Can. J. Math., Vol. XXXIX, No. 1, 1987, pp. 168-221

ON REPRESENTATIONS OF GROTHENDIECK TOPOSES

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Introduction. Results of a representation-theoretic nature have played a major role in topos theory since the beginnings of the subject. For example, Deligne's theorem on coherent toposes, which says that every coherent topos has a continuous embedding into a topos of the form $\mathbf{Set}^I$ for a discrete set I, is a typical result in the representation theory of toposes. (A continuous functor between toposes is the left adjoint of a geometric morphism. For Grothendieck toposes, it is exactly the same as a continuous functor between them, considered as sites with their canonical topologies. By a continuous functor between sites on left exact categories, we mean a left exact functor taking covers to covers.)

A representation-like result for toposes typically asserts that a topos that satisfies some abstract conditions is related to a topos of some concrete kind; the relation between them is usually an embedding of the first topos in the second (concrete) one, for which the embedding satisfies some additional properties (fullness, etc.).

Freyd **[8]** gives important representation-like results for elementary toposes. Barr **[4]** shows that every Grothendieck topos has a continuous embedding in a Boolean Grothendieck topos. More recently, Joyal and Tierney **[12]** have developed representation-like theorems for toposes bounded over any topos, a context that is the proper generalization of the concept of Grothendieck topos.

Several earlier workers in the subject knew from the beginning that results and techniques of model theory were closely related to representation-like results for toposes. For example, Freyd translated the "method of diagrams" in model theory into limit slices and exploited it to great effect in his 1972 paper. The work of Makkai and Reyes **[18]** was intended to make these connections explicit. That monograph used model-theoretic techniques to derive new results in topos theory.

Nonetheless, model-theoretic methods as such remained foreign to the topos literature. Topos theorists tend to ignore results that are proved using model theory. For example, both Johnstone **[10]** and Joyal and Tierney **[12]** show that every **S**-bounded topos has an **S**-spatial open cover, but have ignored the fact that such a result appears for Grothendieck toposes in **[18]**, Theorem 6.3.1.

Received May 26, 1985.

The main results of the present paper are representation-like results for certain kinds of Grothendieck toposes. The methods and some of the results were inspired by model theory. However, in order to avoid the above-mentioned difficulties, we have taken care to translate all our arguments into category theoretic language.

To describe the main results, we require a few definitions. We say that a Grothendieck topos (GT) is *atomic* if all its subobject lattices are complete atomic Boolean algebras [**5**] and that a GT is *prime-generated* if in every subobject lattice every object is the join of a set of join irreducible elements [**14**, **16**]. Clearly, a GT is atomic if and only if it is prime generated and Boolean. We say that a site is *separable* if the underlying category is countable and if there is a countable base for the topology. A GT is *separable* if it has a separable site of definition. A site is *coherent* if every cover has a finite refinement.

We say that a functor $F:\mathscr{A} \rightarrow \mathscr{B}$ is *powerful* if it induces a bijection $\mathrm{Sub}(A) \rightarrow \mathrm{Sub}(FA)$ on subobject lattices for each object A of $\mathscr{A}$. It is easy to prove that a powerful left exact embedding is full and faithful. Notice that if the GT $\mathscr{A}$ has a continuous powerful embedding into an atomic, resp. prime-generated GT, then $\mathscr{A}$ is itself atomic, resp. prime generated.

The simplest kind of atomic GT is the category G-**Set** of G-sets for a group G. It is quite easy to see that a topos has a continuous powerful embedding into G-**Set** if and only if it is equivalent to the category $\hat{G}$-**Set** for a topological group $\hat{G}$ with underlying group G.

The following representation theorems for connected atomic GTs are well known from the point of view of model theory and were given explicitly in [**16**]: Any separable, or any coherent, connected atomic GT has a continuous powerful embedding into a category G-**Set**. Certainly not all atomic GTs can be so represented; the same paper gives an example of a connected atomic GT without any points.

The simplest kind of prime-generated GT is any presheaf category $\mathbf{Set}^{\mathbf{K}}$ for a small category **K**. Thus any GT which has a continuous powerful embedding into a presheaf category is prime-generated. One of our main results is that for a separable GT the converse is true; a separable prime-generated GT has a continuous powerful embedding into a presheaf category. We also show that in the case of sheaves over a regular epimorphism site, the result holds without any size restriction. The fact that a regular epimorphism site has a full embedding into a presheaf category has long been known [**3**, **2**].

In recent but well-established topos theoretic terminology, a geometric morphism whose inverse image is powerful is called hyperconnected. Thus the two above mentioned results can be paraphrased by saying that any separable prime-generated GT and any regular epimorphism sheaf topos has a hyperconnected cover by a presheaf topos.

Another result is that any $\aleph_1$-presentable prime-generated topos has enough points. Here $\aleph_1$-presentable means that it is the category of sheaves on a site whose underlying category has cardinality $\leqq \aleph_1$ and for which the topology has a base of at most that cardinality. This result generalizes the corresponding result for atomic toposes which is essentially equivalent to an unpublished result of Leo Harrington's. The atomic topos with no points mentioned above is $(2^{\aleph_0})^+$ presentable (thus $\aleph_2$-presentable if we assume the continuum hypothesis). Of course, the result mentioned above gives only a continuous conservative embedding into a discrete functor category. It is a very interesting question if any $\aleph_1$-presentable prime-generated (resp. atomic) topos has a powerful (or even full and faithful) embedding into a presheaf category (resp. into G-**Set** for some group G).

We also prove a "conceptual completeness" result of which a weak version says that if a separable geometric morphism between separable atomic toposes induces an equivalence on the categories of points, then it is itself an equivalence. We say that a geometric morphism is *separable* if its left adjoint takes σ-coherent objects into σ-coherent objects. This result is analogous to the conceptual completeness for coherent toposes proved in **[18]**, Theorem 9.2.9. It was, more directly, inspired by an unpublished theorem of Haim Gaifman (which is essentially equivalent to Theorem 3.1.4 of **[15]**).

We also give a very simple proof of the fact that a separable topos is connected atomic if and only if it has, up to isomorphism, exactly one countable point. This simply stated result generalizes the well-known (to model theorists) Ryll-Nardzewski Theorem, as well as its version given by Keisler **[13]** for countable fragments of $L_{\omega,\omega}$.

Besides the results, we consider important the technical tools employed in proving them. Several of them are inspired by model theory, but they are worked out in the language of category theory. In this, we are following the lead of Peter Freyd who invented the technique of limit slices as a conscience translation of the method of diagrams employed in model theory. We hope that these techniques find other uses and that they will help bring category theory and logic closer to each other.

Sections 1 and 2 of the paper contain general material, part of which may be folklore. Section 3 does not rely on Section 2. Sections 3 and 4 contain the main results of the paper.

Both authors gratefully acknowledge support of grants from the Natural Science and Engineering Research Council of Canada, of a team grant from the Ministère de l'Education du Québec and of a grant from the same source to the Centre Interuniversitaire en Etudes Catégoriques. In addition, both authors were guests of the University of Sydney during the summer (winter?) of 1984, thanks to a grant from their research council and in fact our collaboration on this paper began there.

1. General concepts and results. In this paper a *site* always means a left exact category equipped with a Grothendieck topology. If $\mathscr{B} = (\mathbf{B}, \mathbf{J})$ and $\mathscr{C} = (\mathbf{C}, \mathbf{K})$ are sites, then a morphism $I:\mathscr{B} \to \mathscr{C}$ is a left exact functor between the underlying categories that takes covers to covers. Such an I is called $\mathbf{J},\mathbf{K}$-continuous, or simply continuous, when $\mathbf{J}$ and $\mathbf{K}$ are understood. We denote the category of morphisms $\mathscr{B} \to \mathscr{C}$ by $\mathrm{Con}(\mathscr{B}, \mathscr{C})$. Any collection of families $\{A_s \to A : s \in S\}$ in a category $\mathbf{B}$ generates a unique topology $\mathbf{J}$ or equivalently a unique site $\mathscr{B} = (\mathbf{B}, \mathbf{J})$. If $I:\mathbf{B} \to \mathbf{C}$ is left exact, then I is $\mathbf{J},\mathbf{K}$-continuous if and only if I takes the generating sieves into $\mathbf{K}$-covers.

The category **Set** of sets will always be considered a site with its canonical topology: a cover is a surjective family. A morphism $\mathscr{C} \to \mathbf{Set}$ is also called a *model* of $\mathscr{C}$ and the category of models is called $\mathrm{Mod}(\mathscr{C})$.

If $\mathscr{C}$ is a site, we let $\mathbf{Sh}(\mathscr{C})$ denote the category of sheaves on $\mathscr{C}$ and $\epsilon:\mathscr{C} \to \mathbf{Sh}(\mathscr{C})$ the functor that associates to each object of $\mathbf{C}$ the sheaf associated to $\mathrm{Hom}(-, C)$. Unless another topology is explicitly mentioned, we will always consider a Grothendieck topos to be a site with its canonical topology, the one in which the universal effective epimorphic families are the covers. ϵ is characterized by a universal mapping property: for every Grothendieck topos $\mathscr{E}$, the functor

$$\mathrm{Con}(\mathbf{Sh}(\mathscr{C}), \mathscr{E}) \to \mathrm{Con}(\mathscr{C}, \mathscr{E})$$

induced by ϵ is an equivalence of categories. In particular, every continuous functor $\mathscr{C} \to \mathscr{E}$ has an extension, unique up to isomorphism, to a continuous functor $\mathbf{Sh}(\mathscr{C}) \to \mathscr{E}$ ([**6**], Exercise UNIV of Section 7.3). In particular, taking $\mathscr{E} = \mathbf{Set}$, we conclude that ϵ induces an equivalence

$$\mathrm{Mod}(\mathbf{Sh}(\mathscr{C})) \to \mathrm{Mod}(\mathscr{C}).$$

As a matter of fact, it is convenient to view continuous morphisms with arbitrary codomain as generalized models.

A continuous functor from the GT $\mathscr{E}$ to the GT $\mathscr{F}$ is the same as the left adjoint part $p^*:\mathscr{E} \to \mathscr{F}$ of a geometric morphism $p:\mathscr{F} \to \mathscr{E}$ (see [**11**] or [**6**]). The universal property of $\epsilon:\mathscr{C} \to \mathbf{Sh}(\mathscr{C})$ mentioned above implies that there is an equivalence between $\mathrm{Con}(\mathscr{C}, \mathbf{Sh}(\mathscr{D}))$ and the category of geometric morphisms $\mathbf{Sh}(\mathscr{D}) \to \mathbf{Sh}(\mathscr{C})$. The geometric morphism induced by $U:\mathscr{C} \to \mathscr{D}$ will be denoted

$$\mathbf{Sh}(U):\mathbf{Sh}(\mathscr{D}) \to \mathbf{Sh}(\mathscr{C}).$$

Let $\mathbf{C}$ be a category and $D:\mathbf{T} \to \mathbf{C}$ be a diagram in $\mathbf{C}$. There is associated to each such diagram a functor $M:\mathbf{C} \to \mathbf{Set}$ defined as $\mathrm{colim}\ \mathrm{Hom}(DT, -)$. We will call M the functor *represented* by D. It is well known that a set-valued functor on a left exact category is left exact if and only if it can be represented by a cofiltered diagram. The following variation is just as easy to prove and is left to the reader.

PROPOSITION 1.1. *Let* $D:\mathbf{T} \rightarrow \mathscr{C}$ *be a cofiltered diagram in the site* $\mathscr{C}$ *and* $\mathbf{J}_0$ *be a base for the topology. Then the functor it represents is a model of* $\mathscr{C}$ *if and only if for every* $\mathbf{J}_0$*-sieve* $\{C_i \rightarrow C\}$ *and every morphism* $g:DT \rightarrow C$, *there is at least one* $T' \rightarrow T$ *in* $\mathbf{T}$ *and* $DT' \rightarrow C_i$ *such that*

$$\begin{array}{ccc} DT' & \longrightarrow & DT \\ \downarrow & & \downarrow g \\ C_i & \longrightarrow & C \end{array}$$

commutes.

We remark that if $\mathbf{J}_0$ is pullback invariant, in particular if it is a topology, this condition need be verified only when g is the identity morphism.

The following proposition generalizes the "Lemme de comparaison", [**1**], Exposé III, p. 288.

PROPOSITION 1.2. *Let* $U:\mathscr{B} \rightarrow \mathscr{C}$ *be a morphism of sites satisfying the following conditions.*

(i) *U is cocontinuous: if the family* $\{B_i \rightarrow B\}$, $i \in I$ *in* $\mathscr{B}$ *is such that its U-image* $\{UN_i \rightarrow UB\}$, $i \in I$ *is a cover in* $\mathscr{C}$, *then* $\{B_i \rightarrow B\}$, $i \in I$ *is a cover in* $\mathscr{B}$.

(ii) $\mathscr{B}$ *generates* $\mathscr{C}$ *via U: every object of* $\mathscr{C}$ *has a cover by objects of the form UB, B an object of* $\mathscr{B}$.

(iii) *U is locally full: whenever* $g:UB \rightarrow UB'$ *is a morphism in* $\mathscr{C}$, *there is* $\mathscr{B}$*-cover* $\{f_i:B_i \rightarrow B\}$, $i \in I$ *of* $\mathscr{B}$ *such that all the composites* $g \circ Uf_i$ *are of the form* Uh_i *for* $h_i:B_i \rightarrow B'$.

