M)$, then the appropriate changes to the Jacobi structure would be $\Lambda \to f\Lambda$, $E \to fE - \breve{\iota}_{df}\Lambda$, and the transformation for the contact form would be $\theta \to \frac{1}{f}\theta$. Thus it is clear that the freedom to modify the trivializing section of V by a scalar multiple does not change the contact distribution and fully accounts for the freedom to change the contact form by a scalar multiple. Indeed the generalized CR structure is defined intrinsically.

A Appendix: Proof of Proposition 2.7

Suppose that M is a manifold, A is a Lie algebroid over M, V is an A-module over M, and $\mathbb{A}$ is an AV-Courant algebroid over M.

For $X, Y \in \Gamma(A)$, we have the following identities:

- $[\iota_X, \iota_Y] = 0$;
- $[d, \iota_X] = \mathcal{L}_X$;
- $[\mathcal{L}_X, \iota_Y] = \iota_{[X,Y]}$;
- $[d, d] = 0$;
- $[\mathcal{L}_X, d] = 0$;
- $[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}$.

We will provide the proof we promised for Proposition 2.7, which we restate here.

Proposition A.1 *Let $\phi\colon A \to \mathbb{A}$ be an isotropic splitting. Then under the isomorphism $\phi \oplus j\colon A \oplus (V \otimes A^*) \to \mathbb{A}$, the bracket is given by*

$$[\![X + \xi, Y + \eta]\!]_\phi = [X, Y] + \mathcal{L}_X \eta - \iota_Y d\xi + \iota_X \iota_Y H_\phi,$$

where $X, Y \in \Gamma(A)$, $\xi, \eta \in \Gamma(V \otimes A^*)$, and $H_\phi \in \Gamma(V \otimes \wedge^3 A^*)$, with $dH_\phi = 0$.

Furthermore, if $\psi\colon A \to \mathbb{A}$ is a different choice of isotropic splitting, then $\psi(X) = \phi(X) + j(\iota_X \beta)$, and $H_\psi = H_\phi - d\beta$, where $\beta \in \Gamma(V \otimes \wedge^2 A^*)$.

Proof The proof will follow immediately from the following lemmas. ■

Lemma A.2 *If $\xi \in \Gamma(V \otimes A^*)$ and $e \in \Gamma(\mathbb{A})$, then $[\![e, j(\xi)]\!] = j(\mathcal{L}_{\pi(e)}\xi)$.*

Proof Let $e_1, e_2 \in \Gamma(\mathbb{A})$, $\xi \in \Gamma(V \otimes A^*)$,

$$\begin{aligned}\langle [\![e_1, j(\xi)]\!], e_2\rangle &= \mathcal{L}_{\pi(e_1)}\langle j(\xi), e_2\rangle - \langle j(\xi), [\![e_1, e_2]\!]\rangle \\ &= \mathcal{L}_{\pi(e_1)}\iota_{\pi(e_2)}\xi - \iota_{\pi([e_1,e_2])}\xi = \mathcal{L}_{\pi(e_1)}\iota_{\pi(e_2)}\xi - \iota_{[\pi(e_1),\pi(e_2)]}\xi \\ &= \mathcal{L}_{\pi(e_1)}\iota_{\pi(e_2)}\xi - [\mathcal{L}_{\pi(e_1)}, \iota_{\pi(e_2)}]\xi = \iota_{\pi(e_2)}\mathcal{L}_{\pi(e_1)}\xi = \langle j(\mathcal{L}_{\pi(e_1)}\xi), e_2\rangle. \blacksquare\end{aligned}$$

Lemma A.3 *If $\xi \in \Gamma(V \otimes A^*)$ and $e \in \Gamma(\mathbb{A})$, then $[\![j(\xi), e]\!] = -j(\iota_{\pi(e)} d\xi)$.*

Proof

$$\begin{aligned}[\![j(\xi), e]\!] &= D\langle j(\xi), e\rangle - [\![e, j(\xi)]\!] = j(d\iota_{\pi(e)}\xi) - j(\mathcal{L}_{\pi(e)}\xi) \\ &= j(d\iota_{\pi(e)}\xi - (\iota_{\pi(e)}d\xi + d\iota_{\pi(e)}\xi)) = -j(\iota_{\pi(e)}d\xi).\end{aligned}$$ ■

Lemma A.4 *If $\phi\colon A \to \mathbb{A}$ is an isotropic splitting and if $X, Y \in \Gamma(A)$, then*

$$[\![\phi(X), \phi(Y)]\!] - \phi([X, Y]) = j(\iota_X \iota_Y H),$$

where $H \in \Gamma(V \otimes \wedge^3 A^*)$.

