Mathematical Structures in Computer Science (2024), **34**, pp. 244–257
doi:10.1017/S0960129524000069

PAPER

A set-theoretic approach to algebraic L-domains*

Juan Zou, Yuhan Zhao, Cuixia Miao and Longchun Wang

School of Mathematical Sciences, Qufu Normal University, Qufu, China
Corresponding author: Longchun Wang; Email: longchunw@163.com

(Received 28 December 2022; revised 24 January 2024; accepted 23 February 2024; first published online 11 April 2024)

Abstract

In this paper, the notion of locally algebraic intersection structure is introduced for algebraic L-domains. Essentially, every locally algebraic intersection structure is a family of sets, which forms an algebraic L-domain ordered by inclusion. It is shown that there is a locally algebraic intersection structure which is order-isomorphic to a given algebraic L-domain. This result extends the classic Stone's representation theorem for Boolean algebras to the case of algebraic L-domains. In addition, it can be seen that many well-known representations of algebraic L-domains, such as logical algebras, information systems, closure spaces, and formal concept analysis, can be analyzed in the framework of locally algebraic intersection structures. Then, a set-theoretic uniformity across different representations of algebraic L-domains is established.

Keywords: Stone's representation theorem; domain theory; algebraic L-domain; algebraic intersection structure

1. Introduction

The development of re-framing algebraic structures and order structures within the theory of sets can be traced back to Stone's representation theorem for Boolean algebras (Stone, 1936) and Birkhoff's representation theorem for finite distributive lattices (Birkhoff, 1937). Their results show that every Boolean algebra or finite distributive lattice can be represented as a family of sets. So far, many scholars have pointed out that more structures such as groups, rings, lattices, and semilattices can be better understood through the theory of sets.

Algebraic L-domains, as a class of special order structures, are introduced by Jung (1989). The category of algebraic L-domains with Scott continuous function as morphisms, denoted by **AL**, is cartesian closed. So, it is a good candidates for denotational semantics of programming languages and has been widely applied in theoretical computer science, especially in domain theory. Establishing concrete and comprehensible descriptions for various domain structures is an important issue in domain theory research. In different developments, a large number of representations of domain theory have been presented (He and Xu, 2019; Jung et al., 1999; Rounds, and Zhang, 2001; Vickers, 2004; Wang and Li, 2022).

Algebraic lattices are a proper subclass of algebraic L-domains, which have an elementary set-theoretic representation as topped algebraic intersection structures. An algebraic intersection structure $\mathcal{L}$ on a set X is a non-empty family of subsets of X which satisfies

*Supported by Shandong Provincial Natural Science Foundation (ZR2023MA051, ZR2022MA022).

(*a*) $\bigcap_{i\in I} A_i \in \mathcal{L}$ for every non-empty family $\{A_i\}_{i\in I}$ in $\mathcal{L}$,
(*b*) $\bigcup_{i\in I} A_i \in \mathcal{L}$ for every directed family $\{A_i\}_{i\in I}$ in $\mathcal{L}$.

If $\mathcal{L}$ also satisfies

(*c*) $X \in \mathcal{L}$,

then it is called a topped algebraic intersection structure. Moreover, algebraic intersection structures can be used to characterize Scott domains, another important subclass of algebraic L-domains.

As far as we know, algebraic lattices and Scott domains are the only two subclasses of domains which have purely set-theoretic representations. The main purpose of this paper is to provide a set-theoretic representation for algebraic L-domains. An algebraic domain L is called an algebraic L-domain if it satisfies the local property that for all x in L, the principal ideals $\downarrow x$ are algebraic lattices. Motivated by this observation and the set-theoretic representation of algebraic lattices, we define a locally algebraic intersection structure for every algebraic L-domain. The notion of a locally algebraic intersection structure generalizes an algebraic intersection structure by simply changing condition (*a*), within which the local property of an algebraic L-domain is easily characterized. Although we illustrate that an algebraic L-domain consisting of a family of sets with set inclusion may not be a locally algebraic intersection structure, we will prove that every algebraic L-domain can be rewritten as a locally algebraic intersection structure. This enables us to perform algebraic L-domain in a pure set-theoretic form.

We realize that there are at least four different representations for algebraic L-domains, ranging from Chen and Jung's disjunctive propositional logics (Chen and Jung, 2006), over Wu et al.'s algebraic L-information systems (Wu et al., 2016) and algebraic L-closure spaces (Wu et al., 2021), to Guo et al.'s LCF contexts (Guo et al., 2018). We will prove that all of the four representations of algebraic domains are locally algebraic intersection structures. So these known representations for algebraic L-domains have a unified set-theoretic form.

The paper is organized as follows. Section 2 focuses on preliminary of this paper. In Section 3, we introduce the notion of a locally algebraic intersection structure and give the representation theorem of algebraic L-domains. This section also establishes the category of locally algebraic intersection structures with Scott continuous functions, which is not only a subcategory of **AL** but also is equivalent to **AL**. In Section 4, we give a brief review of the four existing representations of algebraic L-domains and show that they are all of locally algebraic intersection structures.

2. Preliminaries

This section recalls some domain theoretical terminology that will be used in this paper, and we refer to Davey and Priestly (2002) and Gierz et al. (2003) for the standard notions of domain theory.

Let $(P, \leq)$ be a poset and let A be a subset of P. We denote by $\downarrow A$ the set of all elements $x \in P$ with $x \leq a$ for some $a \in A$, that is,

$$\downarrow A = \{x \in P \mid (\exists a \in A) x \leq a\}.$$

For the case that A is a singleton $\{a\}$, we write $\downarrow a$ for $\downarrow \{a\}$. Note that the principal ideals are just the sets $\downarrow a$ for all $a \in P$. The supremum of A, if it exists, is the least element of the set of all upper bounds of A in P. We denoted it by $\bigvee A$. The infimum $\bigwedge A$ of A is defined dually. In the case of pairs of elements, it is customary to write $x \wedge y$ for the infimum $\bigwedge \{x, y\}$ when it exists.

Definition 2.1. (Gierz et al. 2003) Let $(P, \leq)$ be a poset.

(1) The poset P is said to be a complete lattice if every subset of it has an infimum.
(2) If $x \wedge y$ exists for all $x, y \in P$, then P is called a semilattice.
(3) A non-empty subset D of P is said to be directed if for every $x, y \in D$, there is some $z \in D$ such that $x \leq z$ and $y \leq z$.
(4) The poset P is said to be a dcpo if every directed subset of it has a supremum.
(5) An element $k \in P$ is said to be compact, if whenever $D \subseteq P$ is directed for which $\bigvee D$ exists and $k \leq \bigvee D$, then $k \leq d$ for some $d \in D$.
(6) If P is a dcpo and every element in P is a directed supremum of compact elements, then P is called an algebraic domain.

We denote by $\mathcal{K}(P)$ the set of all compact elements of a poset P.

Definition 2.2. (Jung 1989) An algebraic domain D is said to be an algebraic L-domain if for every element x in D, the principal ideal

$$\downarrow x = \{y \in D \mid y \leq x\}$$

is a complete lattice.

Definition 2.3. (Gierz et al. 2003) Let P and Q be algebraic domains. A monotonic function $f : P \to Q$ is called Scott continuous if it maps suprema of directed sets to the corresponding suprema.