Then the geometric morphism $\mathbf{Sh}(U):\mathbf{Sh}(\mathscr{B}) \rightarrow \mathbf{Sh}(\mathscr{C})$ *induced by U is an equivalence. As a consequence, for every* GT $\mathscr{E}$, *the functor*

$$U^*:\mathrm{Con}(\mathscr{C}, \mathscr{E}) \rightarrow \mathrm{Con}(\mathscr{B}, \mathscr{E})$$

induced by U, is an equivalence of categories. The quasi-inverse of U^* *is* $U_!$ *for which* $U_!(M)$ *is the left Kan-extension of M along U, for any* $M \in \mathrm{Con}(\mathscr{B}, \mathscr{E})$.

Sketch of proof. Let us first note that for any site $\mathscr{C}$, the canonical continuous functor $\epsilon:\mathscr{C} \rightarrow \mathbf{Sh}(\mathscr{C})$ satisfies the three conditions of the proposition; this is proved in [**1**], and the proof is reproduced in [**18**]; see Lemma 1.3.8 in [**18**].

By direct and routine arguments, one can prove two lemmas as follows. Let us say, temporarily, that a morphism of sites is "nice" if it satisfies the conditions of the proposition above. Suppose $U:\mathscr{B} \rightarrow \mathscr{C}$, $V:\mathscr{C} \rightarrow \mathscr{D}$ are morphisms of sites. Then, if U and V are both nice, so is $V \circ U$ (first lemma), and if $V \circ U$ and U are both nice, so is V (second lemma). Now, the diagram of functors

$$\begin{array}{ccc} \mathscr{B} & \xrightarrow{\quad U \quad} & \mathscr{C} \\ {\scriptstyle \epsilon_{\mathscr{B}}}\downarrow & & \downarrow{\scriptstyle \epsilon_{\mathscr{C}}} \\ \mathbf{Sh}(\mathscr{B}) & \xrightarrow[\mathbf{Sh}(U)^*]{} & \mathbf{Sh}(\mathscr{C}) \end{array}$$

commutes up to isomorphism. Supposing that U is nice, it follows from the two lemmas stated above that $\mathbf{Sh}(U)^*$ is nice. Finally, the main assertion of the proposition will be proved by showing that a nice continuous functor between GTs is necessarily an equivalence.

The verification of the last assertion can conveniently be broken up into two steps. First, one has

LEMMA 1.3. *Suppose the continuous functor $U:\mathscr{E} \to \mathscr{F}$ between* GT*s satisfies the following conditions*:

(i) *U is localic: for any X in $\mathscr{F}$ there is a diagram of the form*

$$\begin{array}{ccc} Y & \overset{\text{mono}}{\rightarrowtail} & UA \\ {\scriptstyle \text{epi}}\downarrow & & \\ X & & \end{array}$$

(ii) *U is full on subobjects: U induces a surjection*

$$\text{Sub}_{\mathscr{E}}(A) \to \text{Sub}_{\mathscr{F}}(UA)$$

for every A in $\mathscr{E}$.

(iii) *U is conservative (reflects isomorphisms).*

Then U is an equivalence of categories.

The last lemma is a direct consequence of Lemma 7.1.7 or Lemma 1.4.9 of [**18**] which, in turn, are restatements of a lemma in [**1**]. (Lemma 1.3 will be used again in this section.) Finally, one should show that a nice functor between GTs satisfies the conditions in Lemma 1.3. This is a routine verification, and it is left to the reader.

For the last assertion of Proposition 1.2, we have to show, first of all, that for any $M \in \text{Con}(\mathscr{B}, \mathscr{E})$, the Kan-extension $\hat{M}$ of M along U, $\hat{M}:\mathscr{C} \to \mathscr{E}$, is in $\text{Con}(\mathscr{C}, \mathscr{E})$. Suppose we have shown this. Then, since in

$$\mathbf{Cat}(\mathbf{C}, \mathbf{E}) \underset{U_!}{\overset{U^{\cdot}}{\rightleftarrows}} \mathbf{Cat}(\mathbf{B}, \mathbf{E})$$

https://doi.org/10.4153/CJM-1987-009-x Published online by Cambridge University Press

where $U^{\cdot}$ is defined by composition, $U^{\cdot}(N) = N \circ U$, and $U_!(M)$ is the left Kan-extension of M, we have that

$$U_! \dashv U^{\cdot}$$

and both $U^{\cdot}$ and $U_!$ take the full subcategories of the continuous functors into each other, the restrictions of $U^{\cdot}$ and $U_!$ to the continuous functors are adjoint to each other. But we also know that the restriction of $U^{\cdot}$, which is U^*, is an equivalence. It follows that the restriction of $U_!$ to $\mathrm{Con}(\mathscr{B}, \mathscr{E})$ is the quasi-inverse of U^*.

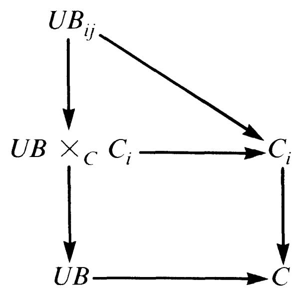
It remains to show that $\hat{M}$, the left Kan-extension of $M \in \mathrm{Con}(\mathscr{B}, \mathscr{E})$, is in $\mathrm{Con}(\mathscr{C}, \mathscr{E})$. Recall that $\hat{M}$ is given by

$$\hat{M}C = \operatorname*{colim}_{(B, IB \to C)} MB.$$

(The index category of this colimit is $(I/C)^{\mathrm{op}}$.) We need to show, first of all, that $\hat{M}$ is left exact. This turns out to be a surprisingly difficult task. For the case $\mathscr{E} = \mathbf{Set}$ (the only one we need), it is easy: first of all, one verifies that the Kan-extension of the representable functor $\mathbf{B}(B, -)$ is $\mathbf{C}(UB, -)$; secondly, since M is left exact, it is a filtered colimit of representable functors; since taking left Kan-extension preserves colimits, it follows that $\hat{M}$ is a filtered colimit of representables, hence a left exact functor. Since we will need the last assertion of the proposition only for the case of $\mathscr{E} = \mathbf{Set}$, we will not give the proof for the general case; we remark only that one can use the case $\mathscr{E} = \mathbf{Set}$ to handle the case of any $\mathscr{E}$ having enough points. The general case could be proved by a refinement of the argument used in [**6**], Proposition 7.3.1.

Finally, we have to show that $\hat{M}$ takes a $\mathbf{C}$-cover into a canonical cover in $\mathscr{E}$.

Let $\{C_i \to C\}$ be a cover in $\mathscr{C}$, and let $(B, UB \to C)$ be an index of the colimit. Consider the following commutative diagram



in which the square is a pullback and, for each i, the family

$$\{UB_{ij} \to UB \times_C C_i\}$$

is a $\mathscr{C}$-cover and, for all i and j, the vertical composite comes from a map

$B \to B_{ij}$ (conditions (ii) and (iii) in 1.2). Clearly,

$$\{UB_{ij} \to UB\}_{i,j}$$

is a $\mathscr{C}$-cover, hence $\{B_{ij} \to B\}_{i,j}$ is a cover in $\mathscr{B}$ (condition (i) in 1.2), hence

$$\{MB_{ij} \to MB\}_{i,j}$$

is a cover in $\mathscr{E}$. For any $(B, UB \to C)$, we have constructed commutative diagrams

$$\begin{array}{ccc} MB_{ij} & \longrightarrow & MB \\ \downarrow & & \downarrow \\ \hat{M}C_i & \longrightarrow & \hat{M}C \end{array}$$

in which the vertical arrows are given by the transition maps from the elements of the cone whose colimit is $\hat{M}C_i$, resp. $\hat{M}C$. Since the colimit $\hat{M}C$ in the GT $\mathscr{E}$ is covered by the family of the canonical transition arrows, and for each $(B, UB \to C)$, the arrows $MB_{ij} \to MB$ cover MB, it follows that $\{\hat{M}C_i \to \hat{M}C\}_i$ is a cover as desired.

We are going to be constructing functors by constructing the representing diagrams and we must consider various kinds of restrictions on the diagram, some on its size and some on the objects that make it up.

Let M:**C** $\to$ **Set** be a left exact functor. We say that a family of elements $\{(A_i, a_i \in MA_i): i \in I\}$ *generates* M if for all objects C of **C** and all elements $c \in MC$ there is an $i \in I$ and an $f:A_i \to C$ such that

$$Mf(a_i) = c.$$

It is trivial to see that this is equivalent to the assertion that no proper subfunctor of M contains all the a_i.

For a cardinal μ, we will say that M is *μ-generated* if there is a family of elements of cardinality at most μ that generates M.

Given a functor M:**C** $\to$ **Set** and an element $c \in MC$, the *type* of c, denoted $\mathbf{t}_C^M(c)$, or more simply $\mathbf{t}(c)$, is the set of all subobjects $X \subseteq C$ such that c is in the image of $MX \to MC$. We will usually write, by abuse of notation, $c \in MX$ when $X \in \mathbf{t}(c)$.

For a cardinal μ we will say that M is *μ-presented* if it is μ-generated and for every element $c \in MC$, $\mathbf{t}(c)$ has a filter base of cardinality at most μ.

A class **P** of objects of $\mathscr{C}$ is called *dense* if for every object C there is a cover $\{P_i \to C\}$ in which every $P_i \in$ **P**.

PROPOSITION 1.4. *Suppose* **C** *is a left exact category which has a factorization system of extremal epis followed by monos. Then*

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(i) *Every representable set-valued functor is* $\aleph_0$*-presentable.*

(ii) *A colimit of a diagram of size at most* μ *of* μ*-generated* (*resp.* μ*-presented*) *set-valued functors is again* μ*-generated* (*resp.* μ*-presented*).

(iii) *Conversely, every* μ*-presented left exact functor is a filtered colimit of a diagram of size at most* μ *of representable functors.*

(iv) *More precisely, let* $\mathbf{P}$ *be a dense class of objects of the site* $\mathscr{C}$. *Then if* M *is a* μ*-presentable model of* $\mathscr{C}$, *it is the filtered colimit of a diagram of size at most* μ *of representables, each represented by an object of* $\mathbf{P}$.

(v) *If* $\mu = \aleph_0$, *then any colimit of a countable filtered diagram is a colimit of a chain of type* ω *which can be taken to be a subdiagram of the given diagram.*

Proof. (i) $\mathrm{Hom}(C, -)$ is generated by a single element, the identity of C; if $f:C \to A$ is an element of $M(A)$ (for $M = \mathrm{Hom}(C, -)$), then $\mathbf{t}(f) \subseteq \mathrm{Sub}(A)$ is generated by the single element $\mathrm{Im}(f)$ (defined by assumption).

(ii) This is left to the reader.

(iii) This is a special case of (iv).

(iv) In this argument, we will use P, P_i, etc. to denote objects from the set $\mathbf{P}$. Begin by choosing, for each of the generators $c_i \in C_i$, a morphism $g_i:P_i \to C_i$ and an element $p_i \in MP_i$ such that

$$Mg_i(p_i) = c_i.$$

Let $\mathbf{T}_0$ be the discrete diagram with vertices (P_i, p_i) and $D_0:\mathbf{T}_0 \to \mathscr{C}$ take (P_i, p_i) to P_i. For each pair of vertices (P_1, p_1), (P_2, p_2) there is a $P \in \mathbf{P}$, a morphism

$$g:P \to P_1 \times P_2$$

and a $p \in MP$ for which

$$Mg(p) = (p_1, p_2).$$

For each such pair add the object (P, p) to $\mathbf{T}_0$ along with an arrow to each of (P_1, p_1) and (P_2, p_2). D_0 is extended by letting its value on these arrows be the composites of the product projections and g. Do this for each pair of objects in $\mathbf{T}_0$ and call the resultant diagram

$$D_1:\mathbf{T}_1 \to \mathscr{C}.$$

At the next stage in the construction, for each object (P, p) in the diagram D_1, choose a basis of $\mathbf{t}(p)$ of cardinality at most μ and for each subobject in that basis, let (P', p') and $f:P' \to P$ be chosen so that f factors through X and

$$Mf(p') = p.$$

Add these vertices and these arrows to the diagram and call the resultant diagram

https://doi.org/10.4153/CJM-1987-009-x Published online by Cambridge University Press

$$D_2:\mathbf{T}_2 \to \mathscr{C}.$$

The third step in our process is similar to the first, except that we will suppose inductively that each object in the diagram has only a finite number of arrows out of it. Then replace the product in that construction by the finite limit of the diagram consisting of P_1, P_2 and all the arrows in the diagram out of them. The element (p_1, p_2) must belong to that inverse limit. Continuing in this way, alternating these two processes, we build up a diagram $D:\mathbf{T} \to \mathscr{C}$ which is filtered, is of size at most μ and has the additional property that if (P, p) is a vertex and $C \subseteq P$ is a subobject with $p \in MC$, then there is a $(P', p') \to (P, p)$ in the diagram for which the image of $P' \to P$ lands inside C. Let $N:\mathscr{C} \to \mathbf{Set}$ be the functor represented by D. There is an obvious natural transformation $N \to M$, which takes the element of NC represented by the morphism

$$f:(P, p) \to C$$

to the element $Mf(p)$. This is surjective since the image includes the generators. Suppose then that (P_1, p_1) and (P_2, p_2) are two vertices in the diagram and that

$$f_1:P_1 \to C \quad \text{and} \quad f_2:P_2 \to C$$

are two morphisms with

$$Mf_1(p_1) = Mf_2(p_2).$$

Then we can find a vertex (P, p) and arrows

$$g_i:(P, p) \to (P_i, p_i), \quad i = 1, 2.$$

Then let C_0 be the equalizer of $f_1 \circ g_1$ and $f_2 \circ g_2$. By hypothesis, $C_0 \in \mathbf{t}(p)$, so there is an $h:(P', p') \to (P, p)$ whose image is contained in C_0. But this means that

$$f_1 \circ g_1 \circ h = f_2 \circ g_2 \circ h$$

which means that f_1 and f_2 represent the same element of NC, whence $N \to M$ is also injective.

