Exact structures for operator modules

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Abstract. We demonstrate how exact structures can be placed on the additive category of right
operator modules over an operator algebra in order to discuss global dimension for operator
algebras. The properties of the Haagerup tensor product play a decisive role in this.

1 Introduction

Among the most important operator space modules over C^*-algebras are the Hilbert
C^*-modules, the operator modules and the matrix normed modules. The first class
became prominent through the work of Paschke and Rieffel and was intensively
studied by Blecher, among others, see, e.g., [5, 6]. The main difference between
the second and the third class lies in the kind of complete boundedness which is
required of the bilinear mappings that give the module action. For operator mod-
ules (we will follow the terminology of [6] in this paper), one demands (multi-
plicatively) completely bounded bilinear mappings, and the associated tensor prod-
uct is the Haagerup module tensor product $\otimes_{hA}$ over the C^*-algebra A. Jointly
completely bounded bilinear mappings and the module operator space projective
tensor product $\hat{\otimes}_A$ govern the class of matrix normed modules; for details, we
refer to [6, Chapter 3]. Both these classes have been put to good use and found a
range of interesting applications; we only mention the recent papers [4, 10–12] as
samples.

For historical reasons the terminology is (still) not uniform; we will follow the
convention in [6], see also [28, Appendix A]. Suffice it to say at this point that operator
modules form a full subcategory of the matrix normed modules (over any operator
algebra). These categories are not abelian and therefore the usual homological algebra
does not apply directly. Nevertheless, homological methods have been successfully
developed in this framework, for example by Helemskii and his school. In [15, Theo-
rem III.5.17 and Corollary IV.4.12] Helemskii proved that a C^*-algebra A is classically
semisimple (i.e., finite dimensional) if and only if all objects in the category $\mathscr{B}an_A^\infty$ of
Banach A-modules are projective with respect to the class of epimorphisms that split
as morphisms in $\mathscr{B}an^\infty$. See also [2, 3, 16].

For an operator algebra A (on a Hilbert space) it is pertinent to use operator
modules to build a cohomology theory (for the definitions, see Section 2.2); of the

numerous contributions, we only mention [7, 16, 24, 32, 34] here. In this paper, we focus on an appropriate definition of cohomological dimension and, in particular, answer a question raised by Helemskii [16] whether quantized global dimension zero is equivalent to the algebra being classically semisimple; see Theorem 5.3. In contrast to the situation in ring theory, it appears necessary to limit ourselves to a relative cohomology theory since, otherwise, there exist too many monomorphisms (equivalently, epimorphisms) and the concepts of injectivity (respectively, projectivity) become too restrictive. For example, the canonically defined injective global dimension of any C^*-algebra with regard to the category of its operator modules always is at least 2, as shown in [28]. Paulsen discussed relative cohomology in [24] and related it to completely bounded Hochschild cohomology. His notions of relative injectivity and projectivity coincide with ours (defined in Section 4) and Helemskii's approach is also compatible.

The main novelty in this paper is the systematic use of exact categories enabling us to bring category theory to the foreground in order to take advantage of its unifying strength. Typically, categories that appear in analysis are not abelian which has hindered the full application of homological algebra methods. Exact structures in the sense of Quillen [27] (to be defined in Section 4) on additive categories like $\mathscr{B}an^\infty$ were employed by Bühler in [9], see also [8] for a very nice introduction. The use of exact structures for categories of operator modules, and indeed sheaves of operator modules over C^*-ringed spaces, was initiated in [1], see also the survey articles [19, 20], and further studied in [28]. Exact categories are ideally suited for the important tools of homological algebra such as (short) exact sequences, diagram lemmas, and derived functors.

Not assuming an in-depth knowledge of category theory (as this article is written for functional analysts), we have included the main categorical concepts deployed throughout this paper in Section 2.1. In Section 2.2, our assumptions on operator spaces, operator algebras. and the type of operator space modules we work with are stated.

In Section 3, we show how the categorical notions of kernels and cokernels can be expressed, using language and concepts of operator space theory, in the additive category $\mathscr{O}\mathscr{M}od_A^\infty$ of nondegenerate right operator A-modules over an operator algebra A. This prepares Section 4 in which we explore the concept of exact categories and some techniques for working in this setting. We show that there is a canonical exact structure that can be placed on our main category $\mathscr{O}\mathscr{M}od_A^\infty$ (Theorem 4.4) and discuss how a global dimension arises in exact categories with enough injectives.

Section 5 deals with another exact structure that can be placed on $\mathscr{O}\mathscr{M}od_A^\infty$. This structure is related to the study of "relative homological algebra" for operator modules as it is done in [3, 16, 24], for example. Our main Theorem 5.3 states that, for a unital operator algebra A, the global dimension of this exact category is zero if and only if A is classically semisimple, hence a finite direct sum of full matrix algebras over the complex numbers. We also discuss similarities and differences between our preferred category, $\mathscr{O}\mathscr{M}od_A^\infty$, and $mn\mathscr{M}od_A^\infty$, the category of matrix normed modules over an operator algebra A in this section.

2 Terminology, notation, and conventions

2.1 Category theory

Let $\mathscr{A}$ be a category. To indicate that E belongs to the class of objects in $\mathscr{A}$, we write $E \in \mathscr{A}$. The set of morphisms from $E \in \mathscr{A}$ to $F \in \mathscr{A}$ is denoted $\mathrm{Mor}_{\mathscr{A}}(E, F)$, with the $\mathscr{A}$ sometimes dropped if the category we are working in is obvious. If $\mathscr{M}$ is a class of morphisms in $\mathscr{A}$, we denote the subset of $\mathrm{Mor}_{\mathscr{A}}(E, F)$ that consists only of the morphisms in $\mathscr{M}$ by $\mathscr{M}(E, F)$. Two of the most important classes of morphisms in a category are the classes of monomorphisms and of epimorphisms. Recall that a morphism f in $\mathscr{A}$ is a *monomorphism* if it is "left cancellable," that is, if g, h are morphisms in $\mathscr{A}$, composable with f, such that $fg = fh$, then we must have $g = h$. The "right cancellable" morphisms in a category are *epimorphisms*. An important class of epimorphisms are the retractions. A morphism $r \in \mathrm{Mor}_{\mathscr{A}}(E, F)$ is a *retraction* if there exists a morphism $s \in \mathrm{Mor}_{\mathscr{A}}(F, E)$ such that $rs = \mathrm{id}_F$, the identity morphism of F. In this case, we say s is a *section* of r and F is a *retract* of E.

The categories we work in will contain a zero object and, therefore, zero morphisms. Thus, we can talk about kernels and cokernels.

Definition 2.1 Let $\mathscr{A}$ be a category with a zero object. Suppose $E, F \in \mathscr{A}$ and $f \in \mathrm{Mor}(E, F)$.

A *kernel* of f is a pair (K, μ), where $K \in \mathscr{A}$ and $\mu \in \mathrm{Mor}(K, E)$ with $f\mu = 0$ such that, when $G \in \mathscr{A}$ and $g \in \mathrm{Mor}(G, E)$ satisfies $fg = 0$, there exists a unique morphism $\overline{g} \in \mathrm{Mor}(G, K)$ making the following diagram commutative

(2.1)

A *cokernel* of f is a pair (C, π), where $C \in \mathscr{A}$ and $\pi \in \mathrm{Mor}(F, C)$ such that $\pi f = 0$ and, whenever $G \in \mathscr{A}$ and $g \in \mathrm{Mor}(F, G)$ satisfy $gf = 0$, there exists a unique morphism $\overline{g} \in \mathrm{Mor}(C, G)$ making the following diagram commutative

(2.2)

Remark 2.1 It is easy to see that kernel morphisms must be monomorphisms. We also note that the property of being a kernel of a morphism is universal. That is, in Diagram 2.1, (G, g) is a kernel for f if and only if $\overline{g}$ is an isomorphism.

Similarly, cokernel morphisms are epimorphisms and cokernel objects are unique up to isomorphism.

It is not difficult to show that, in a category $\mathscr{A}$ where every morphism has a kernel and a cokernel, a morphism is a kernel if and only if it is the kernel of its cokernel and is a cokernel if and only if it is the cokernel of its kernel. Moreover, let the following diagram in $\mathscr{A}$ be commutative.

(2.3)
$$
\begin{array}{ccccc}
K & \xrightarrow{\;\mu\;} & E & \xrightarrow{\;\pi\;} & C \\
\downarrow & & \downarrow & & \downarrow \\
F & \xrightarrow{\;f\;} & E' & \xrightarrow{\;g\;} & G
\end{array}
$$

Suppose μ is a kernel of π and π is a cokernel of μ and that all of the vertical arrows are isomorphisms. Then, by the universal properties of kernels and cokernels, f is a kernel of g and g is a cokernel of f.

Definition 2.2   Let $\mathscr{A}$ be a category.

Suppose $\mathscr{M}$ is a class of monomorphisms in $\mathscr{A}$, closed under composition and such that every isomorphism in $\mathscr{A}$ is in $\mathscr{M}$. An object $I \in \mathscr{A}$ is $\mathscr{M}$-*injective* if, when given $\mu \in \mathscr{M}(E, F)$ and $f \in \mathrm{Mor}_{\mathscr{A}}(E, I)$, for objects $E, F \in \mathscr{A}$, there exists a morphism $g \in \mathrm{Mor}_{\mathscr{A}}(F, I)$ making the following diagram commutative

(2.4)
$$
\begin{array}{ccc}
E & \xrightarrow{\;\mu\;} & F \\
{\scriptstyle f}\downarrow & \swarrow{\scriptstyle g} & \\
I & &
\end{array}
$$

Suppose $\mathscr{P}$ is a class of epimorphisms in $\mathscr{A}$, closed under composition and such that every isomorphism in $\mathscr{A}$ is in $\mathscr{P}$. An object $P \in \mathscr{A}$ is $\mathscr{P}$-*projective* if, when given $\pi \in \mathscr{P}(E, F)$ and $f \in \mathrm{Mor}_{\mathscr{A}}(P, F)$, for objects $E, F \in \mathscr{A}$, there exists a morphism $g \in \mathrm{Mor}_{\mathscr{A}}(P, E)$ making the following diagram commutative

(2.5)
$$
\begin{array}{ccc}
& & P \\
& {\scriptstyle g}\swarrow & \downarrow{\scriptstyle f} \\
E & \xrightarrow{\;\pi\;} & F
\end{array}
$$

We say $\mathscr{A}$ has *enough $\mathscr{M}$-injectives* (respectively, *enough $\mathscr{P}$-projectives*) if, for every $E \in \mathscr{A}$, there exists an $\mathscr{M}$-injective object I (respectively, $\mathscr{P}$-projective object P) such that $\mathscr{M}(E, I) \neq \varnothing$ (respectively, $\mathscr{P}(P, E) \neq \varnothing$).

Remark 2.2   Fix a category $\mathscr{A}$ and classes $\mathscr{M}$ and $\mathscr{P}$ of morphisms. It is easy to see that any retract of an $\mathscr{M}$-injective object must be $\mathscr{M}$-injective and every retract of a $\mathscr{P}$-projective object must be $\mathscr{P}$-projective. Moreover, if I is $\mathscr{M}$-injective and there exists a morphism $\mu \in \mathscr{M}(I, E)$, then μ must be a section of some retraction $r \in \mathrm{Mor}_{\mathscr{A}}(E, I)$. If P is $\mathscr{P}$-projective and $\pi \in \mathscr{P}(E, P)$, then π must be a retraction.

We will need the notion of kernels and cokernels when we talk about exact categories and the notion of injectives and projectives when we discuss their global dimensions in Section 4.