3. Representation Theorem of L-Domains

We begin our theory with the following definition.

Definition 3.1. A non-empty family $\mathcal{C}$ of subsets of a set X is said to be a locally algebraic intersection structure provided that

(L1) for every directed family $\{C_i \in \mathcal{C} \mid i \in I\}$, the directed union $\bigcup_{i \in I} C_i \in \mathcal{C}$,
(L2) for every $C \in \mathcal{C}$ and non-empty family $\{C_j \in \mathcal{C} \mid C_j \subseteq C, j \in J\}$, the intersection $\bigcap_{j \in J} C_j$ belongs to $\mathcal{C}$.

Clearly, every algebraic intersection structure, especially every topped algebraic intersection structure, is a locally algebraic intersection structure.

Example 3.2. *Let $X = \{a, b, c, d, e\}$ and $\mathcal{C} = \{\{a\}, \{a, b\}, \{a, c\}, \{a, b, c, d\}, \{a, b, c, e\}\}$. Then $\mathcal{C}$ is a locally algebraic intersection structure on X. However, $\mathcal{C}$ is not an algebraic intersection structure. In addition, the poset $(\mathcal{C}, \subseteq)$ is defined as in Fig. 1.*

Remark 3.3. Let $\mathcal{C}$ be a locally algebraic intersection structure on a set X.

(1) By condition (L1), $\mathcal{C}$ forms a dcpo with respect to inclusion, in which the supremum of every directed family is given by set union.
(2) For every $B \in \mathcal{C}$ and every subset A of B, the set $\{C \in \mathcal{C} \mid A \subseteq C \subseteq B\}$ is non-empty. We denote by $\Gamma_B(A)$ the intersection of members of the set $\{C \in \mathcal{C} \mid A \subseteq C \subseteq B\}$, that is,

https://doi.org/10.1017/S0960129524000069 Published online by Cambridge University Press

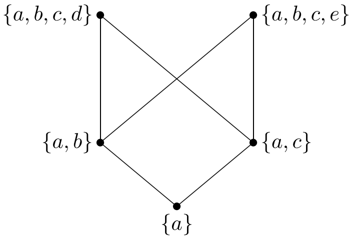

Figure 1. The poset $(\mathcal{C}, \subseteq)$ in Example 3.2.

$$\Gamma_B(A) = \bigcap\{C \in \mathcal{C} \mid A \subseteq C \subseteq B\}. \tag{1}$$

Then $\Gamma_B(A) \in \mathcal{C}$, by condition (L2). Moreover, it is easy to see that $\Gamma_B(A) = A$ whenever $A \in \mathcal{C}$.

The following properties are simple but useful in our theory.

Proposition 3.4. *Let $\mathcal{C}$ be a locally algebraic intersection structure on a set X, and let M be a finite subset of X.*

(1) *If $B \in \mathcal{C}$ and $M \subseteq B$, then $M \subseteq \Gamma_B(M) \subseteq B$.*
(2) *If $B_1, B_2 \in \mathcal{C}$ and $M \subseteq B_1 \subseteq B_2$, then $\Gamma_{B_1}(M) = \Gamma_{B_2}(M)$.*
(3) *If $B, B_1, B_2 \in \mathcal{C}$, $M \subseteq B \subseteq B_1$ and $M \subseteq B \subseteq B_2$, then $\Gamma_{B_1}(M) = \Gamma_{B_2}(M)$.*

Proof. (1) It is straightforward by Equation (1).

(2) Suppose that $B_1, B_2 \in \mathcal{C}$ and $M \subseteq B_1 \subseteq B_2$. Then,

$$\{C \in \mathcal{C} \mid M \subseteq C \subseteq B_1\} \subseteq \{C \in \mathcal{C} \mid M \subseteq C \subseteq B_2\}.$$

Take

$$C' \in \{C \in \mathcal{C} \mid M \subseteq C \subseteq B_2\} - \{C \in \mathcal{C} \mid M \subseteq C \subseteq B_1\}.$$

Noting that

$$\{C \in \mathcal{C} \mid M \subseteq C \subseteq B_1\} = \{C \in \mathcal{C} \mid M \subseteq C \subseteq B_1 \subseteq B_2\},$$

it follows that

$$C \subseteq B_1 \subseteq C' \subseteq B_2$$

for all $C \in \{C \in \mathcal{C} \mid M \subseteq C \subseteq B_1\}$. Therefore,

$$\bigcap\{C \in \mathcal{C} \mid M \subseteq C \subseteq B_1\} = \bigcap\{C \in \mathcal{C} \mid M \subseteq C \subseteq B_2\}.$$

Thus by Equation (1), we have $\Gamma_{B_1}(M) = \Gamma_{B_2}(M)$.

(3) Suppose that $B, B_1, B_2 \in \mathcal{C}$. If $M \subseteq B \subseteq B_1$ and $M \subseteq B \subseteq B_2$, then $\Gamma_B(M) = \Gamma_{B_1}(M)$ and $\Gamma_B(M) = \Gamma_{B_2}(M)$, using part (2) twice. Therefore, $\Gamma_{B_1}(M) = \Gamma_{B_2}(M)$. □

In what follows, we use $M \subseteq_{fin} X$ to denote that M is a finite subset of X.

Lemma 3.5. *Let $\mathcal{C}$ be a locally algebraic intersection structure on a set X.*

(1) *For every $C \in \mathcal{C}$, we write $\mathcal{D}_C$ for the set $\{\Gamma_C(M) \mid M \subseteq_{fin} C\}$. Then,*

$$\mathcal{D}_C = \{\Gamma_C(M) \mid M \subseteq_{fin} C\} \tag{2}$$

is directed under set inclusion and $C = \bigcup \mathcal{D}_C$.

https://doi.org/10.1017/S0960129524000069 Published online by Cambridge University Press

(2) *The set of compact elements of the dcpo* $(\mathcal{C}, \subseteq)$ *is given by:*

$$\mathcal{K}(\mathcal{C}) = \{\Gamma_C(M) \mid C \in \mathcal{C} \text{ and } M \subseteq_{fin} C\}. \tag{3}$$

Proof. (1) The family $\mathcal{D}_C$ is non-empty since $\emptyset \subseteq_{fin} C$ and $\Gamma_C(\emptyset) \in \mathcal{D}_C$. Let $\Gamma_C(M_1), \Gamma_C(M_2) \in \mathcal{D}_C$, where M_1 and M_2 are finite subsets of C. Then, $M_1 \cup M_2$ is a finite subset of C and $\Gamma_C(M_1 \cup M_2) \in \mathcal{D}_C$. By Equation (1), it is clear that $\Gamma_C(M_1) \subseteq \Gamma_C(M_1 \cup M_2)$ and $\Gamma_C(M_2) \subseteq \Gamma_C(M_1 \cup M_2)$. This shows that the family $\mathcal{D}_C$ is directed.

By part (1) of Proposition 3.4, $\Gamma_C(M) \subseteq C$ for every finite subset $M \subseteq C$. Thus, $\bigcup \mathcal{D}_C \subseteq C$. Conversely, for every $x \in C$, we have

$$x \in \{x\} \subseteq \Gamma_C(\{x\}) \subseteq \bigcup \mathcal{D}_C,$$

which implies that $C \subseteq \bigcup \mathcal{D}_C$.