(v) is trivial.

Remark. The restriction above that the category must have a factorization can be avoided by modifying the definition of filter base slightly. Simply say that a class of morphisms $\{C_i \to C\}$ is a *base* for the filter if for any $C_0 \subseteq C$ which belongs to the filter, there is an i such that $C_i \to C$ factors through C_0. However, we have no need here for such a modification as all our categories have factorization systems.

Let α be an infinite regular cardinal and $\mathscr{E}$ be a GT. An object E of $\mathscr{E}$ is called *α-compact* if every cover of E has a refinement with (strictly) fewer than α morphisms. The α-compact object E is called *α-coherent* if in any pullback square

$$\begin{array}{ccc} E_1 & \longrightarrow & E_2 \\ \downarrow & & \downarrow \\ E_3 & \longrightarrow & E \end{array}$$

if E_2 and E_3 are α-compact, so is E_1. These notions generalize in an obvious way those of compact and coherent of [**1**] from $\aleph_0$ to an arbitrary regular cardinal. The following proposition is proved in the case $\alpha = \aleph_0$ in [**1**] and reproduced with some changes in [**18**], Chapter 9, Section 2. The proof given there for the countable case goes through in the general case without change.

PROPOSITION 1.5. *Suppose the topology on the left exact site $\mathcal{C}$ is generated by covering families of size $<\alpha$. Then* $\mathrm{Coh}_\alpha(\mathcal{E})$, *the full subcategory of α-coherent objects of $\mathcal{E} = \mathbf{Sh}(\mathcal{C})$ is closed under finite limits, universal disjoint sums of families of fewer than α objects and coequalizers of universal effective equivalence relations. Moreover, $\epsilon:\mathcal{C} \to \mathcal{E}$ factors through* $\mathrm{Coh}_\alpha(\mathcal{E})$.

We say that a site $\mathcal{C} = (\mathbf{C}, \mathbf{J})$ has *size at most* μ if the underlying category has cardinality at most μ and if there is a $\mathbf{J}_0 \subseteq \mathbf{J}$ of cardinality at most μ that generates $\mathbf{J}$. The GT $\mathcal{E}$ is *μ-presentable* if it has a defining site of size at most μ. An $\aleph_0$-presentable GT is called *separable*. We also say "separable site" or "site of size at most $\aleph_0$".

By a *regular* site we mean a site on a left exact category in which every cover is an extremal family (that is, every cover $\{A_i \to A\}$ has the property that if $B \subseteq A$ is a subobject for which every $A_i \to A$ factors through B, then $B = A$ (as a subobject of A)) and in which every morphism is of the form $i \circ q$ with i a monomorphism and q a cover (hence, in particular, an extremal epi). In a regular site, a single morphism is a cover if and only if it is an extremal epi. Also, the subcollection of the topology consisting of the single extremal epis and the covers by monomorphisms generate the topology.

The covering families of subobjects are not supposed effective. The reason is that we will be dealing mainly with countable sites and an infinite effective family would likely lead to uncountably many morphisms. These families will become effective in the category of sheaves.

If $(\mathcal{C}, \mathbf{J})$ is a site, we will use the notation

$$\bigvee^{(\mathbf{J})}_{i \in I} A_i = A$$

(read $\mathbf{J}$-join) to indicate that $\{A_i\}$ is a collection of subobjects of A which cover A in the topology. Similarly, if $\{A_i\}$ is a collection of subobjects of A, all included in the subobject A_0 and covering A_0, we will write

$\bigvee^{(\mathbf{J})}_{i \in I} A_i = A_0.$

PROPOSITION 1.6. *Let* μ *be an infinite cardinal. A* μ*-presentable* GT *has a regular site of definition of size at most* μ.

Proof. Let $\mathscr{C}$ be a defining site of size at most μ and let

$$\epsilon' : \mathscr{C} \to \mathrm{Coh}_{\leqq \mu}(\mathscr{E})$$

factor ϵ. We construct a (non-full) subcategory

$$\mathbf{D} \subseteq \mathrm{Coh}_{\leqq \mu}(\mathscr{E})$$

such that the size of $\mathbf{D}$ is $\leqq \mu$ and such that the following hold:

(i) ϵ' factors through $\mathbf{D}$.

(ii) $\mathbf{D}$ has finite limits and the inclusion

$$\mathbf{D} \to \mathrm{Coh}_{\leqq \mu}(\mathscr{E})$$

preserves and reflects them.

(iii) For any object D of $\mathbf{D}$, there is a covering family $\{\epsilon C_i \to D\}$ of size $\leqq \mu$.

(iv) If f is a morphism of $\mathbf{D}$ and $f = m \circ e$ with m a mono in $\mathscr{E}$ and e a regular epi in $\mathscr{E}$, then up to isomorphism, both e and m lie in $\mathbf{D}$.

Since every object in $\mathbf{D}$ is μ-compact, and $\mathrm{Coh}_{\leqq \mu}(\mathscr{E})$ is closed under finite limits, $\mathbf{D}$ is easy to construct as required. Let $\mathbf{K}$ be the topology on $\mathbf{D}$ generated by all the ϵ' images of the covers in a base of $\mathbf{J}$ of size $\leqq \mu$ together with all single extremal epimorphisms in D. Then $\mathscr{D} = (\mathbf{D}, \mathbf{K})$ is a site of size at most μ. The functor $U : \mathbf{C} \to \mathbf{D}$ given by (i) is left exact by (ii). By the definition of the topology $\mathbf{K}$, U is continuous. We can apply Proposition 1.2 to U. The facts that U is continuous and locally full both follow from the respective properties of $\epsilon : \mathscr{C} \to \mathscr{E}$. That $\mathscr{D}$ is generated by $\mathscr{C}$ via U is assured by condition (iii) of the construction. Thus by 1.2, $\mathscr{D}$ is a defining site for $\mathscr{E}$. Also every cover Φ in $\mathscr{D}$ is a cover in $\mathscr{E}$, hence an effective epi family in $\mathscr{E}$, which implies that Φ is an extremal family in $\mathbf{D}$ (although not that Φ is an effective epi family in $\mathscr{D}$). Together with the factorization property (iv) of the construction, the last fact implies that $\mathscr{D}$ is a regular site.

An α-coherent geometric morphism $p : \mathscr{F} \to \mathscr{E}$ between GTs is one for which p^* takes α-coherent objects of $\mathscr{E}$ into α-coherent objects of $\mathscr{F}$. Suppose that $\mathscr{C}$ is a defining site for $\mathscr{E}$ and that $\mathscr{C}$ satisfies the hypotheses of Proposition 1.5; hence $\mathscr{E}$ satisfies the conclusions of 1.5. Suppose also that the GT $\mathscr{F}$ satisfies the conclusions of 1.5. Then the continuous $U : \mathscr{C} \to \mathscr{F}$ induces an α-coherent geometric morphism

$$\mathbf{Sh}(U) : \mathscr{F} \to \mathscr{E}$$

if and only if U factors through $\mathrm{Coh}_\alpha(\mathscr{F})$; this is not hard to verify.

Set is a coherent topos with the sets of cardinality $< \alpha$ being the α-coherent objects.

A countable model of $\mathscr{E}$ is one that corresponds to a σ-coherent ($\kappa = \aleph_1$) geometric morphism **Set** $\rightarrow \mathscr{E}$; that is to a continuous $M:\mathscr{E} \rightarrow$ **Set** for which $M(E)$ is a countable set for every E in $\mathrm{Coh}_{\aleph_1}(\mathscr{E})$. In particular, if $\mathscr{C}$ is a separable site, $\mathscr{E} = \mathbf{Sh}(\mathscr{C})$, then a countable model M of $\mathscr{E}$ induces a countable model of $\mathscr{C}$ in the sense that $M(C)$ is a countable set for each C of $\mathscr{C}$. Writing $\mathrm{Mod}_\sigma(\mathscr{C})$ for the category of countable models of $\mathscr{C}$ and $\mathrm{Mod}_\sigma(\mathscr{E})$ for the category of countable models of $\mathscr{E}$, we conclude that

$$\mathrm{Mod}_\sigma(\mathscr{C}) \cong \mathrm{Mod}_\sigma(\mathbf{Sh}(\mathscr{C}))$$

for a separable site $\mathscr{C}$.

PROPOSITION 1.6′. *Suppose $p:\mathscr{F} \rightarrow \mathscr{E}$ is a $\leqq\mu$-coherent geometric morphism between μ-presentable* GTs. *Then there are regular defining sites $\mathscr{C}$ and $\mathscr{D}$ for $\mathscr{E}$ and $\mathscr{F}$ respectively, both of size $\leqq\mu$, and a continuous $U:\mathscr{C} \rightarrow \mathscr{D}$ such that U induces p.*

Proof. The proof is a straightforward variant of that of 1.6.

A class K of models (points) of a GT $\mathscr{E}$ is *sufficient* if the induced continuous functor $\mathscr{E} \rightarrow \mathbf{Set}^K$ is conservative (reflects isomorphisms). In [**18**], it was proved (see also [**6**], Theorem 13 of Section 7.6) that any separable GT has enough countable models (i.e., the class of countable models is sufficient). A proof of this result can be given that is very similar to that of Proposition 3.5 below.

We now give a short account of the so-called Omitting Types Theorems (see [**7**], and [**13**]) in our setting. It turns out to be a simple matter involving the double negation topology.

Let $\mathscr{C} = (\mathbf{C}, \mathbf{J})$ be an arbitrary site. An object A of $\mathbf{C}$ is called *empty* (or, more specifically, **J**-empty) if the empty family is a **J**-cover of A; otherwise, A is (**J**-)*non-empty*. A family $\{A_i \rightarrow A\}$, $i \in I$ of morphisms is called (**J**-)*dense* if for all $B \rightarrow A$ if $A_i \times_A B$ is empty for all $i \in I$, then B is empty. A family of subobjects of a given object is dense if the family of representing monomorphisms is dense. In a regular site, $\{X_i\}$, $i \in I$ is dense in $\mathrm{Sub}(A)$ just in case for any $Y \in \mathrm{Sub}(A)$, $Y \wedge A_i$ being empty ($= 0_A$) for all $i \in I$ implies that Y is empty.

The collection of all **J**-dense sieves forms a topology, denoted $\mathbf{J}_{\neg\neg}$. Any **J**-nonempty object is $\mathbf{J}_{\neg\neg}$-nonempty; in other words, $\mathbf{J}_{\neg\neg}$ does not produce any more empty objects than **J** does. Thus we have:

PROPOSITION 1.7. *If the topology* **J** *on the left exact category* **C** *is consistent* (1 *is not* **J**-*empty*), *so is* $\mathbf{J}_{\neg\neg}$.

As a consequence, and by the fact that there are enough countable models for separable toposes, we have

PROPOSITION 1.8 ("Omitting Types Theorem"). *Suppose $\mathscr{C}$ is a consistent separable site, and Φ is a countable collection of* **J**-*dense sieves. Then $\mathscr{C}$ has a model that takes every sieve in Φ into a surjective family in* **Set**.

An *inclusion* of GTs (or, in general, of toposes) is a geometric morphism which (whose right adjoint part) is full and faithful. We are going to need a characterization of inclusions in terms of their left adjoint parts.

PROPOSITION 1.9. *A geometric morphism* $p:\mathscr{E}' \to \mathscr{E}$ *between* GT*s is an inclusion if and only if it satisfies the following two conditions*:

(i) p *is localic*; *i.e.*, *for any object* X *of* $\mathscr{E}'$ *there is a diagram of the form*

$$\begin{array}{ccc} Y & \xrightarrow{\quad\text{mono}\quad} & p^*(A) \\ {\scriptstyle \text{epi}}\big\downarrow & & \\ X & & \end{array}$$

(ii) p^* *is full on subobjects*, *i.e. it induces a surjection*

$$\text{Sub}_{\mathscr{E}}(A) \to \text{Sub}_{\mathscr{E}'}(p^*A)$$

for every A *in* $\mathscr{E}$.

Proof. If p is an inclusion, then for a (unique) topology $\mathbf{j}$ in $\mathscr{E}$, $\mathscr{E}'$ is $\mathbf{Sh}_{\mathbf{j}}(\mathscr{E})$ and p is the canonical geometric morphism $\mathbf{Sh}_{\mathbf{j}}(\mathscr{E}) \to \mathscr{E}$ (see [**11**], 4.15 Proposition, or [**6**]). It is easy to conclude that any inclusion satisfies (i) and (ii).