2.2 Operator spaces

In the background, we will be using Ruan's Representation Theorem [13, Theorem 2.3.5]. Hence, we will not distinguish between spaces arising as subspaces of $B(H)$, the bounded operators on a Hilbert space H, and matrix normed spaces satisfying Ruan's axioms (see, for example, [6, 1.2.12]). However, when we refer to *operator spaces*, we will mean spaces of this type that we also assume to be complete. If E is an operator space we will write $x = [x_{ij}] \in M_n(E)$ to say that x is an $n \times n$ matrix with entries x_{ij}, $i, j \in \{1, \dots, n\}$ in E, and $\|x\|_n$ is the norm of x in the Banach space $M_n(E)$, whose norm is inherited by the matrix norm of E.

When E, F are operator spaces, we will denote the operator space consisting of all completely bounded linear maps from E to F by $CB(E, F)$. The completely bounded norm of an element $\phi \in CB(E, F)$ is $\|\phi\|_{\mathrm{cb}}$.

By A being an *operator algebra* we will mean A is a closed subalgebra of $B(H)$ of some Hilbert space H such that A contains a contractive approximate identity. By the Blecher–Ruan–Sinclair Theorem [6, Theorem 2.3.2], we need not distinguish between A being an operator algebra and A being an operator space which is also a Banach algebra with a contractive approximate identity such that the map $A \otimes_h A \to A$ from the Haagerup tensor product of A with itself to A induced by the multiplication on A is completely contractive.

A complex algebra is said to be *classically semisimple* if it is a direct sum of minimal right ideals and if it is finitely generated, finitely many minimal right ideals suffice. Hence, combining the Artin–Wedderburn theorem with the Gelfand–Mazur theorem it follows that a unital complex Banach algebra is classically semisimple if and only if it is the direct sum of finitely many full matrix algebras over the complex numbers; in particular, it is finite dimensional.

Definition 2.3 Let A be an operator algebra. We say that a right A-module E that is also an operator space is a *right operator A-module* if the map $E \otimes_h A \to E$, induced by the module action, is completely contractive.

It follows from the associativity of the Haagerup tensor product that, for any operator space E, $E \otimes_h A$ is a right operator A-module; this works analogous to [6, 3.1.5 (3)].

Definition 2.4 Let E be a right Banach A-module. If the linear span of elements of the form $x \cdot a$, where $x \in E$, $a \in A$, is dense in E then we say E is *nondegenerate*. By Cohen's Factorization Theorem [6, Theorem A.6.2], this happens if and only if, for each $x \in E$, there exist $x' \in E$, $a \in A$ such that $x = x' \cdot a$.

We will always restrict ourselves to categories of nondegenerate operator modules, and unital modules if the algebra is unital. If E is an nondegenerate right operator A-module and F is a closed submodule of E then F and E/F become nondegenerate right operator A-modules when equipped with the standard induced operator space structures and module actions [6, Section 3.1].

For a Banach space E and some $\gamma > 0$, we will denote $\{x \in E \mid \|x\| < \gamma\}$ by $E_{\|\cdot\| < \gamma}$. Recall that a linear map $f : E \to F$ between Banach spaces is an open map if and only there exists $\gamma > 0$ such that $f(E_{\|\cdot\| < 1}) \supseteq F_{\|\cdot\| < \gamma}$. If E and F are, moreover, operator spaces we say that f is *completely open* if there exists some common γ such that each amplification f_n is open and for each $n \in \mathbb{N}$, $f_n(M_n(E)_{\|\cdot\| < 1}) \supseteq M_n(F)_{\|\cdot\| < \gamma}$. These maps can be characterized in the following way.

Proposition 2.3 *Let E and F be operator spaces. Then $f \in CB(E, F)$ is completely open if and only if there exists $\lambda > 1$ such that, for each $n \in \mathbb{N}$, every $y \in M_n(F)$ is equal to $f_n(x)$ for some $x \in M_n(E)$ with $\|x\|_n \leq \lambda \|y\|_n$.*

For a completely open map f, we will refer to λ as in Proposition 2.3 as an *openness constant* for f.

Example 2.4 When E is an operator space and F is a closed subspace, then the canonical projection $\pi \in CB(E, E/F)$ is a completely open map and any $\lambda > 1$ is an openness constant.

Injectivity and self-duality of the Haagerup tensor product provide us with the following useful result.

Lemma 2.5 *Let E and F be operator spaces and let $u \in E \otimes_h F$ be nonzero. There exist bounded linear functionals $\alpha \in E^*$, $\beta \in F^*$ such that $(\alpha \otimes \beta)(u) \neq 0$.*

This follows immediately from the completely isometric embeddings

$$E \otimes_h F \hookrightarrow E^{**} \otimes_h F^{**} \hookrightarrow (E^* \otimes_h F^*)^*$$

given by $(x \otimes y)(\alpha \otimes \beta) = \alpha(x)\beta(y)$ for $x \in E$, $y \in F$, $\alpha \in E^*$ and $\beta \in F^*$, see [13, Proposition 9.2.5 and Theorem 9.4.7].

3 The additive category of operator modules

It is well known how the notion of the global dimension for module categories extends to the setting of abelian categories with enough injectives (or enough projectives). See, e.g., [18, 22] or [23]. In this section, we will see to what extent the canonical additive category of nondegenerate (right) operator A-modules over an operator algebra A fails to be abelian. In order to remedy this by introducing an exact structure in the next section, we need to study the kernels and cokernels in this category in detail.

We first recall the definition of an additive category.

Definition 3.1 A category $\mathscr{A}$ is *additive* if $\mathscr{A}$ has a zero object; morphism sets have the structure of abelian groups; composition is distributive with respect to this abelian group structure; and a product exists for each pair of objects.

For $E, F \in \mathscr{A}$, where $\mathscr{A}$ is additive, we denote by $\mathrm{Hom}_{\mathscr{A}}(E, F)$ the morphism set equipped with the abelian group structure. A functor $\mathsf{F} : \mathscr{A} \to \mathscr{B}$ is *additive* if, when $E, F \in \mathscr{A}$, $\mathsf{F}(f + g) = \mathsf{F}(f) + \mathsf{F}(g)$ for all morphisms $f, g \in \mathrm{Hom}_{\mathscr{A}}(E, F)$. Note that, in an additive category, $E, F \in \mathscr{A}$ have a product $G \in \mathscr{A}$ if and only if G is also their coproduct. Moreover, this happens precisely when G is a *direct sum* of E and F, that is, there exists a quintuplet $(G, \iota_E, \iota_F, \pi_E, \pi_F)$, where $\iota_E \in \mathrm{Hom}(E, G)$, $\iota_F \in \mathrm{Hom}(F, G)$,

and $\pi_E \in \mathrm{Hom}(G, E)$, $\pi_F \in \mathrm{Hom}(G, F)$, such that $\pi_E \iota_E = \mathrm{id}_E$, $\pi_F \iota_F = \mathrm{id}_F$ and $\iota_E \pi_E + \iota_F \pi_F = \mathrm{id}_G$.

For the remainder of this section, we fix an operator algebra A. We will use $\mathscr{O}\mathscr{M}od_A^\infty$ to denote the category whose objects are the nondegenerate right operator A-modules and whose morphisms are the completely bounded A-module maps. Instead of $\mathscr{O}\mathscr{M}od_{\mathbb{C}}^\infty$, we write $\mathscr{O}p^\infty$. For $E, F \in \mathscr{O}\mathscr{M}od_A^\infty$, we denote the set $\mathrm{Mor}_{\mathscr{O}\mathscr{M}od_A^\infty}(E, F)$ by $CB_A(E, F)$, and by $CB(E, F)$ if $A = \mathbb{C}$. It is clear that this is an abelian group.

Proposition 3.1 *Let A be an operator algebra. The category $\mathscr{O}\mathscr{M}od_A^\infty$ is additive.*

The only part of Proposition 3.1 that is perhaps not immediately apparent is the existence of a product for each pair of objects in $\mathscr{O}\mathscr{M}od_A^\infty$. For any operator spaces E_1, E_2, let $E_1 \oplus E_2$ be the vector space direct sum equipped with the norm $\|(x, y)\| = \|x\| + \|y\|$ for each $x \in E_1, y \in E_2$. For each $n \in \mathbb{N}$, the obvious identifications $M_n(E_1 \oplus E_2) \cong M_n(E_1) \oplus M_n(E_2)$ yield an operator space structure on $E_1 \oplus E_2$. Moreover, if $E_1, E_2 \in \mathscr{O}\mathscr{M}od_A^\infty$, then we equip $E_1 \oplus E_2$ with the module action $(x, y) \cdot a := (x \cdot a, y \cdot a)$ for all $x \in E_1, y \in E_2, a \in A$. With this, we have $E_1 \oplus E_2 \in \mathscr{O}\mathscr{M}od_A^\infty$ and the quintuplet $(E_1 \oplus E_2, \iota_1, \iota_2, \pi_1, \pi_2)$, where for each $i \in \{1, 2\}$, $\iota_i : E_i \to E_1 \oplus E_2$ denotes the inclusion and $\pi_i : E_1 \oplus E_2 \to E_i$ is the projection, makes $E_1 \oplus E_2$ a direct sum of E_1 and E_2.

The next result is elementary.

Proposition 3.2 *Let $E, F \in \mathscr{O}\mathscr{M}od_A^\infty$. For every $f \in CB_A(E, F)$, we have*

(i) *f is a monomorphism in $\mathscr{O}\mathscr{M}od_A^\infty$ if and only if it is injective.*
(ii) *f is an epimorphism in $\mathscr{O}\mathscr{M}od_A^\infty$ if and only if it has dense range.*

Kernels and cokernels are fundamental tools in our approach.

Proposition 3.3 *Let A be an operator algebra. Suppose $E, F \in \mathscr{O}\mathscr{M}od_A^\infty$ and $f \in CB_A(E, F)$. Then f has a kernel and a cokernel.*

Proof We first deal with the kernel of f. Let K be $\ker f = f^{-1}(0)$; this is clearly an object in $\mathscr{O}\mathscr{M}od_A^\infty$. Let $\mu : K \to E$ be the inclusion. Then $\mu \in CB_A(E, F)$ and $f\mu = 0$. Suppose $G \in \mathscr{O}\mathscr{M}od_A^\infty$ and there exists $g \in CB_A(G, E)$ with $fg = 0$. Then $g(G) \subseteq K$, so we let $\overline{g} \in CB_A(G, K)$ just be the A-module map g. As morphisms we have $g = \mu \overline{g}$. That this is the only such morphism making Diagram (2.1) commutative, follows from the fact that μ is a monomorphism.

To prove that f has a cokernel, we let $C = F/\overline{f(E)}$ and $\pi \in CB_A(F, C)$ be the canonical projection. Obviously, $\pi f = 0$. Suppose there exist $G \in \mathscr{O}\mathscr{M}od_A^\infty$ and $g \in CB_A(F, G)$ such that $gf = 0$. For each $y \in F$, let $\overline{g}(\pi(y)) = g(y)$. If $\pi(y) = 0$ then $y \in \overline{F(E)}$ and, by continuity, $g(y) = 0$. Hence $\overline{g} : C \to G$ is a well-defined map and is clearly an A-module map. In fact, it is completely open with openness constant $\lambda > 1$ since $M_n(C) \cong M_n(F)/M_n(\overline{F(E)})$ [6, 1.2.14]. For each $n \in \mathbb{N}$ and $c \in M_n(C)$ choose $y \in M_n(F)$ such that $\pi_n(y) = c$ and $\|y\|_n \leq \lambda \|c\|_n$. Then $\|\overline{g}(c)\|_n = \|g(y)\|_n \leq \|g\|_{\mathrm{cb}} \lambda \|c\|_n$. Hence $\overline{g}$ is completely bounded, i.e., $\overline{g} \in CB_A(C, G)$ such that $\overline{g}\pi = g$. Lastly, we note since π is an epimorphism, $\overline{g}$ is the only morphism $C \to G$ making Diagram (2.2) commutative. ∎

Proposition 3.4 *Let A be an operator algebra. Let $E, F \in \mathscr{O}\mathscr{M}od_A^\infty$ and $\mu \in CB_A(E, F)$. Then μ is the kernel of some morphism $\pi \in CB_A(F, G)$, $G \in \mathscr{O}\mathscr{M}od_A^\infty$ if*

and only if μ has closed range and is an isomorphism in $\mathcal{OMod}_A^\infty$ (that is, a completely bounded bijective A-module map, with completely bounded inverse) when viewed as a map onto its range.