(2) Suppose that $C \in \mathcal{C}$ and $M \subseteq_{fin} C$. If $\Gamma_C(M) \subseteq \bigcup \mathcal{D}$ for some directed subfamily of $\mathcal{D}$, then $M \subseteq \Gamma_C(M) \subseteq C$ and $M \subseteq \Gamma_C(M) \subseteq \bigcup \mathcal{D}$. Thus, $\Gamma_{\bigcup \mathcal{D}}(M) = \Gamma_C(M)$, by part (3) of Proposition 3.4. Because $\mathcal{D}$ is a directed set and M is a finite subset of $\bigcup \mathcal{D}$, there is some $D_0 \in \mathcal{D}$ such that $M \subseteq D_0$. Hence, $\Gamma_{\bigcup \mathcal{D}}(M) \subseteq \Gamma_{\bigcup \mathcal{D}}(D_0)$. But $\Gamma_{\bigcup \mathcal{D}}(D_0) = D_0$, since $D_0 \in \mathcal{C}$. So $\Gamma_{\bigcup \mathcal{D}}(M) \subseteq D_0$, which indicates that $\Gamma_C(M)$ is a compact element in $(\mathcal{C}, \subseteq)$.

Conversely, suppose that C is a compact element in $(\mathcal{C}, \subseteq)$. By part (1), we have

$$C = \bigcup \{\Gamma_C(M) \mid M \subseteq_{fin} C\}$$

and the family $\{\Gamma_C(M) \mid M \subseteq_{fin} C\}$ is directed. Invoke the compactness of C in $\mathcal{C}$ to find a finite set $M_0 \subseteq C$ such that $C \subseteq \Gamma_C(M_0)$. Thus, $C = \Gamma_C(M_0)$, since the reverse inclusion holds from part (1) of Proposition 3.4. □

Now we will see that every locally algebraic intersection structure forms an algebraic L-domain.

Theorem 3.6. *Let* $\mathcal{C}$ *be a locally algebraic intersection structure on a set* X. *Then the dcpo* $(\mathcal{C}, \subseteq)$ *is an algebraic L-domain.*

Proof. By Lemma 3.5, the dcpo $(\mathcal{C}, \subseteq)$ is an algebraic domain in which the compact elements are of the form $\Gamma_C(M)$, where $C \in \mathcal{C}$ and M is a finite subset of C. So it suffices to show that the family:

$$\downarrow B = \{C \in \mathcal{C} \mid C \subseteq B\}$$

with set inclusion is a complete lattice for every $B \in \mathcal{C}$.

Condition (L2) implies that the family $\downarrow B$ is closed under non-empty intersections, and $B \in \downarrow B$ guarantees that $\downarrow B$ is closed under the empty intersection. Then, $\downarrow B$ is a complete lattice in which the infimum of every subfamily is given by set intersection. □

The converse of Theorem 3.6 does not hold.

Example 3.7. *Let* X *denote the set* $\mathbb{N} \cup \{-1\}$, *where* $\mathbb{N}$ *is the set of natural numbers. Define a family* $\{A_i \mid i \in \mathbb{N}\}$ *by:*

$$A_0 = \{0\}, A_1 = \{0, 1\}, A_2 = \{0, 1, 2\}, \cdots, A_n = \{0, 1, 2, \cdots, n\}, \cdots.$$

Then it is ready to see that $(\mathcal{C}, \subseteq)$ *is an algebraic L-domain, where the family* $\mathcal{C}$ *is given by:*

$$\mathcal{C} = \{A_i \mid i \in \mathbb{N}\} \cup \{X\}.$$

However, $\mathcal{C}$ *is not a locally algebraic intersection structure, since the union of the directed family* $\{A_0, A_1, A_2, \cdots, A_n, \cdots\}$ *is equal to* $\mathbb{N}$, *which is not an element of* $\mathcal{C}$.

Theorem 3.8. (Representation theorem). *Let* $(L, \leq)$ *be an algebraic L-domain. Then there is a locally algebraic intersection structure that is order-isomorphic to* L.

https://doi.org/10.1017/S0960129524000069 Published online by Cambridge University Press

Proof. For every $a \in L$, we denote by D_a the set of all compact elements k of L with $k \leq a$, that is

$$D_a = \{k \in \mathcal{K}(L) \mid k \leq a\}. \tag{4}$$

Define $\mathcal{C}_L$ by

$$\mathcal{C}_L = \{D_a \mid a \in L\}. \tag{5}$$

We first verify that the family $\mathcal{C}_L$ ordered by set inclusion is order-isomorphic to the algebraic L-domain $(L, \leq)$.

Define a function $f : L \to \mathcal{C}_L$ by

$$f(a) = D_a. \tag{6}$$

Obviously, the function f is well defined and surjective. Assume $D_a \subseteq D_b$, where $a, b \in L$. As L is an algebraic L-domain, it follows that $a = \bigvee D_a$ for every $a \in L$. Thus,

$$a = \bigvee D_a \leq \bigvee D_b = b.$$

Conversely, suppose that $a \leq b \in L$. Then, it is clear that $D_a \subseteq D_b$, by Equation (4). Therefore, the function f is an order isomorphism.

We next show that $\mathcal{C}_L$ is a locally algebraic intersection structure.

Let $\{D_{a_i} \in \mathcal{C}_L \mid i \in I\}$ be a directed subfamily in $\mathcal{C}_L$, where D_{a_i} are defined by Equation (4). As we have seen that the function $f : L \to \mathcal{C}_L$ defined by Equation (6) is an order isomorphism, the indexing set $\{a_i \in L \mid i \in I\}$ is a directed subset of L. Take $a = \bigvee_{i\in I} a_i$. Then, we have

$$\begin{aligned} k \in D_a &\Leftrightarrow k \in \mathcal{K}(L) \text{ and } k \leq a = \bigvee{}_{i\in I} a_i \\ &\Leftrightarrow k \leq a_i \text{ for some } i \in I \\ &\Leftrightarrow k \in \bigcup{}_{i\in I} D_{a_i}. \end{aligned}$$

This implies that $\bigcup_{i\in I} D_{a_i}$ is equal to D_a and hence is an element of $\mathcal{C}_L$.

Let $D_a \in \mathcal{C}_L$, where $a \in L$. Suppose that $\{D_{a_j} \in \mathcal{C}_L \mid j \in J\}$ is a subfamily of $\mathcal{C}_L$ such that $D_{a_j} \subseteq D_a$ for every $j \in J$. Then $\{a_j \mid j \in J\}$ is a subset of L and $a_j \leq a$ for every $j \in J$. Since $(L, \leq)$ is an algebraic L-domain, the set $\downarrow a = \{x \in L \mid x \leq a\}$ is a complete lattice in the induced ordering. This implies that the set $\{a_j \mid j \in J\}$ has an infimum in $\downarrow a$, say a_0. We claim that $\bigcap_{j\in J} D_{a_j} = D_{a_0}$. Indeed,

$$\begin{aligned} k \in \bigcap{}_{j\in J} D_{a_j} &\Leftrightarrow k \in \mathcal{K}(L) \text{ and } k \in D_{a_j} \text{ for all } j \in J \\ &\Leftrightarrow k \in \mathcal{K}(L) \text{ and } k \leq a_j \text{ for all } j \in J \\ &\Leftrightarrow k \in \mathcal{K}(L) \text{ and } k \leq a_0 \\ &\Leftrightarrow k \in D_{a_0}. \end{aligned}$$

Therefore, $\bigcap_{j\in J} D_{a_j} \in \mathcal{C}_L$. □

In the rest of this section, we will extend the representation of algebraic L-domains to a categorical equivalence. We have seen that every locally algebraic intersection structure ordered by set inclusion is an algebraic L-domain in which directed suprema are just directed unions. Let **LS** denote the category of locally algebraic intersection structures with Scott continuous functions between them. Then, **LS** is a full subcategory of **AL** (the category of algebraic L-domains with Scott continuous functions between them).