Proof Let ϕ be an isotropic splitting, and $X, Y, Z \in \Gamma(A)$. Then

$$\pi\big([\![\phi(X), \phi(Y)]\!] - \phi([X, Y])\big) = 0,$$

so by exactness of the sequence (2.2), $[\![\phi(X), \phi(Y)]\!] - \phi([X, Y]) \in j(\Gamma(V \otimes A^*))$. We define H by

$$H(X, Y, Z) = \big\langle \phi(Z), [\![\phi(X), \phi(Y)]\!] - \phi([X, Y])\big\rangle = \big\langle \phi(Z), [\![\phi(X), \phi(Y)]\!]\big\rangle$$

where the second equality follows since ϕ is an isotropic splitting. It is obvious that H is tensorial in Z. Furthermore, making repeated use of the fact that ϕ is an isotropic splitting, we check that H is skew-symmetric:

$$\begin{aligned}\langle \phi(Z), [\![\phi(X),\phi(Y)]\!]\rangle &= \big\langle \phi(Z), -[\![\phi(Y),\phi(X)]\!] + D\langle \phi(X),\phi(Y)\rangle\big\rangle \\ &= -\big\langle \phi(Z), [\![\phi(Y),\phi(X)]\!]\big\rangle\end{aligned}$$

and

$$0 = \mathfrak{L}_X\langle \phi(Z),\phi(Y)\rangle = \big\langle [\![\phi(X),\phi(Z)]\!],\phi(Y)\big\rangle + \big\langle \phi(Z), [\![\phi(X),\phi(Y)]\!]\big\rangle$$

It follows that $H \in \Gamma(V \otimes \wedge^3 A^*)$. ■

Lemma A.5 *Using the notation of the previous lemmas,* $dH = 0$.

Proof Using the fact that $[\mathfrak{L}_X, \iota_Y] = \iota_{[X,Y]}$, it is easy to show that

$$d\iota_Z\iota_Y\iota_X + \iota_Z\iota_Y\iota_X d = \mathfrak{L}_Z\iota_Y\iota_X + \mathfrak{L}_Y\iota_X\iota_Z + \mathfrak{L}_X\iota_Z\iota_Y + \iota_Z\iota_{[Y,X]} + \iota_Y\iota_{[X,Z]} + \iota_X\iota_{[Z,Y]}.$$

Let $\phi\colon A \to \mathbb{A}$ be an isotropic splitting. We shall use the identification

$$A \oplus (V \otimes A^*) \xrightarrow{\phi\oplus j} \mathbb{A}$$

explicitly throughout this section. We have, for $X, Y, Z \in \Gamma(A)$,

$$[\![X,Y]\!]_\phi = [X,Y] + \iota_X\iota_Y H.$$

Then using Axiom (AV-1) from the definition of an AV-Courant algebroid, we see that