Suppose, conversely, that p satisfies the conditions. By [**11**], 4.14, or [**6**], p can be factored in the form $p = q \circ i$, where i is an inclusion, and q is surjection (i.e., q^* is conservative). It is now routine to verify that the fact that p and i satisfy the two conditions implies the same for q. Thus, by Lemma 1.3, q^* is an equivalence of categories; therefore $p = q \circ i$ is an inclusion.

Let $(\mathbf{C}, \mathbf{J})$ be a regular site, and suppose that 0 is a strict initial object of $\mathbf{C}$ (0 is initial and every morphism with codomain 0 is an isomorphism). Then, the unique morphism $0 \to A$ is a monomorphism, for any A and represents the least element, denoted 0_A, of the set $\text{Sub}(A)$. Let $X \in \text{Sub}(A)$. $X' \in \text{Sub}(A)$ is a $\mathbf{J}$-*complement* of X if

$$X \vee^{(\mathbf{J})} X' = 1_A \quad \text{and} \quad X \wedge X' = 0_A.$$

Since the site is regular, $X \vee^{(\mathbf{J})} X' = 1_A$ implies that $X \vee X' = 1_A$ in the subobject poset of A. (However, it is not necessarily true that $\vee^{(\mathbf{J})}$ or $\vee$ are fully defined operations on $\text{Sub}(A)$.) Thus, a $\mathbf{J}$-complement is, in particular, a Boolean complement in the subobject poset of A. Because of the pullback axiom on the topology $\mathbf{J}$, it is easy to check that "we have enough distributivity" to have that the $\mathbf{J}$-complement, if it exists, is unique. We call $(\mathbf{C}, \mathbf{J})$ *Boolean* if it is regular, has a strict initial object, and every subobject has a **J**-complement. One realizes (exercise) that in a Boolean site, finite **J**-union, $\vee^{(\mathbf{J})}$, and hence also finite union, is a fully defined operation, and thus every subobject poset Sub(A) is in fact a Boolean algebra. Moreover, a continuous functor takes a complement into a complement; thus a continuous functor between Boolean sites induces Boolean homomorphisms on the subobject lattices.

A GT is Boolean if and only if every subobject lattice in it is a (necessarily complete) Boolean algebra. It is by no means true that $\mathbf{Sh}(\mathscr{C})$ for a Boolean site $\mathscr{C}$ is Boolean. However, we have

PROPOSITION 1.10. *A μ-presentable Boolean* GT *has a Boolean site of definition of size at most μ. Similarly, in* 1.6′, *if $\mathscr{E}$ and $\mathscr{F}$ are Boolean, then $\mathscr{C}$ and $\mathscr{D}$ can be chosen to be Boolean too.*

Proof. The proof is a modification of that of Proposition 1.6. First of all, one readily verifies that the Boolean complement of a $\leqq\mu$-coherent subobject of a $\leqq\mu$-coherent object is again $\leqq\mu$-coherent. Then, in the construction of D, one adds another clause, namely that Boolean complements of subobjects represented by monomorphisms in **D** are represented by monomorphisms in **D**. The modified construction can obviously be carried out, and it clearly leads to the desired Boolean site. The version of 1.6′ is proved similarly.

Finally, let us recall a well known fact concerning the double negation topology. For a GT $\mathscr{E}$, with **J** the canonical topology, we may form $\mathbf{J}_{\neg\neg}$ as above; it is readily seen that

$$\mathbf{Sh}(\mathscr{E}, \mathbf{J}_{\neg\neg}) = \mathscr{E}_{\neg\neg}$$

is a Boolean topos. Proposition 1.7 implies that if $\mathscr{E}$ is non-degenerate, so is $\mathscr{E}_{\neg\neg}$. Of course, the canonical geometric morphism $\mathscr{E}_{\neg\neg} \to \mathscr{E}$ is an inclusion, making $\mathscr{E}_{\neg\neg}$ a subtopos of $\mathscr{E}$.

2. A categorical formulation of the method of diagrams of model theory. In model theory, the method of diagrams plays an important role. In a simple instance, the method appears as follows. Given a model M, one can produce a set of sentences, called Diag M, that axiomatizes the class of elementary extensions of M. Diag M uses the language of M plus a new individual constant for each element of M; Diag M is the set of all sentences in the extended language which are true in M, with the new constants interpreted in the obvious way. The models of Diag M are in an essentially one-to-one correspondence with arrows $M \to N$ that are elementary embeddings.

In applications of the construction, besides the "universal property" of Diag M of axiomatizing elementary extensions of M, the syntactical properties of the construction are also crucial. We will use the method of diagrams in a categorical context; the model will become a functor, Diag M a category, and the syntactical properties of Diag M will be translated into a categorical language.

The actual set-up we have to deal with is as follows. There are given a functor $I:\mathbf{B} \to \mathbf{C}$ and a functor $M:\mathbf{C} \to \mathbf{Set}$, both possibly with additional properties. The "diagram" we are interested in will have models in an essentially one-to-one correspondence with pairs $\langle N, M \circ I \to N \rangle$ with $N:\mathbf{C} \to \mathbf{Set}$, with possibly additional conditions on N. The above simple situation is the special case when I is an identity functor.

In the following proposition, we assert the existence of an appropriate "diagram". As it happens, the full universal property of the diagram (stated separately is the last clause) will not be used in our application. We will outline two proofs of the proposition the first ignoring the last clause, and giving, in fact, a different "diagram" from the one given by the other proof. At the end of this section, we will return to the "universal property" in the context of sites.

PROPOSITION 2.1. *Let $I:\mathbf{B} \to \mathbf{C}$ be a left exact functor between left exact categories and $M:\mathbf{C} \to \mathbf{Set}$ be a left exact functor. Then there exist a left exact category $\mathbf{D}$ and left exact functors $F:\mathbf{C} \to \mathbf{D}$ and $\hat{M}:\mathbf{D} \to \mathbf{Set}$ such that*

(i) $\hat{M}F \cong M$.

(ii) *For every object B of $\mathbf{B}$, every element $b \in \hat{M}(FIB)$ is "definable in $\hat{M}$": i.e., there is a global section $\bar{b}:1 \to FIA$ such that $\hat{M}(\bar{b})$ is (picks out) b.*

(iii) *For every object D of $\mathbf{D}$, there is a monomorphism $D \rightarrowtail FC$.*

(iv) *For every morphism $f:D \to FC$ in $\mathbf{D}$, there are objects B of $\mathbf{B}$ and C' of $\mathbf{C}$ and morphisms $f':D \to FC'$, $g:C' \to C$, $h:C' \to IB$ and $b:1 \to FIB$ such that*

$$\begin{array}{ccc} D & \xrightarrow{\quad f \quad} & FC \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle id \times \bar{b}} \\ FC' & \xrightarrow{\langle Fg,\, Fh\rangle} & FC \times FIB \end{array}$$

is a pullback. Moreover, if f is mono, so is $\langle g, h \rangle$.

(v) *The category* $\mathrm{LEX}(\mathbf{D}, \mathbf{Set})$ *is equivalent to the category* $\mathbf{A}(M)$ *whose objects are pairs $\langle N, MI \to NI \rangle$ with $N:\mathbf{C} \to \mathbf{Set}$ left exact and for which a morphism*

$$\langle N, MI \to MI \rangle \to \langle N', MI \to N'I \rangle$$

is a natural transformation $N \to N'$ making the obvious diagram commute.

Proof (of all except (v)). We may and do replace $\mathbf{C}$ by an equivalent category in which pullbacks are uniquely determined by their factors; that

https://doi.org/10.4153/CJM-1987-009-x Published online by Cambridge University Press

is, if $f_i:C_i \to C$ are given for $i = 1, 2$, then the pullback object $C_1 \times_C C_2$ is uniquely determined by f_1 and f_2. Begin by choosing a correspondence between the elements of the disjoint union of all the values of MI and an initial segment of ordinals. Let the ordinal α correspond to $b_\alpha \in MIB_\alpha$. We now construct inductively categories $\mathbf{D}_\alpha$ and functors $F_{\alpha\beta}:\mathbf{D}_\beta \to \mathbf{D}_\alpha$ for $\beta < \alpha$ and $M_\alpha:\mathbf{D}_\alpha \to \mathbf{Set}$ such that $F_{\alpha\beta} \circ F_{\beta\gamma} = F_{\alpha\gamma}$, for $\gamma < \beta < \alpha$ and $M_\alpha \circ F_{\alpha\beta} = M_\beta$ for $\beta < \alpha$. We begin by letting $\mathbf{D}_0 = \mathbf{C}$ and $M_0 = M$. If α is a limit ordinal, let $\mathbf{D}_\alpha$ be the colimit of the $\mathbf{D}_\beta$ and the $F_{\beta\gamma}$ for $\gamma < \beta < \alpha$, let $F_{\alpha\beta}$ be the transition map into the colimit and let M_α be the unique functor for which $M_\alpha \circ F_{\alpha\beta} = M_\beta$ for $\beta < \alpha$. Next consider a nonlimit ordinal, say $\alpha + 1$. Let

$$\mathbf{D}_{\alpha+1} = \mathbf{D}_\alpha/F_\alpha IB_\alpha,$$

where we write F_α for

$$F_{\alpha 0}:\mathbf{D}_0 = \mathbf{C} \to \mathbf{D}_\alpha.$$

Define

$$F_{\alpha+1,\alpha}:\mathbf{D}_\alpha \to \mathbf{D}_{\alpha+1}$$

to be the usual morphism of crossing with $F_\alpha IB_\alpha$ and define

$$F_{\alpha+1,\beta} = F_{\alpha+1,\alpha} \circ F_{\alpha\beta} \quad \text{for } \beta < \alpha.$$

Since pullbacks in $\mathbf{D}_{\alpha+1}$ are pullbacks in $\mathbf{D}_\alpha$, it is clear that we can show inductively that the supposition that pullbacks be uniquely determined by their arguments remains valid in $\mathbf{D}_\alpha$. Then we can define $M_{\alpha+1}$ on an object

$$f:D \to F_\alpha IB_\alpha$$

of $\mathbf{D}_\alpha$ so that

$$\begin{array}{ccc} M_{\alpha+1}(D \to F_\alpha IB_\alpha) & \longrightarrow & M_\alpha D \\ \big\downarrow & & \big\downarrow M_\alpha f \\ 1 & \xrightarrow{\ \bar{b}_\alpha\ } & M_\alpha F_\alpha IB_\alpha \end{array}$$

is a pullback. Of course, we have freedom in choosing a pullback square in the category of sets. These choices can be made at will, except that in the case that $D \to F_\alpha IB_\alpha$ is the projection of $D_1 \times F_\alpha IB_\alpha$, we choose $M_\alpha D_1$ which can easily be seen to make the square into a pullback. This is the reason for our making the hypothesis that pullbacks (and hence products) are uniquely determined by their factors. This choice forces the equation

$$M_{\alpha+1} \circ F_{\alpha+1,\alpha} = M_\alpha$$

and the remaining identities are a simple calculation. When we exhaust the supply of elements of MI, the category we have is $\mathbf{D}$ and the final $F_{\alpha 0}$ and M_α are F and $\hat{M}$, resp.

Now we want to verify the first four conclusions of the proposition. The first is clear (in fact, according to our construction, we get actual equality). The second is also easy for the element b_α is definable in $\mathbf{D}_{\alpha+1}$ using the diagonal

$$F_\alpha IB_\alpha \to F_\alpha IB_\alpha \times F_\alpha IB_\alpha$$

in $\mathbf{D}_\alpha$. As for the third, we will suppose by induction that the corresponding property is valid in $\mathbf{D}_\alpha$ and prove it in $\mathbf{D}_{\alpha+1}$ (there is no problem at limit ordinals). An object D of $\mathbf{D}_{\alpha+1}$ corresponds to a morphism

$$D_1 \to F_\alpha IB_\alpha$$

in $\mathbf{D}_\alpha$. Supposing a monomorphism $D_1 \rightarrowtail F_\alpha C$ in $\mathbf{D}_\alpha$, we get a monomorphism

$$D_1 \rightarrowtail F_\alpha C \times F_\alpha IB_\alpha$$

in $\mathbf{D}_\alpha$ which corresponds to a monomorphism $D \rightarrowtail F_{\alpha+1}C$ in $\mathbf{D}_{\alpha+1}$.