Theorem 3.9. *The categories **LS** and **AL** are equivalent.*

Proof. Since the category **LS** is a full subcategory of **AL**, the inclusion functor $\mathfrak{F}$ from **LS** to **AL** is full and faithful. With Theorem 3.8, it follows that the category **LS** is equivalent to that of **AL**. □

https://doi.org/10.1017/S0960129524000069 Published online by Cambridge University Press

4. Other Representations

In this section, we give an overview of representations of algebraic L-domains, relating locally algebraic intersection structures with some well-known formalisms from logic, information systems, closure spaces, and formal concept analysis.

4.1 Logical algebras

The most conspicuous of characterizing domains as logical theory includes the work of such scholars as Abramsky, Zhang, Chen, and Jung (Abramsky, 1991; Chen and Jung, 2006; Zhang, 1991). In Chen and Jung (2006), Chen and Jung built a framework of disjunctive propositional logic and showed how to use its Lindenbaum algebras to represent algebraic L-domains.

Definition 4.1. (Chen and Jung 2006) Let $(L, \wedge, 0_L, 1_L)$ be a semilattice with least element 0_L and greatest element 1_L.

1. (1) $x, y \in L$ are said to be disjoint if $x \wedge y = 0_L$.
2. (2) A subset $A \subseteq L$ is called disjoint provided that $x \wedge y = 0_L$ for all distinct elements x, y in A.
3. (3) The semilattice is called a D-semilattice provided that every disjoint subset $A \subseteq L$ has a supremum $\dot{\bigvee} A$, where we denote by $\dot{\bigvee} A$ the supremum of a disjoint set A.
4. (4) The semilattice is called a dD-semilattice provided that it is a D-semilattice and satisfies

$$x \wedge (\dot{\bigvee} A) = \dot{\bigvee}_{a \in A} (x \wedge a)$$

for all elements x in L and disjoint subsets A of L.

Definition 4.2. (Chen and Jung 2006) Consider a dD-semilattice $(L, \wedge, 0_L, 1_L)$.

1. (1) An element $a \in L$ is said to be coprime provided that, for every disjoint subset A of L, if $a \leq \dot{\bigvee} A$ then $a \leq x$ for some $x \in A$. We denote by $\mathrm{Cp}(L)$ the set of coprime elements.
2. (2) The dD-semilattice is said to be coprime-generated if for each $x \in L$, there is a unique disjoint subset $A \subseteq \mathrm{Cp}(L)$ such that $x = \dot{\bigvee} A$.

Definition 4.3. (Chen and Jung 2006) Let L be a coprime-generated dD-semilattice. A proper subset F of L is said to be a disjunctive completely prime filter provided that

- (pt1) $x \in F$ and $x \leq y \in L$ implies that $y \in F$;
- (pt2) $x, y \in F$ implies that $x \wedge y \in F$;
- (pt3) if A is a disjoint subset of L with $\dot{\bigvee} A \in F$, then there is some $a \in A$ such that $a \in F$.

The family of disjunctive completely prime filters is denoted by $\mathrm{pt}(L)$.

Theorem 4.4. (Chen 1997; Chen and Jung 2006) *Let L be a coprime-generated dD-semilattice. Then $\mathrm{pt}(L)$ forms an algebraic L-domain ordered by inclusion. Moreover, every algebraic L-domain can be generated in this way up to isomorphism.*

In Chen and Jung (2006), the authors built a logical system that is logical complete with respect to dD-semilattices. Theorem 4.4 therefore provides a logical characterization for algebraic L-domains. In their program, the family of disjunctive completely prime filters for a coprime-generated dD-semilattice plays a central role. Now we show that the family of disjunctive completely prime filters is a locally algebraic intersection structure.

https://doi.org/10.1017/S0960129524000069 Published online by Cambridge University Press

Theorem 4.5. *Let* $(L, \wedge, 0_L, 1_L)$ *be a coprime-generated dD-semilattice. Then* $\mathrm{pt}(L)$ *is a locally algebraic intersection structure.*

Proof. • $\mathrm{pt}(L)$ satisfies condition (L1).

For every directed family $\{F_i \in \mathrm{pt}(L) \mid i \in I\}$, we have to prove that $\bigcup_{i \in I} F_i$ belongs to $\mathrm{pt}(L)$ by verifying the conditions for a disjunctive completely prime filter. We illustrate this for condition pt(3), because the others are similar. Suppose that $\dot{\bigvee} A \in \bigcup_{i \in I} F_i$ for some disjoint subset A of L. Then there exists some $i_0 \in I$ such that $\dot{\bigvee} A \in F_{i_0}$. Using condition pt(3) for the disjunctive completely prime filter F_{i_0}, it follows that $a \in F_{i_0} \subseteq \bigcup_{i \in I} F_i$ for some $a \in A$.

• $\mathrm{pt}(L)$ satisfies condition (L2).

Let $\{F_j \mid j \in J\}$ be a non-empty subfamily of $\mathrm{pt}(L)$ contained in another disjunctive completely prime filter G. We denote by F for the intersection $\bigcap \{F_j \mid j \in J\}$. It is clear that F satisfies conditions pt(1) and pt(2). For condition pt(3), suppose that $\dot{\bigvee} A \in F$ for some disjoint subset A of L. Then $\dot{\bigvee} A \in F_j$ for every $j \in J$, and hence $\dot{\bigvee} A \in G$. This implies that there exists some $a \in A$ such that $a \in G$. We claim that $a \in F_j$ for all $j \in J$. Indeed, fixing $j \in J$, $\dot{\bigvee} A \in F_j$ implies that there is some $a_j \in A$ such that $a_j \in F_j$, since F_j is a disjunctive completely prime filter. If $a_j \neq a$, then $a_j \wedge a = 0_L$. Because both a and a_j are in G, it follows by condition pt(2) that $0_L \in G$. Thus, $G = L$, a contradiction. □

4.2 Information systems

In Scott (1982), D. Scott introduced information systems as a concrete representation for Scott domains which turns out to be of remarkable significance for understanding the relationship between program logic and denotational semantics. Since then, many similar information systems have been presented to capture other domains (Spreen et al., 2008; Spreen, 2021; Wang and Li, 2021; Wang et al., 2022; Wu et al., 2016). The following information systems were proposed by D. Spreen for algebraic domains.