Proof   By Remark 2.1 and Proposition 3.3, we need only consider the case where $G = F/\overline{\mu(E)}$ and π is the canonical quotient mapping. Let $K = \ker \pi$ and $\iota \in CB_A(K, F)$ be the inclusion map. Then μ is a kernel of π if and only if there exists an isomorphism $\phi \in CB_A(K, E)$ making the following diagram commutative.

(3.1)

Note that $\overline{\mu(E)} = \iota(K)$. Suppose μ has closed range and is an isomorphism in $\mathcal{OMod}_A^\infty$ when considered as a map onto its range. Then $\mu(E) = \iota(K)$ and we simply let ϕ be the completely bounded inverse $\mu(E) \to E$ composed with ι. So μ is a kernel.

On the other hand, if μ is the kernel of π then ϕ exists. Since $\mu = \iota \circ \phi^{-1}$ and ι is an isometry, $\mu(E)$ is closed and we are done. ∎

Proposition 3.5   *Let A be an operator algebra. Let $E, F \in \mathcal{OMod}_A^\infty$ and $\pi \in CB_A(E, F)$. Then π is the cokernel of some morphism $\mu \in CB_A(G, E)$, $G \in \mathcal{OMod}_A^\infty$ if and only if π is completely open.*

Proof   By Remark 2.1 and Proposition 3.3, we need only look at the case where $G = \ker \pi$ and μ is the inclusion mapping. Let $C = E/\mu(G)$ and $g \in CB_A(E, C)$ be the quotient map. Then π is cokernel for μ if and only if there exists an isomorphism $\phi: F \to C$ making the following diagram commutative.

(3.2)

Suppose that π is completely open and hence surjective. Note that, if $x \in E$ is such that $\pi(x) = 0$, then $x \in \mu(G)$ and $g(\mu(x)) = 0$, so the map $\phi: F \to C$, $\pi(x) \mapsto g(x)$ is well defined. As π, g are A-module maps so is ϕ. For any $n \in \mathbb{N}$ and $y \in M_n(F)$, we have by Proposition 2.3 that $\|\phi_n(y)\|_n = \|g_n(x)\|_n \leq \lambda \|g\|_{\mathrm{cb}} \|y\|_n$ for some $x \in E$ and openness constant $\lambda > 1$. So $\phi \in CB_A(F, C)$. A similar argument (using the fact that g is completely open) gives that there is a morphism $\psi \in CB_A(C, F)$ defined by setting $\psi(g(x)) = \pi(x)$ for any $x \in E$. By definition, $\phi\pi = g$ and it is clear that ψ is the inverse of ϕ, thus ϕ is an isomorphism. It follows that π is a cokernel of μ.

Conversely, suppose there exists an isomorphism ϕ making Diagram (3.2) commutative. Let $n \in \mathbb{N}$ and $y \in M_n(F)$. By Proposition 2.3, there exists $x \in M_n(E)$ such that $g_n(x) = \phi_n(y)$ and $\|x\|_n \leq \lambda \|\phi_n(y)\|_n \leq \lambda \|\phi\|_{\mathrm{cb}} \|y\|_n$, where λ is an openness

constant for g. Moreover, the commutativity of Diagram (3.2) gives that $\pi_n(x) = \phi_n^{-1} g_n(x) = \phi_n^{-1} \phi_n(y) = y$ and by Proposition 2.3, π is completely open. ∎

An additive category $\mathscr{A}$ is called *abelian* if every morphism in $\mathscr{A}$ has both a kernel and a cokernel; every monomorphism in $\mathscr{A}$ is the kernel of its cokernel; and every epimorphism in $\mathscr{A}$ is the cokernel of its kernel. Comparing the descriptions above of the kernels and cokernels in $\mathscr{OMod}_A^\infty$ with the monomorphisms and epimorphisms in that category, it is perhaps obvious that $\mathscr{OMod}_A^\infty$ fails to be abelian. Here is the probably simplest example for $A = \mathbb{C}$. Let E be an infinite-dimensional Banach space. Recall that the identity map on E is completely contractive when considered as a map $f\colon \mathrm{Max}(E) \to \mathrm{Min}(E)$, where $\mathrm{Max}(E)$ is E equipped with its maximal quantization and $\mathrm{Min}(E)$ is its minimal quantization. Then $f \in CB_\mathbb{C}(\mathrm{Max}(E), \mathrm{Min}(E))$ is surjective and hence an epimorphism in $\mathscr{OMod}_\mathbb{C}^\infty$. However, it cannot be completely open, that is, an isomorphism as E is infinite dimensional (see, e.g., [26, Corollary 3.9]).

As a result, the homological algebra which is highly developed in abelian categories is not directly available in $\mathscr{OMod}_A^\infty$. In the next section, we show how we can get around this issue by introducing exact categories. The following two results will be essential.

Proposition 3.6 *Let $E, F, G \in \mathscr{OMod}_A^\infty$ with $f_E \in CB_A(E, G)$ and $f_F \in CB_A(F, G)$.*

(i) *There exist $L \in \mathscr{OMod}_A^\infty$ and $\ell_E \in CB_A(L, E)$, $\ell_F \in CB_A(L, F)$ making the following diagram a pullback square.*

$$(3.3) \qquad \begin{array}{ccc} L & \overset{\ell_F}{\dashrightarrow} & F \\ {\scriptstyle \ell_E}\big\downarrow & & \big\downarrow{\scriptstyle f_F} \\ E & \underset{f_E}{\longrightarrow} & G \end{array}$$

(ii) *For any such pullback square, if f_E is a cokernel map, then so is ℓ_F.*

Proof (i) Let $L = \{(x, y) \in E \oplus F \mid f_E(x) = f_F(y)\}$. Then L is a closed submodule of $E \oplus F$ so inherits the operator A-module structure of $E \oplus F$. Let ℓ_F and ℓ_E be the restrictions to L of the canonical projections $\pi_F\colon E \oplus F \to F$ and $\pi_E\colon E \oplus F \to E$, respectively. By definition of L, Diagram (3.3) is commutative.

If there exist $L' \in \mathscr{OMod}_A^\infty$ and $\ell_E' \in CB_A(L', E)$, $\ell_F' \in CB_A(L', F)$ such that $f_F \ell_F' = f_E \ell_E'$, then, by the universal property of products, there exists a unique $\phi \in CB_A(L', L)$ such that $\ell_E' = \pi_E \phi$ and $\ell_F' = \pi_F \phi$ and it is clear that $\phi(L') \subseteq L$; hence L must make Diagram (3.3) a pullback square.

(ii) By Remark 2.1 and the universal property of pullbacks, the result holds if and only if it holds for the pullback square defined in (i). Suppose that f_E is a cokernel map. Proposition 3.5 tells us that f_E is completely open and we are done if ℓ_F is completely open.

Let λ be an openness constant for f_E and set $\lambda' = \max\{\lambda \|f_F\|_{\mathrm{cb}}, 1\}$. For $n \in \mathbb{N}$ and $y \in M_n(F)$, we have $(f_F)_n(y) \in M_n(G)$ and, by Proposition 2.3, there exists $x \in M_n(E)$ such that $(f_E)_n(x) = (f_F)_n(y)$ (hence $(x, y) \in L$) with $\|x\|_n \leq \lambda \|(f_F)_n(y)\|_n \leq \lambda' \|y\|_n$. We note that $(\ell_F)_n(x, y) = y$ with $\|(x, y)\|_n \leq 2\lambda' \|y\|_n$. Proposition 2.3 tells us ℓ_F is completely open. ∎

Proposition 3.7   *Let $E, F, G \in \mathcal{O}\mathcal{M}od_A^\infty$ with $f_E \in CB_A(G, E)$ and $f_F \in CB_A(G, F)$.*

(i) *There exist $C \in \mathcal{O}\mathcal{M}od_A^\infty$ and $h_E \in CB_A(E, C), h_F \in CB_A(F, C)$ making the following diagram a pushout square.*

(3.4)
$$
\begin{array}{ccc}
G & \xrightarrow{\ f_E\ } & E \\
{\scriptstyle f_F}\downarrow & & \downarrow{\scriptstyle h_E} \\
F & \dashrightarrow[h_F] & C
\end{array}
$$

(ii) *For any such pushout square, if f_E is a kernel map, then so is h_F.*

Proof   (i) Let $H = \{(f_E(z), -f_F(z)) \mid z \in G\}$ and $C = E \oplus F/\overline{H}$. Let h_E be the composition of the embedding $\iota_E : E \to E \oplus F$ with the canonical projection $\pi : E \oplus F \to C$ and $h_F = \pi \iota_F$, where ι_F is the embedding $F \to E \oplus F$. Clearly $h_E \in CB_A(E, C)$ and $h_F \in CB_A(F, C)$. For any $z \in G$, $(f_E(z), 0) - (0, f_F(z)) = (f_E(z), -f_F(z)) \in H$; this means that Diagram (3.4) is commutative.

Suppose there exists $C' \in \mathcal{O}\mathcal{M}od_A^\infty$ and $h_E' \in CB_A(E, C')$ and $h_F' \in CB_A(F, C')$ with $h_E' f_E = h_F' f_F$. By the universal property of coproducts, there exists $\phi \in CB_A(E \oplus F, C')$ such that $\phi \iota_E = h_E'$ and $\phi \iota_F = h_F'$. For Diagram (3.4) to be a pushout square it remains to show that there exists $\tau \in CB_A(C, C')$ such that $\tau h_E = h_E'$ and $\tau h_F = h_F'$. Suppose $(x, y) \in H$, then there exists $z \in G$ such that $x = f_E(z)$ and $y = -f_F(z)$. Therefore,

$$
\phi(x, y) = h_E'(f_E(z)) + h_F'(-f_F(z)) = h_E'(f_E(z)) - h_E'(f_E(z)) = 0.
$$

That is, the A-module map $\tau : C \to C'$, $\pi(x, y) \mapsto \phi(x, y)$ is well defined. Let $c \in M_n(C)$ and $\lambda > 1$. As in Example 2.4, π is completely open and λ is an openness constant for π. Therefore, there exists $(x, y) \in M_n(E \oplus F)$ such that $c = \pi_n(x, y)$ with $\|\tau(c)\|_n = \|\phi(x, y)\|_n \leq \|\phi\|_{cb}\|(x, y)\|_n \leq \lambda\|\phi\|_{cb}\|c\|_n$. Hence τ is completely bounded.

(ii) By Remark 2.1 and the universal property of pushouts, the result holds if and only if it holds for the pushout square defined in (i). Suppose that f_E is a kernel map in $\mathcal{O}\mathcal{M}od_A^\infty$. That is, $f_E(G)$ is closed in E and there exists $g \in CB_A(f_E(G), G)$ such that $g f_E = \mathrm{id}_G$ and $f_E g = \mathrm{id}_{f_E(G)}$. We will show that h_F is a kernel map too.

Suppose, we have a sequence $(f_E(z_n), -f_F(z_n))_{n \in \mathbb{N}}$ in H with limit $(x, y) \in E \oplus F$. By continuity, $g(x)$ is the limit of $(z_n)_{n \in \mathbb{N}} = (g f_E(z_n))_{n \in \mathbb{N}}$ and $y = -f_F(g(x))$. That is, $(x, y) = (f_E(g(x)), -f_F(g(x))) \in H$. Therefore, H is closed and $C = E \oplus F/H$.