As for (iv), it is easily checked to be equivalent to the somewhat simpler statement that for any $f:D \to FC$, there are objects B of $\mathbf{B}$ and C' of $\mathbf{C}$ and morphisms $f':D \to FC'$, $g:C' \to C$, $h:C' \to IB$ and $\bar{b}:1 \to FIB$ such that $Fg \circ f' = f$ and so that

$$\begin{array}{ccc} D & \longrightarrow & 1 \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle \bar{b}} \\ FC' & \xrightarrow{Fh} & FIB \end{array}$$

is a pullback and that, moreover, if f is mono, so is $\langle g, h\rangle$. We will show inductively that the corresponding statement is true in $\mathbf{D}_\alpha$ for every α. There is no problem at limit ordinals, so we can suppose it is valid in $\mathbf{D}_\alpha$ and show that it remains true in $\mathbf{D}_{\alpha+1}$. So consider

$$f:D \to F_{\alpha+1}C$$

in $\mathbf{D}_{\alpha+1}$ which corresponds to a morphism

$$\langle f_1, f_2\rangle:D_1 \to F_\alpha C \times F_\alpha IB_\alpha$$

in $\mathbf{D}_\alpha$. By induction we can factor $\langle f_1, f_2\rangle$ as

$$(*)\qquad D_1 \xrightarrow{f_1'} F_\alpha C' \xrightarrow{\langle F_\alpha g_1,\, F_\alpha g_2\rangle} F_\alpha C \times F_\alpha IB_\alpha$$

where we have used the fact that F_α preserves products to identify $F_\alpha C \times F_\alpha IB_\alpha$ with $F_\alpha(C \times IB_\alpha)$. Further, we can find an object B_1 of $\mathbf{B}$ and arrows

$$\bar{b}_1 : 1 \to F_\alpha IB_1 \quad \text{and} \quad h_1 : C' \to IB_1$$

such that

$$\begin{array}{ccc}
D_1 & \longrightarrow & 1 \\
{\scriptstyle f'_1}\downarrow & & \downarrow{\scriptstyle \bar{b}_1} \\
F_\alpha C' & \xrightarrow{F_\alpha h_1} & F_\alpha IB_1
\end{array}$$

is a pullback. Moreover, if $\langle f_1, f_2 \rangle$ is mono, so is $\langle g_1, g_2, h_1 \rangle$. It follows from (*) that the left hand square of

$$\begin{array}{ccccc}
D_1 & \xrightarrow{f_2} & F_\alpha IB_\alpha & \longrightarrow & 1 \\
{\scriptstyle f'_1}\downarrow & & \downarrow & & \downarrow{\scriptstyle \bar{b}_1} \\
F_\alpha C' & \xrightarrow[\langle F_\alpha h_1, F_\alpha g_2 \rangle]{} & F_\alpha IB_1 \times F_\alpha IB_\alpha & \xrightarrow[\pi_1]{} & F_\alpha IB_1
\end{array}$$

commutes (here and below π_i denotes the projection on the ith coordinate). Since the outer and right hand squares are pullbacks, so is the left hand square. A similar argument gives that the upper square of

$$\begin{array}{ccc}
D_1 & \xrightarrow{f_2} & F_\alpha IB_\alpha \\
{\scriptstyle \langle f'_1, f_2 \rangle}\downarrow & & \downarrow{\scriptstyle \langle \bar{b}_1, \text{id}, \text{id} \rangle} \\
F_\alpha C' \times F_\alpha IB_\alpha & \xrightarrow{\langle F_\alpha h_1 \circ \pi_1, F_\alpha g_2 \circ \pi_1, \pi_2 \rangle} & F_\alpha IB_1 \times F_\alpha IB_\alpha \times F_\alpha IB_\alpha \\
{\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \langle \pi_1, \pi_2 \rangle} \\
F_\alpha C' & \xrightarrow{\langle F_\alpha h_1, F_\alpha g_2 \rangle} & F_\alpha IB_1 \times F_\alpha IB_\alpha
\end{array}$$

is a pullback. But in $\mathbf{D}_{\alpha+1} = \mathbf{D}_\alpha / F_\alpha IB_\alpha$, that upper square becomes

$$\begin{array}{ccc}
D & \longrightarrow & 1 \\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle \langle \bar{b}_1, \bar{b} \rangle} \\
F_{\alpha+1} C' & \xrightarrow{\langle F_\alpha h_1, F_\alpha g_2 \rangle} & F_{\alpha+1} IB_1 \times F_{\alpha+1} IB_\alpha
\end{array}$$

where $\bar{b}$ is the map in $\mathbf{D}_{\alpha+1}$ whose underlying arrow in $\mathbf{D}_\alpha$ is the diagonal of $F_\alpha IB_\alpha$. It is easily seen that f factors as

$$D_1 \xrightarrow{\langle f'_1, f_2 \rangle} F_\alpha C'_1 \times F_\alpha IB_\alpha \xrightarrow{\langle F_\alpha g_1 \circ \pi_1, \pi_2 \rangle} F_\alpha C \times F_\alpha IB_\alpha$$

and that if f is mono, so is $\langle g_1, g_2, h_1 \rangle$. The conditions are satisfied with $f' = \langle f_1', f_2 \rangle$, $h = \langle h_1, g_2 \rangle$ and $g = g_1$.

Sketch of the proof of (v). In this proof we use some facts about locally finitely presentable (LFP) categories; although these are classical and mostly found in **[9]**, we find it convenient to refer to **[17]**. This part of the Proposition characterizes **D** up to natural equivalence. The functor

$$I^*:\mathrm{Lex}(\mathbf{C}, \mathbf{Set}) \to \mathrm{Lex}(\mathbf{B}, \mathbf{Set}),$$

defined by composing with I, has a left adjoint $I_!$ which is just the left Kan extension (see Proposition 1.2). Then for any functor $N:\mathbf{C} \to \mathbf{Set}$, the adjunction gives a natural bijection between

$$\mathrm{Nat}(MI, NI) = \mathrm{Nat}(I^*M, I^*N)$$

and $\mathrm{Nat}(I_!O^*M, N)$. Thus the category $\mathbf{A}(M)$ defined above, which is evidently the comma category (I^*M, I^*) is naturally equivalent to the comma category $(M_0, \mathrm{Lex}(\mathbf{C}, \mathbf{Set}))$. We have a pair of adjoint functors, $G_! \dashv G$:

$$(M_0, \mathrm{Lex}(\mathbf{C}, \mathbf{Set})) \underset{G_!}{\overset{G}{\rightleftarrows}} \mathrm{Lex}(\mathbf{C}, \mathbf{Set})$$

for which

$$G(M_0 \to N) = N \quad \text{and} \quad G_!(N) = (i_1:M_0 \to M_0 \otimes N).$$

Since $G_!$ also preserves filtered colimits and is conservative, we conclude that $(M_0, \mathrm{Lex}(\mathbf{C}, \mathbf{Set}))$ is LFP. In other words, if we let **D** be the full subcategory of $(M_0, \mathrm{Lex}(\mathbf{C}, \mathbf{Set}))$ consisting of the LFP objects, then

$$(M_0, \mathrm{Lex}(\mathbf{C}, \mathbf{Set})) \cong \mathrm{Lex}(\mathbf{D}, \mathbf{Set})$$

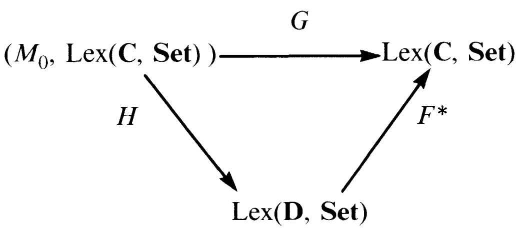
and thus also

$$\mathbf{A}(M) \cong \mathrm{Lex}(\mathbf{D}, \mathbf{Set}).$$

Further, by the Gabriel-Ulmer duality (see **[17]**), G is defined, up to equivalence, as

$$F^*:\mathrm{Lex}(\mathbf{D}, \mathbf{Set}) \to \mathrm{Lex}(\mathbf{C}, \mathbf{Set})$$

for some left exact $F:\mathbf{C} \to \mathbf{D}$. In other words, we have



commuting up to an isomorphism $\varphi:F^*H \cong G$, where H is the canonical equivalence. If

$$\epsilon M:I_! I^* M \rightarrow M$$

is the counit of the adjunction, let $\hat{M}$ be the object $H(\epsilon M)$ of Lex(**D**, **Set**); then

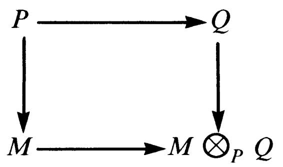
$$\varphi(\epsilon M):\hat{M}F \rightarrow M$$

is an isomorphism.

We are not going to give the detailed verification of the conditions the items we have introduced should satisfy, except to say that in this verification one uses the characterization of the finitely presented objects of $(M_0$, Lex(**C**, **Set**)). These are, up to isomorphism, the same as arrows

$$i_2:M \rightarrow M \otimes_P Q$$

for any pushout diagram

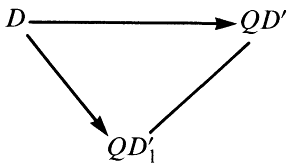


with P and Q finitely presented objects of Lex(**C**, **Set**).

PROPOSITION 2.2. *Under the same hypotheses as Proposition* 2.1, *we can draw the same conclusions, except that* (v) *is replaced by*
(vi) $\hat{M}'$:**D**′ → **Set** *is conservative.*

Proof. Using Proposition 2.1, construct a category **D**′ and functors F':**C** → **D**′ and $\hat{M}'$:**D**′ → **Set** with the properties given there. Factor $\hat{M}'$:**D**′ → **Set** as $\hat{M} \circ Q$ where Q is a quotient functor and $\hat{M}$ is conservative. Q is the universal solution, among left exact functors with domain **D**′, to inverting morphisms inverted by Q. Then both Q and $\hat{M}$ are left exact, $\hat{M}$ is conservative and $\hat{M}Q \cong \hat{M}'$ (see [**15**]). The following characterization of left exact quotients is due to A. M. Pitts and a proof appears in [**15**], Proposition 2.3.2.

A left exact functor Q:**D**′ → **D** is a quotient if and only if for any morphism $D \rightarrow QD'$ in **D**, there is a morphism $f:D'_1 \rightarrow D'$ in **D**′ and an isomorphism $D \rightarrow QD'_1$ so that the triangle



commutes.

Then **D**, $F = QF'$ and $\hat{M}$ are the required categories and functors. We have

$$\hat{M}F = \hat{M}QF' \cong \hat{M}'F' \cong M.$$

Further, $\hat{M}$ is conservative by construction. As for the remaining conditions, condition (ii) is clearly inherited by $\hat{M}$. In view of the characterization of left exact quotients above, so is condition (iii). Condition (iv) in the original proposition, together with the left exactness of Q, implies that any arrow of the form

$$Qf{:}QD'_1 \to QFC$$

will satisfy condition (iv) in **D**. But the characterization says that any arrow of the form $D \to FC$ is isomorphic to one of that special form. Clearly (iv) is invariant under isomorphisms.

PROPOSITION 2.3. *If* $\mathcal{D} = (\mathbf{D}, \mathbf{L})$ *is a site and* $M{:}\mathcal{D} \to \mathbf{Set}$ *is continuous and conservative, then every cover in* **L** *is an extremal family.*

Proof. Suppose $\{f_i{:}D_i \to D\}$, $i \in I$ is a cover in L. Then the Mf_i form a surjective family. Suppose $m{:}D' \to D$ is a monomorphism for which every f_i factors through m. Then Mm is a monomorphism such that every Mf_i factors through it. But in **Set**, every surjective family is extremal. Hence, Mf_i is an isomorphism. Since M is conservative, f_i is an isomorphism.

PROPOSITION 2.4. *Suppose, in addition to the hypotheses of Proposition* 2.2, *that* $\mathcal{B} = (\mathbf{B}, \mathbf{J})$ *and* $\mathcal{C} = (\mathbf{C}, \mathbf{K})$ *are sites, that* I *is* **J**,**K**-*continuous and that* M *is continuous with respect to* **K** *and the canonical topology on* ***Set***. *Then the category* **D** *constructed in* 2.2 *can be endowed with a topology* **L** *so that* $\mathcal{D} = (\mathbf{D}, \mathbf{L})$ *is a regular site and both* F *and* $\hat{M}$ *are continuous. Moreover*

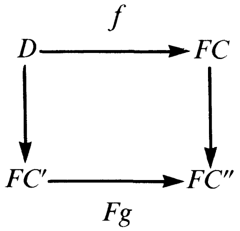
(vii) *If* $\mathcal{B}$ *and* $\mathcal{C}$ *are separable sites and* M *is a countable model, then* $\mathcal{D}$ *can be taken to be separable.*

(viii) *If* $\mathcal{C}$ *is Boolean, then* $\mathcal{D}$ *is Boolean.*

(ix) $\mathcal{D}$ *is* 2-*valued: the terminal object* 1 *of* $\mathcal{D}$ *has just* 2 *subobjects.*

Proof. Let **L** be the smallest topology on **D** that makes F continuous with respect to **K** and **L**. **L** is generated by the F-images of generating covers of **K**. Since $M = \hat{M}F$ and M is **K**-continuous, it is clear that $\hat{M}$ is **L**-continuous. By 2.3, it follows that every cover in **L** is an extremal family. It remains to show that every morphism $f{:}D \to D'$ can be factored as a single cover followed by a mono. By (iii) there is a mono $m{:}D' \to FC$. If we can factor the composite mf in the desired way, this can easily be seen to imply such a factorization for f. Thus we may suppose without loss of generality that D' is of the form FC. But then we have from (iv) that there are objects C' and C'' of **C** and morphisms $g{:}C' \to C''$, $D \to FC'$ and $FC \to FC''$ such that

(1)

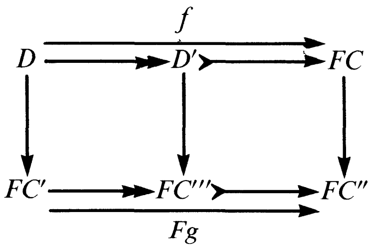


is a pullback. Upon factoring g as

$$C \overset{h}{\twoheadrightarrow} C''' \rightarrowtail C''$$

with $\{h\} \in \mathbf{K}$, we conclude that $\{Fh\} \in \mathbf{L}$. Construct the diagram

(2)



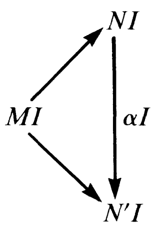
so that the right hand square is a pullback, whence the left hand square is one as well. Then $D \to D'$ belongs to $\mathbf{K}$ which gives the required factorization.