Definition 4.6. (Spreen et al. 2008) Let A be a set, Con a family of finite subsets of A and $\vdash$ a binary relation from Con to A. Then, $(A, \mathrm{Con}, \vdash)$ is called an algebraic information system provided that, for every $M, N \in \mathrm{Con}$:

(I1) $a \in A \Rightarrow \{a\} \in \mathrm{Con}$,
(I2) $M \vdash a \Rightarrow M \cup \{a\} \in \mathrm{Con}$,
(I3) $N \subseteq M$ and $N \vdash a \Rightarrow M \vdash a$,
(I4) $(M \vdash N, N \vdash a) \Rightarrow M \vdash a$,
(I5) $M \vdash N \Rightarrow (\exists M_1 \in \mathrm{Con}) M \vdash M_1 \vdash M_1 \vdash N$,
(I6) $(\forall F \subseteq_{fin} A) M \vdash F \Rightarrow (\exists N \in \mathrm{Con})(M \vdash N, F \subseteq N)$,

where $M \vdash F$ means that $M \vdash b$ for all $b \in F$.

Let $(A, \mathrm{Con}, \vdash)$ be an algebraic information system. For every subset $X \subseteq A$, we define a subset $\overline{X} \subseteq A$ by:

$$\overline{X} = \{a \in A \mid (\exists M \in \mathrm{Con})(M \subseteq X, M \vdash a)\}. \tag{7}$$

Based on Spreen's information systems, Wu et al. defined a kind of information systems for algebraic L-domains.

https://doi.org/10.1017/S0960129524000069 Published online by Cambridge University Press

Definition 4.7. (Wu et al. 2016) An algebraic information system $(A, \mathrm{Con}, \vdash)$ is said to be an algebraic L-information system if, for every $M \in \mathrm{Con}$ and $F \subseteq_{fin} \overline{M}$, there is $N \in \mathrm{Con}$ such that

(IL1) $\overline{F} \subseteq \overline{N}$ and $N \subseteq \overline{M}$
(IL2) for every $M_1 \in Con$, $\overline{F} \subseteq \overline{M_1} \subseteq \overline{M}$ can always implies that $\overline{N} \subseteq \overline{M_1}$.

We call N an M-sup of F and denote the set of all M-sup of F by $\Sigma(M, F)$.

Definition 4.8. (Wu et al. 2016) Let $(A, \mathrm{Con}, \vdash)$ be an algebraic information system, a non-empty subset $S \subseteq A$ is called a state provided that the following conditions hold:

(S1) $\overline{S} \subseteq S$,
(S2) $(\forall F \subseteq_{fin} S)(\exists M \in \mathrm{Con})(M \subseteq S, M \vdash F)$.

As usual, we denote by $|\mathbf{A}|$ the set of all states of an algebraic information system $(A, \mathrm{Con}, \vdash)$.

Lemma 4.9. *(Wu et al. 2016) Let S be a state of an algebraic L-information system $(A, \mathrm{Con}, \vdash)$, $M_1, M_2 \in \mathrm{Con}$ and $F \subseteq_{fin} A$. If $M_1, M_2 \subseteq S$ and $F \subseteq \overline{M_1} \cap \overline{M_2}$, then $\overline{N_1} = \overline{N_2}$ for all $N_1 \in \Sigma(M_1, F)$ and $N_2 \in \Sigma(M_2, F)$.*

In Spreen et al. (2008, Proposition 32), Spreen et al. established a representation of algebraic domains by algebraic information systems; and in Wu et al. (2016, Theorems 3.1 and 3.3), Wu et al. provided a representation of continuous L-domains. As a direct consequence of these results, we have

Corollary 4.10. *If $(A, \mathrm{Con}, \vdash)$ is an algebraic L-information system, then $|\mathbf{A}|$ ordered by inclusion is an algebraic L-domain. Moreover, every algebraic L-domain can be generated in this way up to isomorphism.*

The following theorem tells us that this representation essentially defines a locally algebraic intersection structure.

Theorem 4.11. *Let $(A, \mathrm{Con}, \vdash)$ be an algebraic L-information system. Then $|\mathbf{A}|$ is a locally algebraic intersection structure.*

Proof.

- $|\mathbf{A}|$ satisfies condition (L1).

Suppose that the family $\{S_i \in |\mathbf{A}| \mid i \in I\}$ is directed. For every $a \in \overline{\bigcup_{i\in I} S_i}$, by Equation (7), there is $M \in \mathrm{Con}$ such that $M \subseteq_{fin} \bigcup_{i\in I} S_i$ and $M \vdash a$. Since $\{S_i \in |\mathbf{A}| \mid i \in I\}$ is directed, $M \subseteq_{fin} S_{i_0}$ for some $i_0 \in I$. Thus, $a \in \overline{S_{i_0}} \subseteq S_{i_0} \subseteq \bigcup_{i\in I} S_i$. This implies that $\bigcup_{i\in I} S_i$ satisfies condition (S1). To show that $\bigcup_{i\in I} S_i$ belongs to $|\mathbf{A}|$, it suffices to check that $\bigcup_{i\in I} S_i$ also satisfies condition (S2). For every $F \subseteq_{fin} \bigcup_{i\in I} S_i$, there is some $i_1 \in I$ such that $F \subseteq_{fin} S_{i_1}$. Using condition (S2) for the state S_{i_1}, it follows that $M \subseteq S_{i_1}$ and $M \vdash F$ for some $M \in \mathrm{Con}$. We thus find $M \in \mathrm{Con}$ that satisfies $M \subseteq \bigcup_{i\in I} S_i$ and $M \vdash F$. Condition (S2) follows.

- $|\mathbf{A}|$ satisfies condition (L2).

Suppose that $S \in |\mathbf{A}|$ and the family $\{S_j \in |\mathbf{A}| \mid S_j \subseteq S, j \in J\}$ is non-empty. We show that $\bigcap_{j\in J} S_j \in |\mathbf{A}|$ by checking that $\bigcap_{j\in J} S_j$ is non-empty and satisfies conditions (S1) and (S2).

Note that S_j is non-empty for every $j \in J$. Take $a_j \in S_j$. Then $\{a_j\} \in \mathrm{Con}$ and $\emptyset \subseteq_{fin} \overline{\{a_j\}} \subseteq \overline{S_j} \subseteq S_j$. By Definition 4.7, there is some $N \in \Sigma(\{a_j\}, \emptyset)$ such that $N \subseteq \overline{\{a_j\}}$. Thus, $\overline{N} \subseteq \overline{\{a_j\}} \subseteq \overline{S_j} \subseteq S_j$. This implies that $\overline{N} \subseteq \bigcap_{j\in J} S_j$, by Lemma 4.9. So $\bigcap_{j\in J} S_j$ is non-empty.

https://doi.org/10.1017/S0960129524000069 Published online by Cambridge University Press