For h_F to be a kernel map, we need that $h_F(F)$ is closed in C. Let $(h_F(y_n))_{n \in \mathbb{N}} \subseteq C$ and $c \in C$ be such that $\|h_F(y_n) - c\| \to 0$. There exist $x \in E$, $y \in F$ with $c = \pi(x, y)$ is the limit of $(\pi(0, y_n))_{n \in \mathbb{N}}$; that is, $\|\pi(-x, y_n - y) - \pi(0, y_n)\| \to 0$. Because H is closed in $E \oplus F$ there must exist a sequence $(z_n) \in G$ such that $(f_E(z_n), -f_F(z_n)) \in H$ with $\|(-x - f_E(z_n), y_n - y + f_F(z_n))\| \to 0$. In particular, $\|-x - f_E(z_n)\| \to 0$ and, as $f_E(G)$ is closed in E, there exists some $z \in G$ with $f_E(z) = -x$. By continuity, $z_n = g(f_E(z_n)) \to z$ and $f_F(z_n) \to f_F(z)$. Set $y' = y - f_F(z)$, then $\|y_n - y'\| \to 0$.

Therefore,

$$h_F(y') = \pi(0, y') = \pi((x, y) + (f_E(z), -f_F(z))) = \pi(x, y)$$

and hence, $\pi(x, y) \in h_F(F)$.

Note that h_F is injective. Indeed, if $h_F(y) = 0$, then there exists $z \in G$ such that $(0, y) = (f_E(z), -f_F(z))$ but f_E is injective so $z = 0$ and therefore $y = 0$. So, we certainly have an A-module map $\ell: h_F(F) \to F$ defined by $\ell(h_F(y)) = y$. We are done if ℓ is completely bounded.

Note that, for each $z \in M_n(G)$, we have

$$(3.5) \qquad \|(f_F)_n(z)\|_n \le \|f_F\|_{cb} \|z\|_n \le \|f_F\|_{cb} \|g\|_{cb} \|(f_E)_n(z)\|_n.$$

If $f_F = 0$, the result is obvious, so we can suppose otherwise. Then equation (3.5) tells us that $\|f_E(z)\|_n \ge K \|f_F(z)\|_n$ where $K = \min\{(\|f_F\|_{cb} \|g\|_{cb})^{-1}, 1\}$. Recall that for each $y \in M_n(F)$, $\|(h_F)_n(y)\|_n = \inf \|(0, y) + ((f_E)_n(z), -(f_F)_n(z))\|_n$, where the infimum is over all $z \in M_n(G)$. Then, for each $n \in \mathbb{N}$, there exists $z \in M_n(G)$ such that

$$\begin{aligned}
\|(h_F)_n(y)\|_n &\ge \frac{1}{2} \|(0, y) + ((f_E)_n(z), -(f_F)_n(z))\|_n \\
&= \frac{1}{2} \|((f_E)_n(z), y - (f_F)_n(z))\|_n \\
&= \frac{1}{2} (\|(f_E)_n(z)\|_n + \|y - (f_F)_n(z)\|_n) \\
&\ge \frac{K}{2} (\|(f_F)_n(z)\|_n + \|y - (f_F)_n(z)\|_n) \ge \frac{K}{2} \|y\|_n.
\end{aligned}$$

Therefore, $\|\ell_n((h_F)_n(y))\|_n = \|y\|_n \le \frac{2}{K} \|(h_F)_n(y)\|_n$ for all n and ℓ is completely bounded. ∎

We are now ready to embark on setting up the new framework for homological algebra.

4 Exact categories and global dimension

Exact categories are additive categories equipped with a class of kernel-cokernel pairs that satisfy certain axioms (see Definition 4.2). In this section, we show how exact categories provide a framework to arrive at a more general version of the notion of a global dimension for an abelian category. Our main source for exact categories is the survey article of Bühler [8]. We show that for a general operator algebra A, $\mathcal{OM}od_A^\infty$ has a canonical exact structure.

Definition 4.1 Suppose, we have a pair of composable morphisms

$$K \xrightarrow{\mu_K} E \xrightarrow{\pi_C} C$$

in an additive category $\mathscr{A}$, where μ_K is a kernel of π_C and π_C is a cokernel of μ_K. Then we say that (μ_K, π_C) is a *kernel-cokernel pair*. Suppose $\mathscr{E}$ is a fixed class of kernel-cokernel pairs in $\mathscr{A}$. Then a morphism μ is called an *admissible monomorphism*

if there exists a morphism π such that $(\mu, \pi) \in \mathscr{E}$. A morphism π is called an *admissible epimorphism* if there exists a morphism μ such that $(\mu, \pi) \in \mathscr{E}$. In diagrams admissible monomorphisms (respectively, epimorphisms) will be depicted by $\rightarrowtail$ (respectively, $\twoheadrightarrow$).

We define exact categories using the axioms of an exact structure given in [8]. In Section 2 of that paper, Bühler shows these axioms are equivalent to Quillen's axioms [27].

Definition 4.2   An *exact structure* on $\mathscr{A}$ is a class $\mathscr{E}x$ of kernel-cokernel pairs which is closed under isomorphisms and satisfies the following axioms:

  [E0] $\forall\ E \in \mathscr{A}$: id_E is an admissible monomorphism;

[E0$^{\mathrm{op}}$] $\forall\ E \in \mathscr{A}$: id_E is an admissible epimorphism;

  [E1] the class of admissible monomorphisms is closed under composition;

[E1$^{\mathrm{op}}$] the class of admissible epimorphisms is closed under composition;

  [E2] the pushout of an admissible monomorphism along an arbitrary morphism exists and yields an admissible monomorphism;

[E2$^{\mathrm{op}}$] the pullback of an admissible epimorphism along an arbitrary morphism exists and yields an admissible epimorphism.

An *exact category* is a pair $(\mathscr{A}, \mathscr{E}x)$ consisting of an additive category $\mathscr{A}$ and an exact structure $\mathscr{E}x$ on $\mathscr{A}$.

Definition 4.3   For an additive category $\mathscr{A}$, a kernel-cokernel pair

$$K \xrightarrow{\ \mu_K\ } E \xrightarrow{\ \pi_C\ } C$$

is *split* if there exist morphisms $\mu_C \in \mathrm{Hom}(C, E)$ and $\pi_K \in \mathrm{Hom}(E, K)$ such that the quintuplet $(E, \mu_K, \mu_C, \pi_K, \pi_C)$ makes E a direct sum.

Example 4.1   The class of all split kernel-cokernel pairs forms an exact structure on any additive category. We denote this class $\mathscr{E}x_{min}$ and note that $\mathscr{E}x_{min}$ will be a substructure of any other exact structure placed on $\mathscr{A}$. This is trivial to show, but has useful consequences (see Section 5).

Example 4.2   The class of all kernel-cokernel pairs in an additive category $\mathscr{A}$ is denoted $\mathscr{E}x_{max}$. If $\mathscr{A}$ is an abelian category then $(\mathscr{A}, \mathscr{E}x_{max})$ is an exact category and $\mathscr{E}x_{max}$ is the class of all short exact sequences in $\mathscr{A}$.

For a general additive category $\mathscr{A}$, the class $\mathscr{E}x_{max}$ is closed under isomorphisms (Remark 2.1) and satisfies [E0] and [E0$^{\mathrm{op}}$]. We introduce some conditions that ensure that this class forms an exact structure on $\mathscr{A}$.

Definition 4.4   Let $\mathscr{A}$ be an additive category. We say $\mathscr{A}$ is a *quasi-abelian category* if:

 (i) each morphism in $\mathscr{A}$ has a kernel and a cokernel and

(ii) the class $\mathscr{E}x_{max}$ of all kernel-cokernel pairs satisfies conditions [E2] and [E2$^{\mathrm{op}}$] from Definition 4.2.

The following can be proven by diagram chasing, from the definitions of kernels, cokernels, pushouts, and pullbacks. (See [8, Proposition 4.4].)

Proposition 4.3 [30, Proposition 1.1.7] *Let $\mathscr{A}$ be a quasi-abelian category. Then $\mathscr{E}x_{max}$, the class of all kernel-cokernel pairs, satisfies conditions [E1] and [E1^{op}] from Definition 4.2 and thus forms an exact structure.*

Theorem 4.4 *Let A be an operator algebra. The class of all kernel-cokernel pairs in $\mathcal{OM}od_A^\infty$ forms an exact structure on $\mathcal{OM}od_A^\infty$.*

Proof By Propositions 3.3, 3.6, and 3.7, $\mathcal{OM}od_A^\infty$ is quasi-abelian. The result then follows from Proposition 4.3. Alternatively, one can check [E1] and [E1^{op}] "by hand" [28, Section 3.4]. ∎

In an abelian category, the short exact sequences are the smallest building blocks for homological algebra which then emerges from long exact sequences and so-called diagram lemmas. The kernel-cokernel pairs take the role of short exact sequences, and the axioms of an exact category entail enough of the diagram lemmas to build a viable theory. This is demonstrated in detail in [8] and successfully applied to a functional analytic setting in [9].

Another fundamental tool is the concept of an exact functor.

Definition 4.5 An additive functor $\mathsf{F}\colon (\mathscr{A}, \mathscr{E}x_1) \to (\mathscr{B}, \mathscr{E}x_2)$ between two exact categories is *exact* if $\mathsf{F}(\mathscr{E}x_1) \subseteq \mathscr{E}x_2$.

We will note for later (see Section 5) how exact functors are useful for inducing alternate exact structures on an exact category.

Proposition 4.5 *Let $\mathsf{F}\colon (\mathscr{A}, \mathscr{E}x_1) \to (\mathscr{B}, \mathscr{E}x_2)$ be an exact functor between exact categories. Suppose there is another exact structure $\mathscr{E}x_2'$ on $\mathscr{B}$. Then*

$$\mathscr{E}x_1' = \left\{ (\mu, \pi) \in \mathscr{E}x_1 \,\middle|\, (\mathsf{F}\mu, \mathsf{F}\pi) \in \mathscr{E}x_2' \right\}$$

forms an exact structure on $\mathscr{A}$.

Proposition 4.5 is [17, Proposition 7.3]; the proof also follows easily from [8, Proposition 5.2]. We will call the exact structure $\mathscr{E}x_1'$ appearing in this way a *relative exact structure* as it involves constraints relative to another exact structure.

Injectivity and projectivity can be described using the notion of exact functors. For any object E in an additive category $\mathscr{A}$ we have the following contravariant functor

$$\mathrm{Hom}(-, E)\colon \mathscr{A} \longrightarrow \mathscr{Ab}$$
$$F \longmapsto \mathrm{Hom}_{\mathscr{A}}(F, E)$$
$$\mathrm{Hom}_{\mathscr{A}}(F, G) \ni f \longmapsto f^* = \mathrm{Hom}(f, E),$$

where $f^*(g) = gf$ for $g \in \mathrm{Hom}_{\mathscr{A}}(G, E)$ and $\mathscr{Ab}$ is the category of abelian groups. There is also the similarly defined covariant functor $\mathrm{Hom}(E, -)$, where $f_* = \mathrm{Hom}(E, f)$ is given by postcomposition of f. We call these Hom-*functors*.

Let $\mathscr{M}$ be the class of admissible monomorphisms in an exact category $(\mathscr{A}, \mathscr{E}x)$ and $\mathscr{P}$ be the class of admissible epimorphisms in this category. We shall write $\mathscr{E}x = (\mathscr{M}, \mathscr{P})$ for brevity. Then an object $I \in \mathscr{A}$ is $\mathscr{M}$-injective precisely when $\mathrm{Hom}(-, I)$ is exact as a functor to $(\mathscr{Ab}, \mathscr{E}x_{max})$ and $P \in \mathscr{A}$ is $\mathscr{P}$-projective precisely when $\mathrm{Hom}(P, -)$ is exact.