For the rest of the proposition, (vii) follows because when M is countable, the well ordering construction can be accomplished by a simple induction and a slice of a countable category is clearly countable, while the topology generated by $\mathbf{K}$ will have a countable base if $\mathbf{K}$ does. (viii) is an easy consequence of (iii) and (iv) while (ix) is immediate from the fact that $\hat{M}$ is conservative.

Finally we deduce the universal property of the site $\mathscr{D}$ as constructed in 2.4. Let $\mathbf{A}$ be the category whose objects are pairs $\langle N:MI \to NI\rangle$ with N a model of $\mathbf{C}$ and in which a morphism

$$\langle N, MI \to NI\rangle \to \langle N', MI \to N'I\rangle$$

is a natural transformation $\alpha:N \to N'$ such that



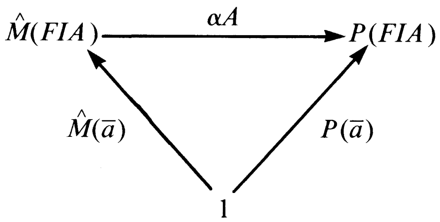
commutes.

PROPOSITION 2.5. *The categories* **A** *and* Mod($\mathscr{D}$) *are equivalent, by a canonically defined functor* G:Mod($\mathscr{D}$) $\rightarrow$ **A**.

Proof. In the definition of **A**, we replace M by the isomorphic model $\hat{M}F$. Of course, up to isomorphism of categories, **A** is unchanged. To define G, we first note that for any model P of $\mathscr{D}$ and any object B of **B**, there is a unique morphism

$$\alpha A:\hat{M}(FIA) \rightarrow P(FIA)$$

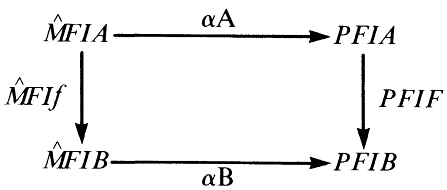
such that



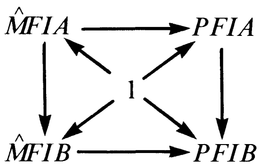
commutes for all $\bar{a}:1 \rightarrow FIA$ in **D**. Indeed, this follows from Proposition 2.1 (ii) and Proposition 2.2. The morphisms αA define a natural transformation

$$\alpha:\hat{M}FI \rightarrow PFI.$$

For any $f:A \rightarrow B$ in **B**, we must show that the diagram



commutes. This follows from the facts that the family of morphisms $\hat{M}\bar{a}$, for all $\bar{a}:1 \rightarrow FIA$ are jointly surjective (2.1 (ii)) and that for every such $\bar{a}$, each triangle in



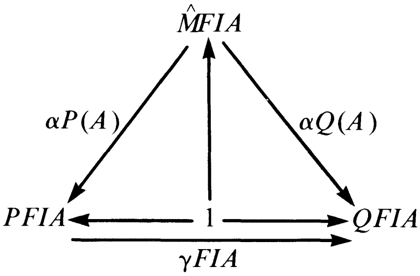
commutes. (Here and in the diagram below, the unlabeled arrows are the respective functors applied to $\bar{a}$.) The object function of the functor G is given by

$$G(P) = \langle PF, \alpha:\hat{M}FI \rightarrow PFI\rangle.$$

Given $\gamma:P \to Q$ in $\text{Mod}(\mathscr{D})$, we have that $\gamma F:PF \to QF$ determines an arrow

$$G(\gamma):G(P) \to G(Q).$$

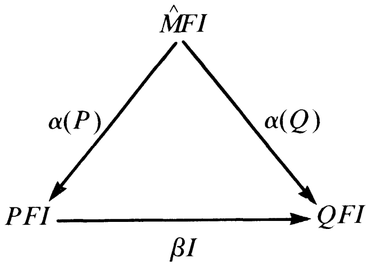
This follows from the commutativity of



for each $\bar{a}:1 \to FIA$. It is clear that G is a functor

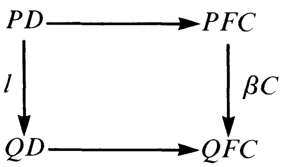
$$\text{Mod}(\mathscr{D}) \to \mathbf{A}.$$

To show that G is full and faithful, let P and Q be models of $\mathscr{D}$ and let $\beta:PF \to QF$ be such that



commutes. We must show that there is a unique $\gamma:P \to Q$ such that $\beta = \gamma F$. We begin with

LEMMA 2.6. *For any monomorphism $D \rightarrowtail FC$, there is a (necessarily unique) arrow $l:PD \to QD$ making the following commute:*



Proof. Start with the data given by Proposition 2.1 (iv) for the monomorphism $D \rightarrowtail FC$. Apply P and Q to the diagram of (iv) to obtain

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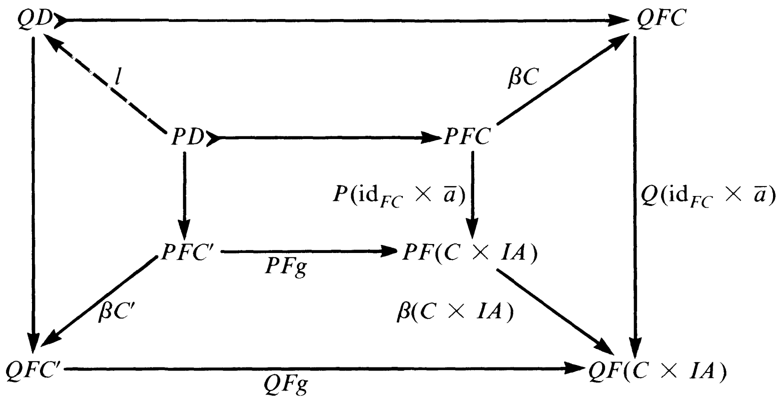
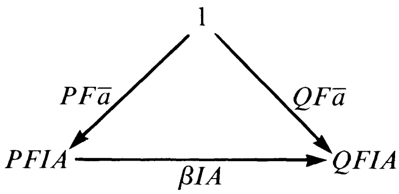
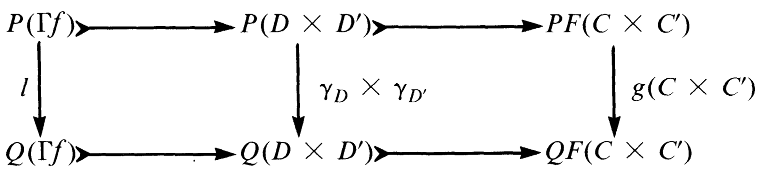


The right hand trapezoid commutes as a consequence of the commutativity of



which is, in turn, a consequence of the assumption on β and the definitions of $\alpha(P)$ and $\alpha(Q)$. The bottom trapezoid commutes by the naturality of β. Both the inner and outer squares are pullbacks, from which the existence of a unique l making the whole diagram commute is assured.

Now we define $\gamma:P \to Q$ as follows. For each object D of $\mathbf{D}$, choose a monomorphism $D \rightarrowtail FC$ (2.1 (iii)). The morphism l just constructed is taken to be γD. To show the naturality of γ, we consider an arbitrary arrow $f:D \to D'$. With the graph of f, $\Gamma f \subseteq D \times D'$ and the specific monomorphisms $D \rightarrowtail FC$ and $D' \rightarrowtail FC'$ that give rise to γD and $\gamma D'$, resp., we obtain the diagram



Lemma 2.5 assures the existence of l to make the outer rectangle commute. The right hand rectangle commutes by the definitions of γD and $\gamma D'$. It follows that the left hand rectangle commutes which implies the naturality of γ with respect to f. It is clear that γ is uniquely determined.

We have shown that G is full and faithful. The proof that it is essentially surjective is left to the reader (that fact is not actually used in our application in Section 4).

3. Atomic and prime generated toposes. An object P of a site $\mathscr{C}$ is called a *prime* if every cover of the object has a refinement by a singleton. In particular, every prime is *non-empty*, that is the empty sieve is not a cover. An object is called an *atom* if it is non-empty and if every morphism to it with non-empty domain is a cover. Clearly every atom is a prime. In a regular site, primes are characterized by the fact that if

$$P = \bigvee_i^{(\mathbf{J})} A_i,$$

then for some i, $P = A_i$, i.e., that P is $\mathbf{J}$-join irreducible. In a Grothendieck topos, a prime is an object which is join irreducible in the usual sense. Similarly, in a regular site, an atom is characterized by being non-empty and having at most two subobjects (exactly two if there is an empty object).

A site is *prime-generated* (resp. *atomic*) if every object has a cover by primes (resp. atoms). Thus a regular site is prime-generated (resp. atomic) if every object is a $\mathbf{J}$-join of primes (resp. atoms). It is easy to see that in a prime-generated (resp. atomic) regular site the $\mathbf{J}$-unions of primes (resp. $\mathbf{J}$-sums of atoms), together with covering epis between primes (resp. atoms) form a base for the topology. It is interesting to note that a regular category has at most one regular atomic topology; there is such a topology if and only if each object is a universal disjoint union (in its own subobject lattice) of atoms and then these, together with the covering epis between atoms are a base for the topology.

Note that in a Boolean (regular) site, every prime is necessarily an atom; thus a Boolean prime-generated site is atomic.

The above terminology applied to a GT with its canonical topology gives us the notions of a *prime-generated* GT and an *atomic* GT. It is easy to see that for $\epsilon:\mathscr{C} \rightarrow \mathbf{Sh}(\mathscr{C})$ the canonical continuous functor, P is a prime (atom) in $\mathscr{C}$, if and only if $\epsilon(P)$ is a prime (atom) in $\mathbf{Sh}(\mathscr{C})$; it follows that the category of sheaves over a prime-generated (atomic) site is prime-generated (atomic).

It is easily seen that a GT is atomic if and only if it is prime-generated and Boolean. The following proposition is verified directly; we state it explicitly for its importance.

PROPOSITION 3.1. *In any (left exact) site, if $P \twoheadrightarrow A$ is a (singleton) cover (A is a quotient of P) then if P is a prime, A is a prime and if P is an atom, A is an atom.*

PROPOSITION 3.2. *Let $\mathscr{C}$ be a regular site and $\mathscr{E}$ the category of sheaves on $\mathscr{E}$ and $\epsilon:\mathscr{C} \rightarrow \mathscr{E}$ be the canonical continuous functor. Then $\mathscr{E}$ is prime-generated (resp. atomic) if and only if $\mathscr{C}$ is and in that case the primes (resp. atoms) of $\mathscr{E}$ are exactly the quotients of the ϵP for P a prime (resp. an atom) of $\mathscr{C}$.*

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Proof. Assume that $\mathscr{E}$ is prime-generated (resp. atomic). Let C be an object of $\mathscr{C}$, and let $\{E_i \rightarrowtail \epsilon C\}$, $i \in I$ be a covering family of monomorphisms with each E_i being a prime (resp. an atom). Since $\mathscr{C}$ generates $\mathscr{E}$ via ϵ, and ϵ is locally full (see Proposition 1.2), we may find objects C_{ij} of $\mathscr{C}$ and covers $\{\epsilon C_{ij} \to E_i\}_j$ such that each composite

$$\epsilon C_{ij} \to E_i \rightarrowtail \epsilon C$$

is of the form $\epsilon(g_{ij})$. Since E_i is a prime, (resp. an atom), for each i there is j such that $\epsilon C_{ij} \to E_i$ is a cover; let us now denote C_{ij} as C_i and g_{ij} as g_i. Let

$$\begin{array}{c}\xrightarrow{\qquad g_i \qquad}\\ C_i \twoheadrightarrow C_i' \rightarrowtail C\end{array}$$

be the regular epi/mono factorization of g_i; comparing it with

$$\begin{array}{c}\xrightarrow{\qquad \epsilon g_i \qquad}\\ \epsilon C_i \twoheadrightarrow E_i \rightarrowtail \epsilon C\end{array}$$

we see that $\epsilon C_i' \cong E_i$. It follows that each C_i' is a prime (resp. an atom), and that we have a cover $\{C_i' \twoheadrightarrow C\}$ of C with primes (resp. atoms), showing that $\mathscr{C}$ is prime generated (resp. atomic).

Finally, suppose $\mathscr{C}$ is prime-generated (resp., atomic) and E is a prime (resp. an atom) in $\mathscr{E}$. Then E has a cover with objects ϵC, hence a cover with objects ϵP with each P a prime, (resp. an atom), hence E is a quotient of an ϵP as desired.