$$\begin{aligned}
0 &= [\![Z,[\![Y,X]\!]_\phi]\!]_\phi - [\![[\![Z,Y]\!]_\phi,X]\!]_\phi - [\![Y,[\![Z,X]\!]_\phi]\!]_\phi \\
&= [\![Z,[Y,X] + \iota_Y\iota_X H]\!]_\phi - [\![[Z,Y] + \iota_Z\iota_Y H, X]\!]_\phi - [\![Y,[Z,X] + \iota_Z\iota_X H]\!]_\phi \\
&= [\![Z,[Y,X]]\!]_\phi + \mathfrak{L}_Z\iota_Y\iota_X H - [\![[Z,Y],X]\!]_\phi + \iota_X d\iota_Z\iota_Y H - [\![Y,[Z,X]]\!]_\phi - \mathfrak{L}_Y\iota_Z\iota_X H \\
&= [\![Z,[Y,X]]\!]_\phi - [\![[Z,Y],X]\!]_\phi - [\![Y,[Z,X]]\!]_\phi \\
&\quad + \mathfrak{L}_Z\iota_Y\iota_X H + \mathfrak{L}_X\iota_Z\iota_Y H + \mathfrak{L}_Y\iota_X\iota_Z H - d\iota_Z\iota_Y\iota_X H \\
&= [Z,[Y,X]] + \iota_Z\iota_{[Y,X]}H - [[Z,Y],X] - \iota_{[Z,Y]}\iota_X H - [Y,[Z,X]] - \iota_Y\iota_{[Z,X]}H \\
&\quad + \mathfrak{L}_Z\iota_Y\iota_X H + \mathfrak{L}_X\iota_Z\iota_Y H + \mathfrak{L}_Y\iota_X\iota_Z H - d\iota_Z\iota_Y\iota_X H \\
&= [Z,[Y,X]] - [[Z,Y],X] - [Y,[Z,X]] + \iota_Z\iota_{[Y,X]}H + \iota_X\iota_{[Z,Y]}H + \iota_Y\iota_{[X,Z]}H \\
&\quad + \mathfrak{L}_Z\iota_Y\iota_X H + \mathfrak{L}_X\iota_Z\iota_Y H + \mathfrak{L}_Y\iota_X\iota_Z H - d\iota_Z\iota_Y\iota_X H \\
&= \iota_Z\iota_Y\iota_X dH.
\end{aligned}$$

■

Lemma A.6 *Let $\phi\colon A \to \mathbb{A}$ and $\psi\colon A \to \mathbb{A}$ be two isotropic splittings, and let H_ϕ and H_ψ be the elements of $\Gamma(V \otimes \wedge^3 A^*)$ associated with the corresponding splittings. Namely, if $X, Y \in \Gamma(A)$, then $[\![\phi(X), \phi(Y)]\!] = \phi([X, Y]) + j\iota_X\iota_Y H_\phi$, and similarly for H_ψ.*

Then there exists $\beta \in \Gamma(V \otimes \wedge^2 A^*)$ such that $\psi(X) = \phi(X) + j(\iota_X\beta)$ and $H_\psi = H_\phi - d\beta$.

Proof Since ϕ and ψ are splittings, we see that

$$\pi((\phi - \psi)(X)) = 0.$$

Thus, by the exactness of the sequence (2.2), $(\phi - \psi)(X) = j \circ S(X)$ for some linear map $S\colon A \to V \otimes A^*$.

However since the splittings are isotropic,

$$\begin{aligned} 0 &= \langle \phi(X), \phi(Y) \rangle \\ &= \langle \psi(X) + j \circ S(X), \psi(Y) + j \circ S(Y) \rangle \\ &= S(X)(Y) + S(Y)(X), \end{aligned}$$

so we can define $\beta \in \Gamma(V \otimes \wedge^2 A^*)$ by $\iota_X\beta = S(X)$. Then, we see that

$$\begin{aligned} \psi([X, Y]) + \iota_X\iota_Y H_\psi &= [\![\phi(X) + j(\iota_X\beta), \phi(Y) + j(\iota_Y\beta)]\!] \\ &= \phi([X, Y]) + j(\mathfrak{L}_X\iota_Y\beta - \iota_Y d\iota_X\beta + \iota_X\iota_Y H_\phi) \\ &= \phi([X, Y]) + j(\iota_X\iota_Y H_\phi) + j(\mathfrak{L}_X\iota_Y\beta - \iota_Y\mathfrak{L}_X\beta + \iota_Y\iota_X d\beta) \\ &= \phi([X, Y]) + j(\iota_X\iota_Y H_\phi) + j(\iota_{[X,Y]}\beta + \iota_Y\iota_X d\beta) \\ &= \phi([X, Y]) + j(\iota_{[X,Y]}\beta) + j(\iota_X\iota_Y H_\phi - \iota_X\iota_Y d\beta) \\ &= \psi([X, Y]) + j(\iota_X\iota_Y H_\phi - \iota_X\iota_Y d\beta), \end{aligned}$$

so we have $H_\psi = H_\phi - d\beta$. ■

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