If $a \in \overline{\bigcap_{j\in J} S_j}$, then there is some $M \in \text{Con}$ such that $M \vdash a$ and $M \subseteq \bigcap_{j\in J} S_j \subseteq S_j$ for every $j \in J$. Thus, $a \in \overline{S_j} \subseteq S_j$ for every $j \in J$, and hence $a \in \bigcap_{j\in J} S_j$. Condition (S1) follows. For condition (S2), let $F \subseteq_{fin} \bigcap_{j\in J} S_j$. Then, $F \subseteq_{fin} S_j$ for every $j \in J$. Using condition (S2) for $F \subseteq_{fin} S_j$, there is some $M_j \in \text{Con}$ such that $M_j \subseteq S_j$ and $M_j \vdash F$ for every $j \in J$. Let us fix $i \in J$ and $N_i \in \Sigma(M_i, F)$. By Lemma 4.9, $\overline{N_i} = \overline{N_j}$ for all $N_j \in \Sigma(M_j, F)$ and all $j \in J$. Note that $\overline{N_j} \subseteq \overline{S_j} \subseteq S_j$. It follows that $\overline{N_i} \subseteq \bigcap_{j\in J} S_j$. Since $N_i \vdash F$, there are $M_1, N_1 \in \text{Con}$ such that $N_i \vdash M_1 \vdash M_1 \vdash N_1$ and $F \subseteq N_1$ by conditions (I6) and (I5). Therefore, $M_1 \in \text{Con}$, $M_1 \subseteq \bigcap_{j\in J} S_j$ and $M_1 \vdash F$. □

4.3 Closure spaces

Closure spaces are often used to restructure lattices. A classical result is that closure spaces generate exactly all of complete lattices, which becomes an inspiring source for many mathematicians. The idea of representing other order structures in terms of closure spaces would be traced back to Birkhoff's representation theorem for finite distributive lattices (Birkhoff, 1937). Recently, Wu et al. (2021) developed the notion of an algebraic closure space to that of an algebraic L-closure space and generalized the representation theorem of finite distributive lattices to that of algebraic L-domains.

Definition 4.12. (Davey and Priestly 2002) Let X be a set. A function γ on $\mathcal{P}(X)$ is called a closure operator on X provided that for every $A, B \subseteq X$,

(C1) $A \subseteq \gamma(A)$,
(C2) $A \subseteq B \Rightarrow \gamma(A) \subseteq \gamma(B)$,
(C3) $\gamma(A) = \gamma(\gamma(A))$.

The set of all fixed points of γ is denoted as $\mathfrak{X}_\gamma$ and the pair $(X, \mathfrak{X}_\gamma)$ is called a closure space. Moreover, if for every $A \subseteq X$,

$$\gamma(A) = \bigcup\{\gamma(F) \mid F \subseteq_{fin} A\}, \tag{8}$$

then we call $(X, \mathfrak{X}_\gamma)$ is algebraic.

Definition 4.13. (Wu et al. 2021) Let $(X, \mathfrak{X}_\gamma)$ be an algebraic closure space. An element $C \in \mathfrak{X}_\gamma$ is called Finset-bounded provide that, for every $M \subseteq_{fin} C$, there is $c \in C$ such that $M \subseteq \gamma(c) \subseteq C$.

We denote by $\mathcal{S}(\mathfrak{X}_\gamma)$ the family of all FinSet-bounded subsets of $(X, \mathfrak{X}_\gamma)$.

Definition 4.14. (Wu et al. 2021) An algebraic closure space $(X, \mathfrak{X}_\gamma)$ is said to be an algebraic L-closure space provided that, for every $x \in X$ and $M \subseteq \gamma(x)$, there is $y \in \gamma(x)$ such that

(LC1) $M \subseteq \gamma(y)$;
(LC2) $z \in \gamma(x)$ and $M \subseteq \gamma(z)$ implies that $\gamma(y) \subseteq \gamma(z)$.

Theorem 4.15. (Wu et al. 2021) *Let $(X, \mathfrak{X}_\gamma)$ be an algebraic L-closure space. Then $\mathcal{S}(\mathfrak{X}_\gamma)$ ordered by inclusion forms an algebraic L-domain. Moreover, every algebraic L-domain can be generated in this way up to isomorphism.*

The above theorem demonstrates the capability of a closure space in representing algebraic L-domains. In fact, the family of all FinSet-bounded subsets of an algebraic L-closure space is a locally algebraic intersection structure.

Theorem 4.16. *Let $(X, \mathfrak{X}_\gamma)$ be an algebraic L-closure space. Then $\mathcal{S}(\mathfrak{X}_\gamma)$ is a locally algebraic intersection structure.*

Proof. • $\mathcal{S}(\mathfrak{X}_\gamma)$ satisfies condition (L1).

Suppose that $\{C_i \in \mathcal{S}(\mathfrak{X}_\gamma) \mid i \in I\}$ is a directed set. Then

$$\begin{aligned}\gamma(\bigcup{}_{i\in I}C_i) &= \bigcup\{\gamma(M) \mid M \subseteq_{fin} \bigcup{}_{i\in I}C_i\}\\ &= \bigcup\{\gamma(M) \mid M \subseteq_{fin} C_i \text{ for some } i \in I\}\\ &\subseteq \bigcup{}_{i\in I}\gamma(C_i).\end{aligned}$$

By condition (C2), it is clear that $\bigcup_{i\in I}\gamma(C_i) \subseteq \gamma(\bigcup_{i\in I}C_i)$. Thus, $\bigcup_{i\in I}C_i \in \mathfrak{X}_\gamma$.

For every $M \subseteq_{fin} \bigcup_{i\in I}C_i$, there is $i \in I$ such that $M \subseteq \gamma(C_i)$ since the set $\{C_i \in \mathcal{S}(\mathfrak{X}_\gamma) \mid i \in I\}$ is directed. Noting that C_i is a Finset-bounded set, it follows that $M \subseteq \gamma(c_i) \subseteq C_i \subseteq \bigcup_{i\in I}C_i$ for some $c_i \in C_i \subseteq \bigcup_{i\in I}C_i$. Therefore, $\bigcup_{i\in I}C_i \in \mathcal{S}(\mathfrak{X}_\gamma)$.

• $\mathcal{S}(\mathfrak{X}_\gamma)$ satisfies condition (L2).

Suppose $C \in \mathcal{S}(\mathfrak{X}_\gamma)$ and the set $\{C_j \in \mathcal{S}(\mathfrak{X}_\gamma) \mid C_j \subseteq C, j \in J\}$ is non-empty. For every $j \in J$, we have

$$\bigcap{}_{j\in J}C_j \subseteq C_j,$$

thus

$$\gamma(\bigcap{}_{j\in J}C_j) \subseteq \gamma(C_j) = C_j,$$

and hence $\gamma(\bigcap_{j\in J}C_j) \subseteq \bigcap_{j\in J} C_j$. By condition (C1), we have $\bigcap_{j\in J} C_j \subseteq \gamma(\bigcap_{j\in J} C_j)$. Therefore, $\bigcap_{j\in J} C_j = \gamma(\bigcap_{j\in J} C_j)$, which means that $\bigcap_{j\in J} C_j \in \mathfrak{X}_\gamma$.

Now let $M \subseteq_{fin} \bigcap_{j\in J} C_j$. Then $M \subseteq_{fin} C_j$ for all $j \in J$. Since C_j is a Finset-bounded set, it follows that $M \subseteq \gamma(c_j) \subseteq C_j$ some $c_j \in C_j$. Thus, there is $y_j \in \gamma(c_j)$ such that

(1) $M \subseteq \gamma(y_j)$;
(2) $\gamma(y_j) \subseteq \gamma(z_j)$, whenever $z_j \in \gamma(c_j)$ and $M \subseteq \gamma(z_j)$.

Take $k \in J$. Then $\gamma(y_k) \subseteq C_k$. We claim that $\gamma(y_i) = \gamma(y_k)$ for every $i \in I$.