COROLLARY 3.3. *A μ-presentable prime-generated, resp. atomic,* GT *has a defining site of size at most μ which is regular and prime-generated, resp. Boolean and atomic.*

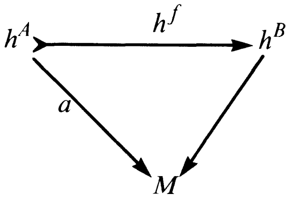
Proof. This follows from 3.2, 1.6, and 1.10.

Atomic sites and toposes were introduced (in slightly different form) in [**5**]. Prime-generated ones were introduced in [**14**] and [**16**]. It is easy to see that a Grothendieck topos is an atomic topos if and only if the subobject lattice of each object is a complete atomic Boolean algebra. There is a similar characterization for a prime-generated Grothendieck topos. Call a lattice *prime-generated* if each element is the join of join irreducibles. Then a Grothendieck topos is prime-generated if and only if the subobject lattice of each object is prime-generated. Since these properties depend only on the subobject lattices, it follows that a topos which has a powerful embedding into a category in one of these classes also belongs to the class.

Recall that a model of a regular site $\mathscr{C} = (\mathbf{C}, \mathbf{J})$ is a **Set**-valued functor on $\mathscr{C}$ that is left exact and **J**-continuous, i.e., that preserves **J**-joins and images. We will say that a *premodel* is a left exact functor that preserves **J**-joins.

PROPOSITION 3.4. (i) *A filtered colimit of premodels is a premodel.*

(ii) *Suppose $\mathscr{C}$ is prime-generated. A premodel M is a model if and only if for every covering* epi $f:B \twoheadrightarrow A$ *between primes and every element $a \in MA$, there is a commutative diagram*



Proof. (i) This is clear.

(ii) Since $\mathscr{C}$ is prime-generated, its topology is generated by families of monomorphisms and single covers of primes by primes. The first kind of covers are preserved by all premodels. Preserving the second kind of cover is equivalently restated in the condition under (ii) (compare 1.1).

For $M:\mathbf{C} \to \mathbf{Set}$ a left exact functor, and X a subobject of C in $\mathbf{C}$, $M(X)$ is a subobject of C, hence determines a definite subset of $M(C)$. Henceforth, in this situation, by $M(X)$ we will mean this subset of $M(C)$. Thus, if $c \in M(C)$, it is meaningful to ask if $c \in M(X)$.

If M is a pre-model of the site $\mathscr{C}$ and A is an object of $\mathscr{C}$, the *type* of an element $a \in MA$, denoted $\mathbf{t}(a)$, consists of the set of all subobjects $A_0 \subseteq A$, for which $a \in M(A_0)$. It is clear that $\mathbf{t}(a)$ is a filter in the subobject lattice of A. If M is a model, then each $\mathbf{t}(a)$ is a **J**-*prime* filter, i.e., it satisfies the following additional condition (we write $\mathbf{t}$ for $\mathbf{t}(a)$):

(*) Whenever $X_i \in \mathrm{Sub}(A)$ $(i \in I)$, $X = \bigvee^{(\mathbf{J})}_{i \in I} X_i \in \mathrm{Sub}(A)$, and $X \in \mathbf{t}$, then for some i, $X_i \in \mathbf{t}$.

The following proposition shows that, in the case of separable sites, being a **J**-prime filter is the same as being a type of an element of a model.

PROPOSITION 3.5. *Suppose* $(\mathbf{C}, \mathbf{J})$ *is a separable site. Suppose A is an object of* $\mathbf{C}$ *and* $\mathbf{t}$ *is a* **J**-*prime filter on* $\mathrm{Sub}(A)$. *Then there is a countable model M, having an element $a \in M(A)$ whose type $\mathbf{t}_M(a)$ is equal to* $\mathbf{t}$.

Proof. Assume first that $A = 1$ is the terminal object of $\mathbf{C}$. Let $\mathbf{J}_0$ be a countable base for $\mathbf{J}$. Recall that the support, denoted $\mathrm{supp}(A)$, of an object A is the smallest subobject of 1 that the terminal map of A factors through. It is constructed as the regular image of the terminal map of A. Enumerate in a sequence $\langle \tau_k \rangle_{k<\omega}$ all triples

$$\tau = \langle i, D \to A, \mathbf{A} \rangle$$

where $i \in \omega$, $D \to A$ is a morphism in $\mathbf{C}$, and $\mathbf{A}$ is a $\mathbf{J}_0$-cover of A, in such a way that each individual τ is repeated infinitely often, which is clearly possible. Also, let $\langle U_n \rangle$, $n \in \omega$ be an enumeration of $\mathbf{t}$. Now, we are going to construct a sequence of arrows

(1) $\quad D_1 \leftarrow D_1 \leftarrow \ldots \leftarrow D_n \leftarrow D_{n+1} \leftarrow \ldots$

such that, among other things, $D_0 = 1$ and $\text{supp}(D_n) \in \mathbf{t}$ for all $n < \omega$.

Suppose we have constructed the sequence up to and including D_n.

Case 1. $n = 2k$ is even. Consider

$$\tau_k = \langle i, D \to A, \mathbf{A} \rangle.$$

If either $i > n$ or $D \neq D_i$, put $D_{n+1} = D_n$, and extend the sequence by

$$\text{id}: D_{n+1} \to D_n.$$

If, however, $i \leqq n$ and $D = D_i$, consider the composite

$$D_n \to \ldots \to D_i \to A.$$

Denote the composite by $D_n \to A$, and with $\mathbf{A} = \{A_s \to A\}$, $s \in S$, consider the pullbacks, one for each $s \in S$:

$$\begin{array}{ccc} D_n^s = A_s \times D_n & \longrightarrow & D_n \\ \downarrow & & \downarrow \\ A_s & \longrightarrow & A \end{array}$$

Since $\{A_s \to A\}$, $s \in S$ is a $\mathbf{J}$-cover of A, we have that $\{D_n^s \to D_n\}$, $s \in S$ is a $\mathbf{J}$-cover of D_n. Hence

$$\text{supp}(D_n) = \textstyle\bigvee^{(\mathbf{J})}_{s \in S} \text{supp}(D_n^s).$$

Since $\text{supp}(D_n) \in \mathbf{t}$, and $\mathbf{t}$ is a $\mathbf{J}$ filter,

$$\text{supp}(D_n^s) \in \mathbf{t} \quad \text{for some } s \in S.$$

Choose such an $s \in S$ and let $D_{n+1} \to D_n$ be $D_n^s \to D_n$.

Case 2. $n = 2k + 1$ is odd. We now look at U_k, the kth member of $\mathbf{t}$, and put $D_{n+1} = D_n \times U_k$. We have

$$\text{supp}(D_{n+1}) = \text{supp}(D_n) \wedge U_n \in \mathbf{t},$$

so the induction hypothesis is satisfied. We take $D_{n+1} \to D_n$ to be

$$\pi_i: D_n \times U_k \to D_n.$$

This completes the construction of the sequence (2). Consider

$$M = \text{colim}_{n<\omega} h^{D_n}.$$

The condition of Proposition 1.1 holds in our case: given any $\mathbf{A}$, a

$\mathbf{J}_0$-cover of A, any $i \in \omega$ and any $D_i \to A$, we may find $n = 2k \geqq i$ such that

$$\tau_k = \langle i, D_i \to A, \mathbf{A} \rangle;$$

the construction at stage $n + 1$ will ensure that the condition holds. Thus, M is a model.

Let $M(1) = \{*\}$ ($= 1$ in **Set**). Then we define the *type* of M, denoted $\mathbf{t}(M)$, to be

$$\mathbf{t}_M(*) = \{U \in \text{Sub}(1) \,|M(U) \neq \emptyset\}$$
$$= \{u \in \text{Sub}(1) \,|U \leqq \text{supp}(D_n) \text{ for some } n \in \omega\}.$$

If $U \in \mathbf{t}(M)$, let $n \in \omega$ be such that $\text{supp}(D_n) \leqq U$. Since $\text{supp}(D_n) \in \mathbf{t}$ by the construction, we have $U \in \mathbf{t}$. We have shown that $\mathbf{t}(M) \subseteq \mathbf{t}$.

Conversely, if $U \in \mathbf{t}$, then $U = U_k$ for some $k < \omega$. Let $n = 2k + 1$; by the construction,

$$\text{supp}(D_{n+1}) \subseteq U_k.$$

It follows that $U = U_k \in \mathbf{t}(M)$.

We have proved that $\mathbf{t}(M) = \mathbf{t}$; this completes the proof for the case when $A = 1$.

In the general case, we pass to the slice category $\mathbf{C}/A$, and the induced topology $\mathbf{J}/A$ (in more detail, see the proof of Theorem 3.8 below). The set $\mathbf{t} \in \text{Sub}_{\mathbf{C}}(A)$ gives rise to a set $\mathbf{t}/A \subseteq \text{Sub}_{\mathbf{C}/A}(1)$ defined as

$$\{ (i{:}X \to A) \subseteq (\text{id}{:}A \to A) \},$$

i.e., just the subobjects in $\mathbf{t}(A)$, but considered in $\mathbf{C}/A$.

We can now show that $\mathbf{t}/A$ is a $\mathbf{J}/A$-filter on $\text{Sub}_{\mathbf{C}/A}(1)$, and that applying the case $A = 1$ to $(\mathbf{C}/A, \mathbf{J}/A)$ we obtain the desired result.

Let M be a premodel and a be an element of M. If the type of a is a principal filter, we will say that a is a *principal* element of M. If the type of a is principal, generated by the subobject A_0 of A, we will often say that A_0 itself is the type of a. If every element of M is principal, we will call M a *principal premodel.*

If a is a principal element of M and M is a model, then the subobject generating the type of a is a prime; this follows from the fact that the type is a $\mathbf{J}$-prime filter. If A is a prime, M is a model, then $a \in M(A)$ is *generic* for A if $\mathbf{t}(a)$ consists exactly of id_A.

PROPOSITION 3.6. *Let $\mathscr{C}$ be a site and M be a model of $\mathscr{C}$.*

(i) *M is principal if and only if it is a filtered colimit of representables in which the transition arrows are regular epis.*

(ii) *A principal model preserves intersections of arbitrary power.*

(iii) *If $\mathscr{C}$ has complete subobject lattices and M preserves arbitrary intersection, then M is principal.*

(iv) *M is principal if and only if its extension to* **Sh**($\mathscr{C}$) *preserves arbitrary intersection.*

Proof. The first three parts are just Theorem 16 of [**3**]. The direct part of (iv) follows from the fact that the extension of M to **Sh**($\mathscr{C}$) is represented by the same diagram that represents M. The converse is trivial.

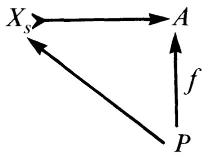
The following is a fundamental observation.

PROPOSITION 3.7. *Let* (**C**, **J**) *be a regular site, and P a prime object of* **C**. *Then the hom-functor*

$$\mathscr{C}(P, -):\mathbf{C} \rightarrow \mathbf{Set}$$

takes **J***-unions into ordinary unions.*

Proof. The assertion says, in other words, that if $X_s \in \mathrm{Sub}(A)$ for $s \in S$; $U^{(\mathbf{J})}_{s \in S} X_s = A$ and P is a prime, then for any $f:P \rightarrow A$, there is $s \in S$ and a factorization



Indeed, $\{i_s:X_s \times_A P \rightarrow P\}$, $s \in S$ is a **J**-cover of P; also each i_s is a monomorphism. Since P is a prime, there is an $s \in S$ such that i_s is a singleton cover, hence, since $\mathscr{C}$ is regular, an extremal epi. It follows that i_s is an isomorphism. The assertion follows.

PROPOSITION 3.8. *Let* $\mathscr{C}$ *be a regular site and let* $D:\mathbf{I} \rightarrow \mathscr{C}$ *be a cofiltered diagram and M be the corresponding functor, i.e.,*

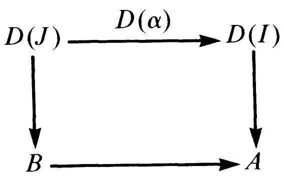
$$MC = \mathrm{colim}\ \mathrm{Hom}(DI, -),$$

the colimit taken over **I**. *Then*

(i) *If* $D(I)$ *is prime for every object* I *of* **I**, *then M is a pre-model.*

(ii) *If, in addition,* $D(\alpha):D(J) \rightarrow D(I)$ *is a regular epi for every morphism* $\alpha:J \rightarrow I$ *of* **I**, *then M is principal.*

(iii) *If, in addition to* (i), *for any object* I *of* **I**, *any morphism* $D(I) \rightarrow A$ *in* $\mathscr{C}$ *and any covering epi* $f:B \rightarrow A$ *in* $\mathscr{C}$, *there is a morphism* $\alpha:J \rightarrow I$ *in* **I** *and a factorization*



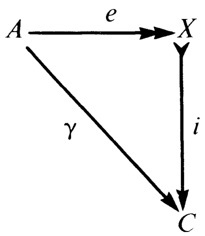
Then M is a model.

https://doi.org/10.4153/CJM-1987-009-x Published online by Cambridge University Press

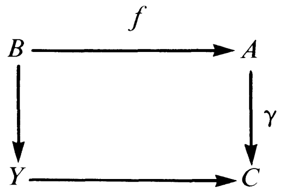
Proof. (i) This is a direct consequence of 3.7.

(ii) This follows from 3.6 (i). One notes that this also may be verified directly as follows.