In an abelian category with enough injectives there is a standard notion of injective dimension of an object. This is defined using injective resolutions. In order for these methods to transfer to the setting of exact categories we need to take extra care in our set up. To this end, we work with resolutions built from particular types of morphisms.

Definition 4.6   Let $(\mathcal{A}, \mathcal{Ex})$ be an exact category and $\mathcal{Ex} = (\mathcal{M}, \mathcal{P})$. A morphism $f \in \mathrm{Hom}(E, F)$ is called *admissible* if there exist an object $G \in \mathcal{A}$ and morphisms $\pi_f \in \mathcal{P}(E, G)$ and $\mu_f \in \mathcal{M}(G, E)$ such that $f = \mu_f \pi_f$. (Such decomposition is unique up to unique isomorphism.)

A sequence of admissible morphisms

$$\cdots \longrightarrow E_1 \xrightarrow{\ \ f_1\ \ } E_2 \xrightarrow{\ \ f_2\ \ } E_3 \xrightarrow{\ \ f_3\ \ } \cdots$$
$$\pi_1 \searrow\ \ G_1\ \ \nearrow \iota_1 \quad \pi_2 \searrow\ \ G_2\ \ \nearrow \iota_2 \quad \pi_2 \searrow\ \ G_3\ \ \nearrow \iota_3$$

is called *exact* if, for each $n \in \mathbb{Z}$, (ι_n, π_{n+1}) is in $\mathcal{Ex}$.

An exact sequence of admissible morphisms of the form

$$E \xrightarrowtail{\ \ \iota\ \ } I^0 \xrightarrow{\ \ d^0\ \ } I^1 \xrightarrow{\ \ d^1\ \ } I^2 \xrightarrow{\ \ d^2\ \ } \cdots,$$

where each I^n is $\mathcal{M}$-injective is called an *$\mathcal{M}$-injective resolution of E*.

It is easy to see that, if $\mathcal{M}$ is the class of admissible monomorphisms in an exact category $(\mathcal{A}, \mathcal{Ex})$ and $\mathcal{A}$ has enough $\mathcal{M}$-injectives, then an injective resolution exists for every object in $\mathcal{A}$. In the same situation, a little more work gives the following analogue of the injective dimension theorem for modules over rings.

Theorem 4.6   *Let $\mathcal{M}$ be the class of admissible monomorphisms in an exact category $(\mathcal{A}, \mathcal{Ex})$. Suppose $\mathcal{A}$ has enough $\mathcal{M}$-injectives. The following are equivalent for $n \geq 1$ and every $E \in \mathcal{A}$.*

(i) *If there is an exact sequence of admissible morphisms*

$$(4.1) \qquad E \xrightarrowtail{\quad\quad} I^0 \longrightarrow \cdots \longrightarrow I^{n-1} \twoheadrightarrow F$$

with each I^m, $0 \leq m \leq n-1$ injective, then F must be injective.

(ii) *There is an exact sequence of admissible morphisms*

$$(4.2) \qquad E \xrightarrowtail{\quad\quad} I^0 \longrightarrow \cdots \longrightarrow I^{n-1} \twoheadrightarrow I^n$$

with each I^m, $0 \leq m \leq n$ injective.

Remark 4.7   That (i) implies (ii) in Theorem 4.6 follows easily from the fact that every object in such a category $\mathcal{A}$ has an $\mathcal{M}$-injective resolution and by the definition of an exact sequence of admissible morphisms. To get (ii) implies (i), one can follow the method of the same result for general abelian categories, or indeed module categories, making slight adjustments when necessary. That is, one goes via the route of Ext groups. These are defined using derived functors arising from Hom-functors. A nice explanation of derived functors from exact categories can be found in [8, Section 12].

The module category versions of the injective dimension theorem can be found in various places, see, for example, [33, Lemma 4.1.8] or [23, Proposition 4.8]. An explicit proof of Theorem 4.6 is given in [28].

Theorem 4.6 can also be seen as a consequence of an injective version of Schanuel's lemma for exact categories; for details see [21]. This approach is useful if one wants to avoid the use of Ext groups.

We now introduce the homological dimension we are after.

Definition 4.7 Let $E \in \mathscr{A}$ and $\mathscr{M}$ be the class of admissible monomorphisms in an exact category $(\mathscr{A}, \mathscr{E}x)$. We say E has *finite $\mathscr{M}$-injective dimension* if there exists an exact sequence of admissible morphisms as in Diagram (4.2) with all I^m $\mathscr{M}$-injective. If E is of finite $\mathscr{M}$-injective dimension we write $\text{Inj}_{\mathscr{M}}-\dim(E) = 0$ if E is $\mathscr{M}$-injective and $\text{Inj}_{\mathscr{M}}-\dim(E) = n$ if E is not $\mathscr{M}$-injective and n is the smallest natural number such that there exists an exact sequence of admissible morphisms as in Diagram (4.2) with all I^m $\mathscr{M}$-injective. If E is not of finite $\mathscr{M}$-injective dimension, we write $\text{Inj}_{\mathscr{M}}-\dim(E) = \infty$.

The *global dimension* of the exact category $(\mathscr{A}, \mathscr{E}x)$ is

$$\sup\left\{\text{Inj}_{\mathscr{M}}-\dim(E) \,|\, E \in \mathscr{A}\right\} \in \mathbb{N}_0 \cup \{\infty\}.$$

Remark 4.8 It is easy to observe, that the $\mathscr{M}$-injective dimension of an object E in an exact category $(\mathscr{A}, \mathscr{E}x)$ is independent of any choice of injective resolutions. Indeed, suppose

$$(4.3) \qquad E \rightarrowtail I^0 \xrightarrow{d^0} \dots \xrightarrow{d^{n-2}} I^{n-1} \xrightarrow{d^{n-1}} I^n \xrightarrow{d^n} \dots$$

is an $\mathscr{M}$-injective resolution of E. Terminate this exact sequence by replacing d^{n-1} and I^n by the cokernel of d^{n-2} to get a sequence as in Diagram (4.1). Then, by Theorem 4.6, $\text{Inj}_{\mathscr{M}}-\dim(E) \leq n$ if and only if the cokernel object of d^{n-2} is $\mathscr{M}$-injective.

If $\mathscr{P}$ is the class of admissible epimorphisms then the dual procedure to the above yields the notion of projective resolutions and projective dimension for an object. There is a dual result to Theorem 4.6 which leads to the notion of the projective dimension of an object. An object will have projective dimension 0 if and only if it is $\mathscr{P}$-projective. We are able to characterize global dimension 0 using projectives by Corollary 4.11.

Remark 4.9 Let $\mathscr{M}$ be the class of admissible monomorphisms and $\mathscr{P}$ be the class of admissible epimorphisms in an exact category. If there are enough $\mathscr{P}$-projectives and enough $\mathscr{M}$-injectives, the supremum of all the $\mathscr{P}$-projective dimensions will coincide with the supremum of all the $\mathscr{M}$-injective dimensions. So we are always able to define the global dimension using only $\mathscr{P}$-projectives provided there are enough of them.

The following proposition is a classical result known as the "Splitting Lemma."

Proposition 4.10 (Splitting Lemma) *Let $\mathscr{A}$ be an additive category with kernels and cokernels and suppose*

$$E \xrightarrow{\mu} F \xrightarrow{\pi} G$$

is a kernel-cokernel pair in $\mathscr{A}$. Then the following are equivalent:

(i) *the kernel-cokernel pair is split;*

(ii) *there exists $\pi_E \in \mathrm{Hom}_{\mathscr{A}}(F, E)$ such that $\pi_E \mu = \mathrm{id}_E$; and*

(iii) *there exists $\mu_F \in \mathrm{Hom}_{\mathscr{A}}(G, F)$ such that $\pi \mu_F = \mathrm{id}_G$.*

Corollary 4.11 *Let $(\mathscr{A}, \mathscr{E}x)$ be an exact category and $\mathscr{E}x = (\mathscr{M}, \mathscr{P})$. Every object in $\mathscr{A}$ is $\mathscr{M}$-injective if and only if every object is $\mathscr{P}$-projective.*

Proof This follows from the Splitting Lemma and Remark 2.2. ∎

5 Operator algebras with global dimension zero

When A is a unital operator algebra and $\mathscr{M}$ is the class of admissible monomorphisms in the exact category $(\mathcal{OM}od_A^\infty, \mathscr{E}x_{max})$, it is unclear whether there are enough $\mathscr{M}$-injectives. The canonical object $CB(A, I)$, where I is injective in $\mathcal{O}p^\infty$, which is the analogue of the canonical injective object in algebraic module categories, lies in the larger category of matrix normed modules [10]. We will discuss this issue at the end of the present section. However, there is a different exact structure we can place on $\mathcal{OM}od_A^\infty$ that has enough projectives. In this section, we discuss this structure and give a characterization of when the global dimension associated to this exact category is zero.

Let $\mathscr{E}x_{rel}$ be the class of kernel-cokernel pairs in $\mathcal{OM}od_A^\infty$ that split in $\mathcal{O}p^\infty$.

Proposition 5.1 *Let A be an operator algebra. Then $(\mathcal{OM}od_A^\infty, \mathscr{E}x_{rel})$ is an exact category.*

Proof By Theorem 4.4, $(\mathcal{OM}od_A^\infty, \mathscr{E}x_{max})$ is an exact category. In particular, this holds for $\mathcal{OM}od_{\mathbb{C}}^\infty = \mathcal{O}p^\infty$. The forgetful functor $(\mathcal{OM}od_A^\infty, \mathscr{E}x_{max}) \longrightarrow (\mathcal{O}p^\infty, \mathscr{E}x_{max})$ is exact and as in Example 4.1, we know that $\mathscr{E}x_{min}$ forms an exact structure on $\mathcal{O}p^\infty$. The result follows from Proposition 4.5. ∎

We next show that $(\mathcal{OM}od_A^\infty, \mathscr{E}x_{rel})$ has enough projectives.

Proposition 5.2 *Let A be a unital operator algebra. For every $E \in \mathcal{O}p^\infty$, the Haagerup tensor product $E \otimes_h A$ is $\mathscr{P}_{rel}$-projective. In particular, $(\mathcal{OM}od_A^\infty, \mathscr{E}x_{rel})$ has enough $\mathscr{P}_{rel}$-projectives.*

Proof Suppose, we have the following diagram of morphisms in $\mathcal{OM}od_A^\infty$,

$$
\begin{array}{c}
E \otimes_h A \\
\downarrow f \\
F \xrightarrow{\quad \pi \quad} G
\end{array}
$$

with $\pi \in \mathscr{P}_{rel}$ and that $\widetilde{\pi} \in CB(G, F)$ satisfies $\pi \widetilde{\pi} = \mathrm{id}_G$. Let $g' : E \otimes_h A \to F$ be the map defined on elementary tensors by $g'(x \otimes a) = (\widetilde{\pi} f(x \otimes 1)) \cdot a$ for $x \in E, a \in A$. Since g' is the composition of

$$
E \otimes_h A \xrightarrow{\iota \otimes \mathrm{id}_A} (E \otimes_h A) \otimes_h A \xrightarrow{\widetilde{\pi} f \otimes \mathrm{id}_A} F \otimes_h A \xrightarrow{\quad m \quad} F
$$

,

where $\iota\colon E \to E \otimes_h A$ is $x \mapsto x \otimes 1$ and m is the completely contractive linearization of the module action of A on F, it is a well-defined completely bounded A-module map. As

$$\pi g'(x \otimes a) = \pi(\widetilde{\pi} f(x \otimes 1) \cdot a) = (\pi \widetilde{\pi} f(x \otimes 1)) \cdot a = f(x \otimes 1) \cdot a = f(x \otimes a),$$

we have $\pi g' = f$.