In fact, according to the fact that C is an Finset-bounded set and $y_i, y_k \in C$, there is $c' \in C$ such that $\{y_i, y_k\} \subseteq \gamma(c') \subseteq C$. Then $M \subseteq_{fin} \gamma(c')$, and this implies that there is $y' \in \gamma(c')$ such that

(3) $M \subseteq \gamma(y')$;
(4) $\gamma(y') \subseteq \gamma(z')$, whenever $z' \in \gamma(c')$ and $M \subseteq \gamma(z')$.

Noting that $y_k \in \gamma(c')$ and $M \subseteq \gamma(y_k)$, by (4), we have $\gamma(y') \subseteq \gamma(y_k)$. Thus, $y' \in \gamma(y_k) \subseteq \gamma(c_k)$. It follows from $M \subseteq \gamma(y')$ that $\gamma(y_k) \subseteq \gamma(y')$ by (2). Therefore, $\gamma(y_k) = \gamma(y')$. Similarly, $\gamma(y_i) = \gamma(y')$.

By the above claim, we have $y_k \in C_i$ for every $i \in J$, and so $y_k \in \bigcap_{j\in J} C_j$ satisfying

$$M \subseteq \gamma(y_k) \subseteq \bigcap{}_{j\in J}C_j.$$

As a result, $\bigcap_{j\in J}C_j \in \mathcal{S}(\mathfrak{X}_\gamma)$. □

https://doi.org/10.1017/S0960129524000069 Published online by Cambridge University Press

4.4 Formal concept analysis

Formal concept analysis was introduced by R. Wille in the 1980s as a mathematical theory for the formalization of conceptual thinking (Ganter and Wille, 1999). A fundamental application of formal concept analysis is to restructure lattice theory, which needs the notion of a formal context.

A *formal context* is a triple $(G_o, G_a, \vDash_P)$, in which $\vDash_P$ is a binary relation from the set G_o to the set G_a. In this case, two operators can be defined as follows:

$$\alpha : \mathcal{P}(G_o) \to \mathcal{P}(G_a), A \mapsto \{n \in G_a \mid \forall m \in A, m \vDash n\}, \tag{9}$$

$$\omega : \mathcal{P}(G_a) \to \mathcal{P}(G_o), B \mapsto \{m \in G_o \mid \forall n \in B, m \vDash n\}. \tag{10}$$

Definition 4.17. (Guo et al. 2018) Let $(G_o, G_a, \vDash_P)$ be a formal context and $\mathcal{F}$ a non-empty family of non-empty finite subset of G_o. Then $(G_o, G_a, \vDash_P, \mathcal{F})$ is said to be a consistent F-augmented context if, for every $M \in \mathcal{F}$, there is a directed family of $\{M_i \in \mathcal{F} \mid i \in I\}$ such that $\omega \circ \alpha(M) = \bigcup_{i \in I} M_i$.

For every $A \subseteq G_o$, we denote by $\langle A \rangle$ the set $\bigcup\{\omega \circ \alpha(M) \mid M \in \mathcal{F}, M \subseteq_{fin} A\}$.

Definition 4.18. (Guo et al. 2018) A consistent F-augmented context $(G_o, G_a, \vDash_P, \mathcal{F})$ is said to be an LCF context if, for every $M \in \mathcal{F}$ and $F \subseteq_{fin} \omega \circ \alpha(M)$, there is $Z \in \mathcal{F}$ such that

(CF1) $\langle F \rangle \subseteq \omega \circ \alpha(Z) \subseteq \omega \circ \alpha(M)$;
(CF2) $Y \in \mathcal{F}$ and $\langle F \rangle \subseteq \omega \circ \alpha(Y) \subseteq \omega \circ \alpha(M)$ implies that $\omega \circ \alpha(Z) \subseteq \omega \circ \alpha(Y)$.

In this case, we call Z an M-cover of F.

Definition 4.19. (Guo et al. 2018) Let $(G_o, G_a, \vDash_P, \mathcal{F})$ be an LCF contexts. A subset X of G_o is said to be an F-approximable extent if for every $F \subseteq_{fin} X$, there is $M \in \mathcal{F}$ such that $F \subseteq \omega \circ \alpha(M) \subseteq X$.

We denote by $\mathfrak{C}(G)$ the family of all F-approximable extents of $(G_o, G_a, \vDash_G, \mathcal{F})$.

Theorem 4.20. (Guo et al. 2018) *If $(G_o, G_a, \vDash_G, \mathcal{F})$ is an LCF context, then $\mathfrak{C}(G)$ ordered by set inclusion forms an algebraic L-domain. Moreover, every algebraic L-domain can be generated in this way up to isomorphism.*

Theorem 4.20 restructures algebraic L-domains in terms of formal concept analysis. This method can also be included into the framework of locally algebraic intersection structures.

Theorem 4.21. *Let $(G_o, G_a, \vDash_G, \mathcal{F})$ be an LCF context. Then $\mathfrak{C}(G)$ is a locally algebraic intersection structure.*

Proof. • $\mathfrak{C}(G)$ satisfies condition (L1).

Suppose that the set $\{X_i \in \mathfrak{C}(G) \mid i \in I\}$ is directed. If $F \subseteq_{fin} \bigcup_{i \in I} X_i$, then $F \subseteq_{fin} X_i$ for some $i \in I$. Since X_i is an F-approximable extent, there is $M \in \mathcal{F}$ such that

$$F \subseteq_{fin} \omega \circ \alpha(M) \subseteq X_i \subseteq \bigcup{}_{i \in I} X_i.$$

Thus, $\bigcup_{i \in I} X_i$ is an F-approximable extent.

• $\mathfrak{C}(G)$ satisfies condition (L2).

Suppose that $X \in \mathfrak{C}(G)$ and the set $\{X_i \in \mathfrak{C}(G) \mid X_i \subseteq X, i \in I\}$ is non-empty. If $F \subseteq_{fin} \bigcap_{i \in I} X_i$, then $F \subseteq_{fin} X_i$ for all $i \in I$. Thus, there are $M_i \in \mathcal{F}$ such that $F \subseteq_{fin} \omega \circ \alpha(M_i) \subseteq X_i$. For every $i \in I$,

https://doi.org/10.1017/S0960129524000069 Published online by Cambridge University Press

let Z_i be an M_j-cover of F. Pick $k \in I$. Clearly, $F \subseteq \langle F \rangle \subseteq_{fin} \omega \circ \alpha(M_k)$. Setting $M_i' = M_k \cup M_i$, we pick an M_i'-cover of F, say Z_i'. Since

$$\langle F \rangle \subseteq \omega \circ \alpha(Z_k) \subseteq \omega \circ \alpha(M_k) \subseteq \omega \circ \alpha(M_i'),$$

it follows that $\omega \circ \alpha(Z_i') \subseteq \omega \circ \alpha(Z_k)$. This implies that

$$\langle F \rangle \subseteq \omega \circ \alpha(Z_i') \subseteq \omega \circ \alpha(Z_k) \subseteq \omega \circ \alpha(M_k).$$

Using condition (CF2) for Z_k, we have $\omega \circ \alpha(Z_k) \subseteq \omega \circ \alpha(Z_i')$. Therefore, $\omega \circ \alpha(Z_i') = \omega \circ \alpha(Z_k)$. Similarly, $\omega \circ \alpha(Z_i') = \omega \circ \alpha(Z_i)$. This implies that $\omega \circ \alpha(Z_k) = \omega \circ \alpha(Z_i) \subseteq X_i$ for all $i \in I$. To sum up, we have found $M_k \in \mathcal{F}$ such that $F \subseteq_{fin} \omega \circ \alpha(M_k) \subseteq \bigcap_{i \in I} X_i$. Thus, $\bigcap_{i \in I} X_i$ is an F-approximable extent. □

5. Conclusion

This paper generalizes the notion of (topped) algebraic intersection structures to that of locally algebraic intersection structures. Just as topped algebraic intersection structures are set-theoretic representations of algebraic lattices and algebraic intersection structures are set-theoretic representations of Scott domains, locally algebraic intersection structures are set-theoretic representations of algebraic L-domains. This gives a new insight into exposing the difference and relationship among algebraic L-domains, Scott domains, and algebraic lattices. What is more important, this makes the mathematical foundation for denotational semantics more comprehensible to a broad audience by showing how the abstract and complex domain theory can be based on an elementary set-theoretic form.