Now let $E \in \mathcal{OMod}_A^\infty$. Let $P\colon E \otimes_h A \to E$ be the canonical complete contraction given by $P(x \otimes a) = x \cdot a$. As each $x = [x_{ij}] \in M_n(E)$ is of the form $P_n([x_{ij} \otimes 1])$ and $\|[x_{ij} \otimes 1]\|_n \le \|[x_{ij}]\|_n$, P is completely open. For each $x \otimes a$ in $E \otimes_h A$ and $a_0 \in A$, we have

$$P(x \otimes a) \cdot a_0 = (x \cdot a) \cdot a_0 = x \cdot a a_0 = P(x \otimes a a_0) = P((x \otimes a) \cdot a_0),$$

that is, P is an A-module map. Define $\widetilde{P} \in CB(E, E \otimes_h A)$ by $\widetilde{P}(x) = x \otimes 1$. Then $P\widetilde{P} = \mathrm{id}_E$, hence $P \in \mathscr{P}_{rel}$ as required. ∎

Here comes the main result of the paper.

Theorem 5.3 *Let A be a unital operator algebra. The following are equivalent:*

(i) *The global dimension of $(\mathcal{OMod}_A^\infty, \mathscr{Ex}_{rel})$ is zero.*
(ii) *Every object in $\mathcal{OMod}_A^\infty$ is $\mathscr{M}_{rel}$-injective, where $\mathscr{M}_{rel}$ is the class of admissible monomorphisms in $\mathscr{Ex}_{rel}$.*
(iii) *Every object in $\mathcal{OMod}_A^\infty$ is $\mathscr{P}_{rel}$-projective, where $\mathscr{P}_{rel}$ is the class of admissible epimorphisms in $\mathscr{Ex}_{rel}$.*
(iv) *A is classically semisimple.*

In keeping with traditional notation we shall write $\mathrm{dg}_{cb} A$ for the global dimension of $(\mathcal{OMod}_A^\infty, \mathscr{Ex}_{rel})$ and call it *the completely bounded global dimension* of the operator algebra A.

Corollary 5.4 *For every unital operator algebra A, its completely bounded global dimension $\mathrm{dg}_{cb} A$ is zero if and only if A is a finite direct sum of full matrix algebras.*

The equivalence of (i) and (ii) in Theorem 5.3 follows from Definition 4.7 and (ii) $\Longleftrightarrow$ (iii) is a consequence of the splitting lemma (Corollary 4.11).

The argument for the implication (iv) $\Rightarrow$ (iii) follows closely the proof of [15, Theorem 3.5.17]; for the details see [28], Section 5.3. A direct argument of (iv) $\Rightarrow$ (ii) is provided by Proposition 5.11. The final implication (iii) $\Rightarrow$ (iv) needs some preparation.

Our approach is based on the following well-known characterization; see, e.g., [23, Theorem 4.40].

Proposition 5.5 *A unital algebra is classically semisimple if and only if each of its maximal submodules is a direct summand.*

In other words, to show that a unital algebra A is classically semisimple, we have to show that, for every maximal right ideal I of A, the exact sequence of right A-modules

$$0 \longrightarrow I \overset{\iota}{\longrightarrow} A \overset{\pi}{\longrightarrow} A/I \longrightarrow 0$$

(5.1)

splits. As A/I is a simple right A-module, this is evidently equivalent to the existence of an A-module map $f: A/I \to A$ such that $f(A/I) \nsubseteq I$ (cf. Lemma 4.10).

Let A be a unital operator algebra; as every maximal right ideal I of A is closed, we can consider the sequence (5.1) above within $\mathcal{OMod}_A^\infty$. The next auxiliary result will enable us to complete the final step in the proof of Theorem 5.3.

Lemma 5.6   *Let I be a closed right ideal in the unital operator algebra A. Set*

$$S_1 = \{f(x) \mid x \in A/I, f: A/I \to A \text{ is an } A\text{-module map}\},$$
$$S_2 = \{ab \mid a, b \in A, \ ay = 0 \text{ for all } y \in I\}.$$

Then $S_1 = S_2$ and its linear span S is a two-sided ideal.

Proof   That the linear span of S_2 is a two-sided ideal is obvious. Let $\pi: A \to A/I$ be the canonical quotient map. Let $a \in A$. We claim that there exists an A-module map $f: A/I \to A$ such that $f(\pi(1_A)) = a$ if and only if $ay = 0$ for all $y \in I$.

Suppose, there exists such an f. Take $y \in I$. Then $ay = f(\pi(1_A))y = f(\pi(y)) = f(0) = 0$. Now suppose $ay = 0$ for all $y \in I$. Define $f: A/I \to A$ by $f(x) = ab$, where $\pi(b) = x$. Clearly, this is well defined and $f(\pi(1_A)) = a$. So the claim is true. The result then follows as an element is of the form $f(x)$ if and only if it is of the form $f(\pi(1_A))b$ for $b \in A$ such that $\pi(b) = x$. ∎

The proof of the next result mirrors [15, Proposition IV.4.4.].

Proposition 5.7   *Let $\phi: A \to B$ be a unital completely contractive homomorphism between unital operator algebras and let $E \in \mathcal{OMod}_A^\infty$ be $\mathscr{P}_{rel}$-projective. Suppose $F \in \mathcal{OMod}_B^\infty$ and $f: E \to F$ is a completely bounded A-module map, where we consider F as an A-module by the restriction of scalars via ϕ. For any $x \in E$ with $f(x) \neq 0$, there exists a completely bounded A-module map $g: E \to A$ such that $\phi g(x) \neq 0$.*

Proof   In the following proof, for any $G \in \mathcal{OMod}_C^\infty$, where C is an operator algebra, we will denote by $P_{G,C}$ the completely contractive C-module map $G \otimes_h C \to G$ defined on elementary tensors by $P_{G,C}(z \otimes c) = z \cdot c$ (where $z \in G$, $c \in C$). If G is a left operator C-module, we denote the similarly defined completely contractive C-module map by $_{C,G}P: C \otimes_h G \to G$.

By our assumptions, we have the following commutative diagram of completely bounded linear maps.

$$
\begin{array}{ccc}
E & \xleftarrow{\quad P_{E,A} \quad} & E \otimes_h A \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f \otimes \phi} \\
F & \xleftarrow{\quad P_{F,B} \quad} & F \otimes_h B
\end{array}
$$

As in the proof of Proposition 5.2, we have $P_{E,A} \in \mathscr{P}_{rel}$. Hence, there exists a completely bounded A-module map $\iota: E \to E \otimes_h A$ such that $P_{E,A}\, \iota = \mathrm{id}_E$. Then it is clear that the above diagram stays commutative if we replace $P_{E,A}$ with ι. Moreover, for any (completely) bounded linear functional $\alpha: F \to \mathbb{C}$ it is easy to see the following

diagram of completely bounded linear maps is commutative:

$$
\begin{array}{ccccccc}
E & \xrightarrow{\ \iota\ } & E \otimes_h A & \xrightarrow{\ \alpha f \otimes \mathrm{id}_A\ } & \mathbb{C} \otimes_h A & \xrightarrow{\ {}_{\mathbb{C},A}P\ } & A \\
{\scriptstyle f}\downarrow & & {\scriptstyle f \otimes \phi}\downarrow & & {\scriptstyle \mathrm{id}_\mathbb{C} \otimes \phi}\downarrow & & \downarrow{\scriptstyle \phi} \\
F & \xleftarrow{\ P_{F,B}\ } & F \otimes_h B & \xleftarrow{\ \alpha \otimes \mathrm{id}_B\ } & \mathbb{C} \otimes_h B & \xrightarrow{\ {}_{\mathbb{C},B}P\ } & B
\end{array}
$$

Let $a \in A$ and $x \otimes a'$ be an elementary tensor in $E \otimes_h A$ and $\lambda \otimes a''$ be an elementary tensor in $\mathbb{C} \otimes_h A$. Then

$$
\big((\alpha f \otimes \mathrm{id}_A)(x \otimes a')\big) \cdot a = \alpha f(x) \otimes a'a = (\alpha f \otimes \mathrm{id}_A)\big((x \otimes a') \cdot a\big)
$$

and ${}_{\mathbb{C},A}P(\lambda \otimes a'') \cdot a = \lambda a''a = {}_{\mathbb{C},A}P\big((\lambda \otimes a'') \cdot a\big)$. By continuity and linearity, $\alpha f \otimes \mathrm{id}_A$ and ${}_{\mathbb{C},A}P$ are A-module maps. So, for any linear functional α, we have that ${}_{\mathbb{C},A}P(\alpha f \otimes \mathrm{id}_A)\iota \colon E \to A$ is a completely bounded A-module map. We now show that there exists an α that makes this the desired A-module map.

Let $x \in E$ such that $f(x) \neq 0$. Then $P_{F,B}(f \otimes \phi)\iota(x) \neq 0$; in particular $u = (f \otimes \phi)\iota(x) \in F \otimes_h B$ is nonzero. By Lemma 2.5, there exist $\alpha \in F^*$, $\beta \in B^*$ such that $(\alpha \otimes \beta)(u) \neq 0$. As $\alpha \otimes \beta$ is the composition $(\mathrm{id}_E \otimes \beta)(\alpha \otimes \mathrm{id}_B)$ we obtain that $(\alpha \otimes \mathrm{id}_B)u \neq 0$ and therefore ${}_{\mathbb{C},B}P(\alpha \otimes \mathrm{id}_B)u \neq 0$.

Let $g = {}_{\mathbb{C},A}P(\alpha(f) \otimes \mathrm{id}_A)\iota$. Commutativity of the diagram gives then that $\phi g(x) \neq 0$ as required. ∎

We can now complete the *Proof* of Theorem 5.3 (iii) $\Rightarrow$ (iv):

Let I be a maximal right ideal of A. Suppose that the image of every A-module map $A/I \to A$ is a subset of I; then, with notation as in Lemma 5.6, $S_1 \subseteq I$, and hence $S \subseteq I$. Put $E = A/I$ and $P = \{a \in A \mid x \cdot a = 0, \forall x \in E\}$, the right annihilator of E. Then P is a closed two-sided ideal in A. Moreover, $S \subseteq P$. Indeed, suppose $a \in S$ and $x \in E$ is of the form $\pi(b)$ for some $b \in A$, where $\pi \colon A \to E$ is the canonical epimorphism. Then $x \cdot a = \pi(b) \cdot a = \pi(ba) = 0$ because S is a left ideal contained in I.

Put $B = A/P$ and let $\phi \colon A \to B$ be the canonical epimorphism of unital operator algebras. Then E is a right B-module with action defined in the following way: for $x \in E$ and $b \in B$, $x \cdot b = \pi(aa')$ where $x = \pi(a)$, $b = \phi(a')$. This is well defined by the definition of P. Then $x \cdot b = x \cdot a'$. In fact, $E \in \mathscr{OM}od_B^\infty$; suppose $x \in M_n(E)$, $b \in M_n(B)$. For every $a' \in M_n(A)$ with $b = \phi_n(a')$, $\|x \cdot b\|_n = \|x \cdot a'\|_n \leq \|x\|_n \|a'\|_n$, so $\|x \cdot b\|_n \leq \|x\|_n \|b\|_n$. Let $F = E$ and $f = \mathrm{id}_E$; then obviously we have that $f(\pi(1_A)) \neq 0$. If E is $\mathscr{P}_{rel}$-projective, by Proposition 5.7, for each $x \notin I$, there exists an A-module map $g \colon E \to A$ such that $\phi(g(\pi(x))) \neq 0$, so $g(\pi(x)) \notin P$.

Since, by (the proof of) Lemma 5.6, $g(\pi(x)) \in S \subseteq P$ it follows that E cannot be $\mathscr{P}_{rel}$-projective. Consequently, if A is not classically semisimple, using Proposition 5.5, we conclude that not all modules in $\mathscr{OM}od_A^\infty$ can be $\mathscr{P}_{rel}$-projective. That is, if (iv) does not hold, then (iii) cannot hold either.

Remark 5.8 Implication (iii) $\Rightarrow$ (iv) in Theorem 5.3 answers Helemskii's question, for operator algebras, in [16, Section 7] in the positive since it is easy to see that his relative structure is equivalent to ours. Relative homological algebra is common in the

ring theory setting, cf., e.g., [18, Chapter IX] or [22, Section V.7]. Paulsen undertook a systematic study in the setting of operator modules in [24], see also [14]. He discovered an intimate interrelation between the cohomology groups that arise in this relative theory (which is equivalently described here by our relative exact structure) and the completely bounded Hochschild cohomology groups. See in particular Propositions 5.5 and 6.4 in [24]. Specialising to bimodules it then follows that all operator A-bimodules over a unital $C^\star$-algebra A are relatively projective if and only if A is finite dimensional which in turn is equivalent to A possessing a diagonal [24, Theorem 6.13 and Corollary 6.14]. For an extension of the latter to unital operator algebras, see [25].

In the remainder of this section, we shall discuss the interrelations between the various types of injectivity within $\mathscr{O}\mathscr{M}od_A^\infty$, but also in comparison to the larger category $mn\mathscr{M}od_A^\infty$. It will become clear that injectivity is not determined by the category but rather the exact structure which one puts on the category. For a discussion of injectivity in general terms, we refer to [20].

Let $\mathscr{X}\mathscr{M}od_A^\infty$ be any additive category whose objects are operator spaces which are right A-modules and whose morphisms are the completely bounded A-module maps between these objects, where A is a unital operator algebra (in the sense of [6, Section 2.1]). Suppose $\mathscr{X}\mathscr{M}od_A^\infty$ is closed under direct sums, closed submodules and quotients. Then the kernels (respectively, cokernels) in $\mathscr{X}\mathscr{M}od_A^\infty$ are as described in Proposition 3.4 (respectively, Proposition 3.5). Moreover, the proofs of Propositions 3.7 and 3.6 still work for $\mathscr{X}\mathscr{M}od_A^\infty$ and therefore, $(\mathscr{X}\mathscr{M}od_A^\infty, \mathscr{E}x_{max})$ and $(\mathscr{X}\mathscr{M}od_A^\infty, \mathscr{E}x_{rel})$ are exact categories, where $\mathscr{E}x_{rel}$ again denotes the kernel-cokernel pairs that split in $\mathscr{O}p^\infty$ (Proposition 5.1).

In particular, this holds for the category $mn\mathscr{M}od_A^\infty$, whose objects are the nondegenerate matrix normed A-modules. An operator space E that is also a right A-module is known as a *matrix normed A-module* if the A-module action induces a completely contractive linear mapping $E \,\hat\otimes\, A \to E$, where $\hat\otimes$ denotes the operator space projective tensor product. Comparing $\hat\otimes$ with $\otimes_h$ immediately tells us that $\mathscr{O}\mathscr{M}od_A^\infty$ is a full subcategory of $mn\mathscr{M}od_A^\infty$. (see [6, Example 3.1.5]). The category $mn\mathscr{M}od_A^\infty$ is, e.g., used in [3, 10, 29].

For each $E \in \mathscr{O}p^\infty$, on $CB(A, E)$ we define the right A-module action by $(T \cdot a)(b) = T(ab)$, for all $T \in CB(A, E)$ and $a, b \in A$. Then $CB(A, E)$ is an object in $mn\mathscr{M}od_A^\infty$ which is just a specific case of [6, 3.5.2].

Proposition 5.9 *Let A be a unital operator algebra and let $\mathscr{N}_{rel}$ be the class of admissible monomorphisms in $(mn\mathscr{M}od_A^\infty, \mathscr{E}x_{rel})$. For any $G \in \mathscr{O}p^\infty$, the matrix normed module $CB(A, G)$ is $\mathscr{N}_{rel}$-injective in $mn\mathscr{M}od_A^\infty$. Moreover, $E \in mn\mathscr{M}od_A^\infty$ is $\mathscr{N}_{rel}$-injective if and only if it is a retract of some $CB(A, G)$.*

Proof Suppose $E, F \in mn\mathscr{M}od_A^\infty, G \in \mathscr{O}p^\infty$. Let $\mu \in \mathscr{N}_{rel}(E, F)$ and $f \in CB_A(E, CB(A, G))$. We show that $CB(A, G)$ is $\mathscr{N}_{rel}$-injective by finding a morphism $g \in CB_A(F, CB(A, G))$ such that $f = g\mu$.

As $\mu \in \mathscr{N}_{rel}$, there exists $\widetilde{\mu} \in CB(F, E)$ such that $\widetilde{\mu}\mu = \mathrm{id}_E$. For $y \in F$ write $f(\widetilde{\mu}(y \cdot a))(1_A)$ as $g(y)(a)$, for each $a \in A$. This defines a completely bounded linear map $g(y) \in CB(A, G)$.

It is routine to verify that, in fact, this yields a morphism $g \in CB_A(F, CB(A, G))$ and moreover, for all $x \in E, a \in A$

$$g(\mu(x))(a) = f(\widetilde{\mu}(\mu(x) \cdot a))(1) = f(\widetilde{\mu}(\mu(x \cdot a))(1) = f(x \cdot a)(1)$$
$$= (f(x) \cdot a)(1) = f(x)(a).$$

So $g\mu = f$ as required.

Now suppose that $E \in mn\mathcal{M}od_A^\infty$ and define $\iota \colon E \to CB(A, E)$ by $\iota(x)(a) = x \cdot a$ for each $x \in E, a \in A$. Clearly, ι is a completely isometric A-module map and thus a kernel map in $mn\mathcal{M}od_A^\infty$. Define $\widetilde{\iota} \colon CB(A, E) \to E$ by $\widetilde{\iota}(T) = T(1_A)$ for all $T \in CB(A, E)$. Then $\widetilde{\iota}$ is a completely bounded linear map such that $\widetilde{\iota}\iota = \mathrm{id}_E$. That is, $\iota \in \mathcal{N}_{rel}$. The result follows by Remark 2.2. ∎

The above proposition is obtained in [10, Section 2] in a similar way but without making the categorical setting explicit. Since we shall compare injectivity in different categories, we need to make sure our arguments fit the correct situation.

Remark 5.10 Let $\mathcal{M}_{rel}$ denote the class of admissible monomorphisms in $(\mathcal{O}\mathcal{M}od_A^\infty, \mathcal{E}x_{rel})$ and $\mathcal{N}_{rel}$ be the admissible monomorphisms in $(mn\mathcal{M}od_A^\infty, \mathcal{E}x_{rel})$. If $E, F, I \in \mathcal{O}\mathcal{M}od_A^\infty$, then $\mathcal{M}_{rel}(E, F) = \mathcal{N}_{rel}(E, F)$ and

$$\mathrm{Mor}_{\mathcal{O}\mathcal{M}od_A^\infty}(F, I) = CB_A(F, I) = \mathrm{Mor}_{mn\mathcal{M}od_A^\infty}(F, I).$$

Hence, if I is $\mathcal{N}_{rel}$-injective it must also be $\mathcal{M}_{rel}$-injective. Therefore, if $E \in \mathcal{O}\mathcal{M}od_A^\infty$ and E is a retract (in $mn\mathcal{M}od_A^\infty$) of $CB(A, E)$, then E will be $\mathcal{M}_{rel}$-injective.

The next result shows that "completely bounded global dimension zero" does not depend on which of the two categories one chooses.

Proposition 5.11 *Suppose A is a classically semisimple unital operator algebra. Then every object in $\mathcal{O}\mathcal{M}od_A^\infty$ is $\mathcal{M}_{rel}$-injective and every object in $mn\mathcal{M}od_A^\infty$ is $\mathcal{N}_{rel}$-injective.*

Proof By Proposition 5.9 and Remark 5.10, it suffices to show that for every $E \in mn\mathcal{M}od_A^\infty$, there exist $r \in CB_A(CB(A, E), E)$ and $s \in CB_A(E, CB(A, E))$ such that $rs = \mathrm{id}_E$. First we will fix some notation:

There exists $n \in \mathbb{N}$ such that $A = M_{m_1}(\mathbb{C}) \oplus M_{m_2}(\mathbb{C}) \oplus \cdots \oplus M_{m_n}(\mathbb{C})$. For each $k \in \{1, \ldots, n\}$ and $i, j \in \{1, \ldots, m_k\}$, let e_{ij}^k denote the n-tuple in A with all zero entries apart from the kth entry which is a matrix in $M_{m_k}(\mathbb{C})$ with 1 for the ijth entry and 0 everywhere else.

Note that A is the linear span of the elements e_{ij}^k. Moreover, $\|e_{ij}^k\| = 1$ and $e_{ij}^k e_{pq}^\ell = 0$ unless $j = p$, and $k = \ell$ in which case $e_{ij}^k e_{pq}^\ell = e_{iq}^k$. Then $1_A = \sum_{k=1}^n \sum_{i=1}^{m_k} e_{ii}^k$ and for any $x \in E$ we have $x = \sum_{k=1}^n \sum_{i=1}^{m_k} x \cdot e_{ii}^k$.

Let $s \colon E \to CB(A, E)$ be the completely bounded A-module map defined by $s(x)(a) = x \cdot a$ for all $x \in E, a \in A$. For each $T \in CB(A, E)$, let

$$r(T) = \sum_{k=1}^n \sum_{i=1}^{m_k} T(e_{i1}^k) \cdot e_{1i}^k.$$

It is clear that this defines a linear mapping $r\colon CB(A, E) \to E$ and so will be an A-module map if for each $T \in CB(A, E)$ we have $r(T) \cdot e_{pq}^{\ell} = r(T \cdot e_{pq}^{\ell})$ for arbitrary $t \in \{1, \ldots, n\}$ and $p, q \in \{1, \ldots, m_k\}$.

We compute the two terms in question:

$$(5.2) \qquad r(T \cdot e_{pq}^{\ell}) = \sum_{k=1}^{n} \sum_{i=1}^{m_k} T(e_{pq}^{\ell} e_{i1}^{k}) \cdot e_{1i}^{k} = T(e_{pq}^{\ell} e_{q1}^{\ell}) \cdot e_{1q}^{\ell} = T(e_{p1}^{\ell}) \cdot e_{1q}^{\ell},$$

as every other term is zero. Similarly, as $e_{1i}^{k} e_{pq}^{\ell} = 0$ unless $k = \ell$ and $i = p$, we have

$$(5.3) \qquad r(T) \cdot e_{pq}^{\ell} = \sum_{k=1}^{n} \sum_{i=1}^{m_k} T(e_{i1}^{k}) \cdot e_{1i}^{k} e_{pq}^{\ell} = T(e_{p1}^{\ell}) \cdot (e_{1p}^{\ell} e_{pq}^{\ell}) = T(e_{p1}^{\ell}) \cdot e_{1q}^{\ell}.$$

Comparing equations (5.2) and (5.3) gives us that r is an A-module map.

Let $x \in E$. Then

$$rs(x) = \sum_{k=1}^{n} \sum_{i=1}^{m_k} s(x)(e_{i1}^{k}) \cdot e_{1i}^{k} = \sum_{k=1}^{n} \sum_{i=1}^{m_k} (x \cdot e_{i1}^{k}) \cdot e_{1i}^{k} = \sum_{k=1}^{n} \sum_{i=1}^{m_k} x \cdot e_{ii}^{k} = x,$$

so $rs = \mathrm{id}_E$ and all that remains is to show r is completely bounded.