This paper also establishes a uniformity across four existing well-known representations of algebraic L-domains, including Chen and Jung's logical algebras, Wu et al.'s information systems, Wu et al.'s closure spaces, and Guo et al.'s formal concept analysis. Beyond these four representations, there are many approaches to representing algebraic L-domains, such as Wang and Li's locally continuous sequent calculi (Wang and Li, 2024), Spreen et al.'s generalized information systems (Spreen, 2021), Wang and Li's interpolative generalized closure spaces (Wang and Li, 2023), and Huang et al.'s formal contexts (Huang et al., 2014). Formally, there is a one-to-one correspondence between the four different representations we discussed in Section 4 and those we mentioned here, respectively. Then by an argument similar to that given in Section 4, it is not difficult to see that all the representations that were mentioned here also have a close relationship with locally algebraic intersection structures. So the four different representations have been selected to investigate in Section 4 for the representativeness and typicalness they provided.

There are many representations of Scott domains and algebraic lattices by logical algebras, formal concept analysis, information systems, and closure spaces, respectively (Abramsky, 1987; Hitzler et al., 2006; Jung et al., 1991; Wang and Li, 2023). According to the discussion of Section 4, it is natural to ask whether these representations are essentially an algebraic intersection structure. This is an interesting topic for further study, and we think it has a huge possibility of being right.

To our knowledge, there are few results on purely set-theoretic representations of various domain structures except for algebraic L-domains and Scott domains. The study of the paper now opens a possibility that one could find more set-theoretic representations for other subclasses of domains. For example, the case of Lawson compact algebraic L-domains may be solved by generalizing the notions of algebraic intersection structures and locally algebraic intersection structures, since they are a subclass of domains between algebraic L-domains and Scott domains.

References

Abramsky, S. (1987). *Domain Theory and the Logic of Observable Properties*. Phd thesis, University of London.
Abramsky, S. (1991). Domain theory in logical form. *Annals of Pure and Applied Logic* **51** 1–77.

https://doi.org/10.1017/S0960129524000069 Published online by Cambridge University Press

Birkhoff, G. (1937). Rings of sets. *Duke Mathematical Journal* **3** (3) 443–454.
Chen, Y. (1997). Stone duality and representation of stable domain. *Computers and Mathematics with Applications* **34** (1) 27–41.
Chen, Y. and Jung, A. (2006). A logical approach to stable domains. *Theoretical Computer Science* **368** 124–148.
Davey, B. A. and Priestly, H. A. (2002). *Introduction to Lattices and Order*, Cambridge, Cambridge University Press.
Ganter, B. and Wille, R. (1999). *Formal Concept Analysis*, Berlin, Springer-Verlag.
Gierz, G., Hofmann, K. H., Keimel, K., Lawson, J. D., Mislove, M. and Scott, D. S. (2003). *Continuous Lattices and Domains*, Cambridge, Cambridge University Press.
Guo, L., Li, Q. and Yao, L. (2018). Locally complete consistent F-augmented contexts: A category-theoretic representation of algebraic L-domains. *Discrete Applied Mathematics* **249** 53–63.
He, Q. and Xu, L. (2019). Weak algebraic information systems and a new equivalent category of DOM of domains. *Theoretical Computer Science* **763** 1–19.
Hitzler, P., Kröetzsch, M. and Zhang, G. (2006). A categorical view on algebraic lattices in formal concept analysis. *Fundamenta Informaticae* **74** 1–29.
Huang, M., Li, Q. and Guo, L. (2014). Formal contexts for algebraic domains. *Electronic Notes in Theoretical Computer Science* **301** 79–90.
Jung, A. (1989). *Cartesian Closed Categories of Domains*, CWI Tracts, vol. 66, Amesterdam, Centrum voor Wiskunde en Informatica.
Jung, A., Kegelmann, M. and Moshier, M. A. (1999). Multi lingual sequent calculus and coherent spaces. *Fundamenta Informaticae* **37** 369–412.
Larsen, K. G. and Winskel, G. (1991). Using information systems to solve recursive domain equations effectively. *Information and Computation* **91** 232–258.
Rounds, W. and Zhang, G. (2001). Clausal logic and logic programming in algebraic domains. *Information and Computation* **171** 183–200.
Scott, D. S. (1982). Domains for denotational semantics. *Lecture Notes in Computer Science* **140** 577–613.
Spreen, D., Xu, L. and Mao, X. (2008). Information systems revisited: The general continuous case. *Theoretical Computer Science* **405** 176–187.
Spreen, D. (2021). Generalised information systems capture L-domains. *Theoretical Computer Science* **869** 1–28.
Stone, M. H. (1936). The theory of representations for Boolean algebras. *Transactions of the American Mathematical Society* **40** 37–111.
Vickers, S. (2004). Entailment systems for stably locally compact locales. *Theoretical Computer Science* **316** 259–296.
Wang, L. and Li, Q. (2021). Representations of stably continuous semi-lattices by information systems and abstract bases. *Information Processing Letters* **165** 106036.
Wang, L. and Li, Q. (2022). Bounded complete domains and their logical form. *Information and Computation* **289** 104958.
Wang, L. and Li, Q. (2023). The categorical equivalence between domains and interpolative generalized closure spaces. *Studia Logica* **111** 187–215.
Wang, L. and Li, Q. (2024). L-domains as locally continuous sequent calculi. *Archive for Mathematical Logic*, http://doi.org/10.1007/s00153-023-00903-4.
Wang, L., Zhou, X. and Li, Q. (2022). Information systems for continuous semi-lattices. *Theoretical Computer Science* **913** 138–150.
Wu, M., Guo, L. and Li, Q. (2016). A representation of L-domains by information systems. *Theoretical Computer Science* **612** 126–136.
Wu, M., Guo, L. and Li, Q. (2021). New representations of algebraic domains and algebraic L-domains via closure systems. *Semigroup Forum* **103** 700–712.
Zhang, G.-Q. (1991). *Logic of Domains*, Boston, Birkhauser.

Cite this article: Zou J, Zhao Y, Miao C and Wang L (2024). A set-theoretic approach to algebraic L-domains. *Mathematical Structures in Computer Science* **34**, 244–257. https://doi.org/10.1017/S0960129524000069