Note that for $T \in CB(A, E)$,

$$r(T) = \sum_{k=1}^{n} \sum_{i=1}^{m_k} r_i^{k}(T),$$

where $r_i^{k}(T)$ is defined to be $T(e_{i1}^{k}) \cdot e_{1i}^{k}$. Each $r_i^{k}\colon CB(A, E) \to E$ is a linear map. Hence, it suffices that each r_i^{k} is completely bounded.

Let $k \in \{1, \ldots, n\}$ and $i \in \{1, \ldots, m_k\}$. For each $N \in \mathbb{N}$, let $e_N \in M_N(A)$ be the matrix with e_{1i}^{k} as every entry in the leading diagonal and zero everywhere else. Note $\|e_N\|_N = \|e_{1i}^{k}\| = 1$. For each $T = [T_{vw}] \in M_N(CB(A, E))$ we have

$$\left\| [(r_i^{k})_N(T_{vw})] \right\|_N = \left\| [T_{vw}(e_{i1}^{k}) \cdot e_{1i}^{k}] \right\|_N \leq \left\| [T_{vw}(e_{i1}^{k})] \right\|_N \|e_N\|_N \leq \|T\|_N \leq \|T\|_{\mathrm{cb}}.$$

So r_i^{k} is completely contractive and r is completely bounded as required. ∎

Corollary 5.12 *Let A be a unital operator algebra. The following are equivalent.*

(i) $\mathrm{dg}_{cb} A = 0$.
(ii) *A is classically semisimple.*
(iii) *The global dimension of $(\mathit{mnMod}_A^{\infty}, \mathscr{E}x_{rel})$ is zero.*

Proof Statements (i) and (ii) are equivalent by Theorem 5.3. That (ii) implies (iii) is Proposition 5.11. Finally, we have the implication (iii) $\Rightarrow$ (i) by Remark 5.10. ∎

It is interesting to compare the relation between "global injectivity," that is, injectivity with respect to the maximal exact structure and relative injectivity. To this end, we record the following general result which is part of the "injective version" of [8, Proposition 11.3] where it is obtained for projective objects.

Lemma 5.13 *Let $(\mathscr{A}, \mathscr{E}x)$ be an exact category and $\mathscr{M}$ be the class of admissible monomorphisms. Then $I \in \mathscr{A}$ is $\mathscr{M}$-injective if and only if it is an absolute $\mathscr{M}$-retract, that is, for every $\mu \in \mathscr{M}(I, F)$ with $F \in \mathscr{A}$ there exists $v \in \mathrm{Mor}(F, I)$ such that $v\mu = \mathrm{id}_I$.*

Proof That every $\mathcal{M}$-injective is an absolute retract is immediate from the definition. Suppose $I \in \mathcal{A}$ is an absolute $\mathcal{M}$-retract and that $E, F \in \mathcal{A}$ with morphisms $\mu \in \mathcal{M}(E, F)$ and $f \in \mathrm{Mor}(E, I)$ are given. By axiom [E2] of Definition 4.2, there exists a (commutative) pushout square:

$$
\begin{array}{ccc}
E & \overset{\mu}{\rightarrowtail} & F \\
{\scriptstyle f}\downarrow & {\scriptstyle vg} \nearrow & \downarrow{\scriptstyle g} \\
I & \underset{\mu'}{\rightarrowtail} & C
\end{array}
$$

such that $\mu' \in \mathcal{M}(I, C)$. As I is an absolute $\mathcal{M}$-retract, there exists $v \in \mathrm{Mor}(C, I)$ such that $v\mu' = \mathrm{id}_I$. Then $vg \in \mathrm{Mor}(F, I)$ with $(vg)\mu = v\mu' f = f$ as required. ∎

The next result can also be obtained in the general setting; however, this would require to fix two categories and four exact structures. In order to avoid such unnecessary generality, we restrict our attention to the two categories of operator space modules, $\mathcal{OMod}_A^\infty$ and mnMod_A^∞. We will suppress the formal forgetful functor from either of these categories to $\mathcal{Op}^\infty$ and simply consider the modules as operator spaces when needed. But it is essential to note that the morphisms in $\mathcal{Op}^\infty$ are the completely bounded linear mappings *and not the complete contractions*. Therefore, $\mathcal{Op}^\infty$-injectivity is *not* what one usually calls "injective operator space."

Let $\mathcal{A}$ denote either $\mathcal{OMod}_A^\infty$ or mnMod_A^∞, for a unital operator algebra A. Since the morphisms, the (global) admissible monomorphisms, and the admissible monomorphisms in the relative structure are the same in both categories, we can simply talk about "injective object" and "relatively injective object" in $\mathcal{A}$ below.

Proposition 5.14 *Let $E \in \mathcal{A}$. Suppose E is relatively injective and $\mathcal{Op}^\infty$-injective. Then E is injective.*

Proof Take $\mu \in \mathcal{M}(E, F)$ for some $F \in \mathcal{A}$. By the lemma above, we are done if there exists $v \in CB_A(F, E)$ such that $v\mu = \mathrm{id}_E$. As μ is an admissible monomorphism there exists a kernel-cokernel pair in $\mathcal{Ex}_{max}$:

$$
(5.4) \qquad\qquad E \overset{\mu}{\rightarrowtail} F \overset{\pi}{\twoheadrightarrow} G,
$$

which gives a kernel-cokernel pair in $(\mathcal{Op}^\infty, \mathcal{Ex}_{min})$

$$
(5.5) \qquad\qquad E \overset{\mu}{\rightarrowtail} F \overset{\pi}{\twoheadrightarrow} G.
$$

Since E is $\mathcal{Op}^\infty$-injective, E is an absolute retract in $\mathcal{Op}^\infty$ (Lemma 5.13); hence there exists $\theta \in CB(F, E)$ such that $\theta\mu = \mathrm{id}_E$. By the Splitting Lemma (Lemma 4.10), $(\mu, \pi) \in \mathcal{Ex}_{min}$ in $\mathcal{Op}^\infty$, in (5.5) and therefore $(\mu, \pi) \in \mathcal{Ex}_{rel}$ in (5.4). In particular, $\mu \in \mathcal{M}_{rel}$. As E is relatively injective, by the other implication in Lemma 5.13, there exists $v \in CB_A(F, E)$ such that $v\mu = \mathrm{id}_E$ as required so that E is injective in $\mathcal{A}$. ∎

With the same caveats as above, we obtain a converse under an additional assumption.

Proposition 5.15 *Suppose that, for every $E \in \mathscr{A}$, there exists $\mu \in CB_A(E, J_E)$ for some $J_E \in \mathscr{A}$ such that J_E is $\mathscr{O}p^\infty$-injective. If $I \in \mathscr{A}$ is injective then I is relatively injective as well as $\mathscr{O}p^\infty$-injective.*

Proof Let $I \in \mathscr{A}$ be injective in $\mathscr{A}$; then it is clearly relatively injective. By assumption, there exist $J_I \in \mathscr{A}$ which is $\mathscr{O}p^\infty$-injective and a kernel cokernel pair in $\mathscr{E}x_{max}$:

$$I \overset{\mu}{\rightarrowtail} J_I \overset{\pi}{\twoheadrightarrow} G.$$

By Lemma 5.13, there exists $\nu \in CB_A(J_I, I)$ such that $\nu\mu = \mathrm{id}_I$. This identity persists in $\mathscr{O}p^\infty$ so that I is a retract of the $\mathscr{O}p^\infty$-injective operator space J_I. Hence I is injective in $\mathscr{O}p^\infty$ too. ∎

For clarity, we formulate the individual statements for the two categories involved separately.

Corollary 5.16 *Let A be a unital operator algebra and $E \in \mathscr{O}\!\mathscr{M}\!od_A^\infty$. Then E is $\mathscr{E}x_{max}$-injective in $(\mathscr{O}\!\mathscr{M}\!od_A^\infty, \mathscr{E}x_{max})$ if and only if E is $\mathscr{E}x_{max}$-injective in $(\mathscr{O}p^\infty, \mathscr{E}x_{max})$ and $\mathscr{E}x_{rel}$-injective in $(\mathscr{O}\!\mathscr{M}\!od_A^\infty, \mathscr{E}x_{rel})$.*

The "if"-part follows directly from Proposition 5.14 and the "only if"-part follows from Proposition 5.15 together with the CES theorem [6, Theorem 3.3.1]: for every $E \in \mathscr{O}\!\mathscr{M}\!od_A^\infty$ there exists a Hilbert space H such that $B(H) \in \mathscr{O}\!\mathscr{M}\!od_A^\infty$ and E is a closed submodule of $B(H)$. The fact that $B(H)$ is the prototypical injective operator space finished the argument.

This result was obtained for $C^\star$-algebras in [28, Proposition 5.11]. For $E = A$ a unital $C^\star$-algebra, the "if"-part is also given by [14, Theorem 3.5].

The next result implies [10, Proposition 2.3], without the estimate on the constants which are irrelevant in the completely bounded category. It follows straight from Proposition 5.14.

Corollary 5.17 *Let A be a unital operator algebra and $E \in mn\mathscr{M}\!od_A^\infty$. Then E is $\mathscr{E}x_{max}$-injective in $(mn\mathscr{M}\!od_A^\infty, \mathscr{E}x_{max})$ if E is $\mathscr{E}x_{max}$-injective in $(\mathscr{O}p^\infty, \mathscr{E}x_{max})$ and $\mathscr{E}x_{rel}$-injective in $(mn\mathscr{M}\!od_A^\infty, \mathscr{E}x_{rel})$.*

The converse direction, however, fails in general: by the operator algebra version of [29, Corollary 4.11], $CB(A, G)$ is always $\mathscr{E}x_{max}$-injective for an injective operator space G and thus automatically $\mathscr{E}x_{rel}$-injective. So we can choose for A a $C^\star$-algebra whose dual $A^* = CB(A, \mathbb{C})$ is not injective in $\mathscr{O}p^\infty$. An example using the (completely contractive) Fourier–Stieltjes algebra $B(\Gamma)$ for a non-amenable discrete group Γ is discussed in [10, p. 1068]; there, $E = C^*(\Gamma)^{**} = CB(B(\Gamma), \mathbb{C})$.

Considered as a $C^*(\Gamma)$-module on the other hand, $C^*(\Gamma)^{**}$ is not injective in $\mathscr{O}\!\mathscr{M}\!od_{C^*(\Gamma)}^\infty$ by [14, Theorem 3.4]. This reveals a subtle difference between the two operator space module categories.

The above results help to understand the issue of the existence of enough injectives in our module categories; that is, for each $E \in \mathscr{A}$, do there exist $I \in \mathscr{A}$ injective and $\mu \in \mathscr{M}(E, I)$? When A is a $C^\star$-algebra, $B(H)$ supplies $(\mathscr{O}\!\mathscr{M}\!od_A^\infty, \mathscr{E}x_{max})$ with enough injectives, however it loses its role when A is a general operator algebra [31, Example

3.5] and in fact, the question seems to be open. For $C^\star$-algebras, [29, Proposition 4.13] answers the question affirmatively for $(mn\mathcal{M}od_A^\infty, \mathcal{E}x_{max})$, and the argument extends to general (unital) operator algebras. The question remains unresolved for either of the two categories with the relative structure $\mathcal{E}x_{rel}$. The expectation seems to be that it fails, compare [2] and [3], for example. By the above corollaries, a module which is not injective in $\mathcal{O}p^\infty$ cannot be embedded into a module which is injective in $\mathcal{E}x_{max}$ which restricts the possible choices to modules which are only $\mathcal{E}x_{rel}$-injective and not injective in $\mathcal{E}x_{max}$ nor in $\mathcal{O}p^\infty$.

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