Suppose $M = \operatorname{colim} D$, with D having the properties described. Let $c \in M(C)$. Then c is represented by an arrow $\gamma A \to C$, with A an object in the diagram D. Consider the factorization of γ



into an effective epi followed by a mono. Then for the subobject X represented by i, we certainly have $c \in M(X) \subseteq M(C)$. On the other hand, if $Y \in \mathrm{Sub}(C)$, and $c \in M(Y)$, then for some arrow $B \to A$ in D, we have a commutative diagram



Since, by hypothesis, f is an effective epi, ef and i provide an effective epi/mono factorization of αf; hence $X \leqq Y$. This shows that the type of c is generated by X.

(iii) This follows from 3.4 (ii).

THEOREM 3.9. *An $\aleph_1$-presentable prime-generated Grothendieck topos has enough points.*

THEOREM 3.10. *A consistent regular prime-generated site of size $\leqq \aleph_1$ has a model.*

We begin by showing how Theorem 3.9 follows from Theorem 3.10. By 3.3, we have a regular prime-generated site $\mathscr{C}$ of size $\leqq \aleph_1$ for the given $\aleph_1$-presentable prime-generated GT. To demonstrate Theorem 3.9, we must show that given an object A and a sieve $\{A_i \to A\}$ of subobjects of A which does not cover, there is a model $M: \mathscr{C} \to \mathbf{Set}$ for which $\bigvee^{(\mathbf{J})} MA_i \neq A$. The first reduction is that we may consider the case $A = 1$, the terminal object. The reason is that we may form the slice category $\mathscr{C}/A$ in which $\mathscr{C}$ has a canonical model $A \times -$. The topology on $\mathscr{C}$ can be easily seen to induce a topology on $\mathscr{C}/A$ in which a sieve is a cover in $\mathscr{C}/A$ if and only if it is when the forgetful functor to $\mathscr{C}$ is applied. It is immediately verified that for a prime B in $\mathscr{C}$ the object $B \to A$ of $\mathbf{C}/A$ is a prime in $\mathscr{C}/A$ and that $\mathscr{C}/A$ is regular and prime-generated. Thus the task is reduced to a similar task for $\mathscr{C}/A$ where the codomain of the sieve, A above, is 1. Thus we are reduced to showing that if $\{U_i \to 1\}$ is a non-covering sieve in C, there is a model of $\mathscr{C}$ whose value at each U_i is empty. This is done by extending the topology $\mathbf{J}$ to a topology $\mathbf{J}'$ by making each of the U_i empty. That is, we add to the topology the empty sieve over each U_i. This is done by saying that a sieve $\{C_j \to C\}_j$ is a $\mathbf{J}'$-cover if and only if

$$\{C_j \to C\}_j \cup \{C \times U_i \to C\}_i$$

is a $\mathbf{J}$-cover. This topology is consistent for the empty sieve is not a $\mathbf{J}'$-cover of 1. It is easily checked that the new site is regular (resp. $\aleph_1$-presentable, resp. prime-generated) if the original one was. In particular, every prime for $\mathbf{J}$ that is non-empty for $\mathbf{J}'$ is a prime for $\mathbf{J}'$.

By Theorem 3.9, the new site $(\mathbf{C}, J')$ has a model; this is a model of $\mathscr{C}$ that makes every U_i empty as desired.

"Countably presented" for left exact functors was defined before Proposition 1.4. We will abbreviate "countably presented premodel" as CPPM. We assume throughout this proof that $\mathscr{C}$ is a regular prime-generated site of size $\leqq \aleph_1$. For the proof of 3.10, we need the following in which we denote the hom functor $\mathrm{Hom}(A, -)$ by h^A.

LEMMA 3.10′. *Let F be a* CPPM *and $B \twoheadrightarrow A$ be a covering epi. Then the diagram in* $\mathrm{Lex}(\mathscr{C}, \mathbf{Set})$

$$\begin{array}{ccc} h^A & \rightarrowtail & h^B \\ \downarrow & & \\ F & & \end{array}$$

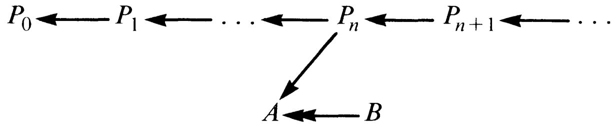
can be completed to a diagram

$$\begin{array}{ccc} h^A & \rightarrowtail & h^B \\ \downarrow & & \downarrow \\ F & \rightarrowtail & G \end{array}$$

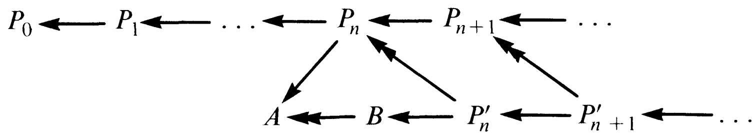
in which G is a CPPM.

Note. The fact that $B \twoheadrightarrow A$ implies that $h^A \rightarrowtail h^B$.

Proof. By applying Proposition 1.4 (iv) to the site on $\mathbf{C}$ whose topology is generated by the $\mathbf{J}$-families consisting of monomorphisms (so that F is a model for this site), we conclude that F is a countable filtered colimit of representables, each represented by a prime; by 1.4 (v), this means that F is represented by an ω-chain of primes P_n, $n < \omega$. Thus the given diagram in the lemma is equivalent to a diagram in $\mathscr{C}$,



which we will embed in the diagram



defined as follows. Since $A \twoheadrightarrow B$, we have also that

$$B \times_A P_n \twoheadrightarrow P_n.$$

It follows that for at least one prime subobject of $B \times_A P_n$, call it P'_n, the composite

$$P'_n \to B \times_A P_n \to P_n$$

is a regular epi. Similarly, let P'_{n+1} be a prime subobject of $P_{n+1} \times_{P_n} P'_n$ that maps by a regular epi to P_{n+1}. In this way, we inductively build up the desired sequence. Then G is the functor represented by the diagram

$$P_0 \leftarrow P_1 \leftarrow \ldots \leftarrow P'_n \leftarrow P'_{n+1} \leftarrow \ldots .$$

The map $F \to G$ is induced by epis and is easily seen to be a mono.

Proof of theorem 3.10. For **Set**-valued functors F and F' we write $F \subseteq F'$ to indicate that F is a subfunctor of F'. We will define an ordinal chain of CPPMs

$$F_0 \subseteq F_1 \subseteq \ldots \subseteq F_\omega \subseteq \ldots \subseteq F_\alpha \subseteq \ldots$$

indexed by all the countable ordinals and whose union will be the model we want. If F is a CPPM, let $\Phi(F)$ denote the set of all pairs (f, a) where $f:B \to A$ is a covering epi of $\mathscr{C}$ and $a \in FA$. It is clear that the cardinality of $\Phi(F)$ is at most $\aleph_1$. Moreover, if $F \subseteq F'$, then $\Phi(F) \subseteq \Phi(F')$. Begin by dividing the set $\aleph_1$ into a set of $\aleph_1$ disjoint subsets X_α, $\alpha \in \aleph_1$, each of size $\aleph_1$. Along with the F_α, we will construct a surjective indexing

$$g_\alpha: \cup_{\beta \leqq \alpha} X_\alpha \twoheadrightarrow \Phi(F_\alpha)$$

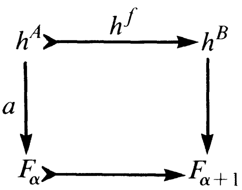
such that for every element of $\Phi(F_\alpha)$ there are $\aleph_1$ many distinct indices i that map to the element by g_α, and such that for $\beta < \alpha$, we have that g_α extends g_β. We will define the F_α and g_α simultaneously by an induction on the ordinals $\alpha < \aleph_1$.

We begin by letting F_0 be the functor represented by the terminal object 1. For the indexing g_0, simply choose any suitable correspondence between X_0 and the set of object of $\mathscr{A}$ (which is what $\Phi(F_0)$ is).

Next suppose that F_α and g_α have been chosen. There are two cases to consider. If α is not in dom g_α, let $F_{\alpha+1} = F_\alpha$ and $g_{\alpha+1}$ any function extending g_α with the right domain and codomain. Otherwise, let $a \in F_\alpha(A)$ and $f:B \to A$ be such that

$$g_\alpha(\alpha) = (f, a).$$

In this case, apply Lemma 3.10′ to get a diagram



It is easy to see that the monomorphism $F_\alpha \rightarrowtail F_{\alpha+1}$ can be chosen, by a suitable choice of $F_{\alpha+1}$, to be an inclusion; thus $F_\alpha \subseteq F_{\alpha+1}$. To obtain $g_{\alpha+1}$, we extend g_α by choosing a suitable correspondence between the elements of $\Phi(F_{\alpha+1}) - F_{\alpha+1}$ and the set $X_{\alpha+1}$. If α is a limit ordinal, let

$$F_\alpha = \text{colim}_{\beta<\alpha} F_\beta = \cup_{\beta<\alpha} F_\beta$$

and

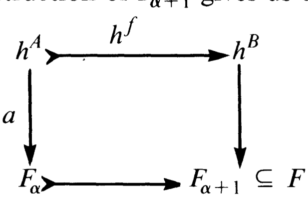
$$g_\alpha = \cup_{\beta<\alpha} g_\beta.$$

Then F_α is a CPPM by 1.4 (ii) and 3.4 (i).

Finally,

$$F = \text{colim}_{\alpha<\omega_1} F_\alpha = U_{\alpha<\omega_1} F_\alpha$$

is the required model. Indeed, by 3.4 (i), it is a premodel. We may verify the condition in 3.4 (ii) as follows. Suppose $f:B \to A$ is a covering epi, and $a \in F(A)$. Then $a \in F_\beta(A)$ for some $\beta < \omega_1$. Then for some $\alpha \in \text{dom}(g_\beta)$, $\alpha \geqq \beta$, we have $g_\beta(\alpha) = (f, a)$; therefore $g_\alpha(\alpha) = (f, a)$. Now, the construction of $F_{\alpha+1}$ gives us the commutative diagram



that shows what we want.

Problem. Can we force the model in 3.10 to be a principal one? We can in the separable case.

THEOREM 3.11. *Suppose $\mathscr{C}$ is a separable, prime-generated site. Then for any prime P, there is a principal model M of $\mathscr{C}$ which contains a generic element for P.*

Proof. The proof is similar to, and simpler than, that of 3.10. The ω_1-sequence of CPPMs is replaced by an ω-sequence of representable premodels, each represented by a prime. In this situation, we are able to make the morphisms in the diagram to be covering epis which is the key to the principalness.

To start, let us enumerate all triples

$$\tau = \langle i, D \rightarrow A, B \twoheadrightarrow A \rangle$$

such that $i < \omega$, $D \rightarrow A$ is an arrow in $\mathbf{C}$ and $B \twoheadrightarrow A$ is a covering epi, in a sequence $\langle \tau_n \rangle$, $k < \omega$ in which every τ appears infinitely often. We construct a sequence

(3) $D_0 \twoheadleftarrow D_1 \twoheadleftarrow \ldots \twoheadleftarrow D_n \twoheadleftarrow D_{n+1} \twoheadleftarrow \ldots$

for $n \in \omega$ such that each D_n is a prime, $D_0 = P$, and each $D_{n+1} \twoheadrightarrow D_n$ is a covering epi, as follows. Suppose we have constructed all items, up to and including D_n. Consider

$$\tau_n = \langle i, D \rightarrow A, B \twoheadrightarrow A \rangle.$$

If either $i > n$, or $D \neq D_i$, put $D_{n+1} \rightarrow D_n$ to be equal to $\mathrm{id}: D_n \rightarrow D_n$. If $i \leqq n$ and $D = D_i$, let $D_n \rightarrow A$ be the composite

$$D_n \rightarrow \ldots \rightarrow D_i \rightarrow A,$$

and consider the pullback:

$$\begin{array}{ccc} C = B \times_A D_n & \twoheadrightarrow & D_n \\ \downarrow & & \downarrow \\ B & \twoheadrightarrow & A \end{array}$$

Of course, $C \twoheadrightarrow D_n$ is a covering epi. Let D_{n+1} be the underlying object of a prime subobject X of C such that $X \rightarrow D_n$ is a covering epi (such X exists since $C = \bigvee_s^{(\mathrm{J})} X_s$ for prime $X_s \in \mathrm{Sub}(C)$, and since D_n is prime). Thus we have the commutative diagram

$$\begin{array}{ccc} D_{n+1} & \twoheadrightarrow & D_n \\ \downarrow & & \downarrow \\ B & \twoheadrightarrow & A \end{array}$$

and we have obtained $D_{n+1} \twoheadrightarrow D_n$ as wanted.

This completes the construction of the sequence (3). Let

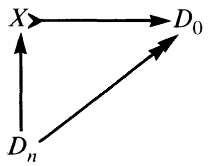
$$M = \operatorname{colim}_n h^{D_n}.$$

By 3.8 (i) and (ii), M is a principal premodel, and by 3.8 (iii), and the construction, M is a model. For if $D_k \to A$ and a covering epi $f:B \twoheadrightarrow A$ are given, let $n \geqq k$ be such that

$$\tau_n = \langle k, D_k \to A, f \rangle;$$

then in the construction of D_{n+1}, we make sure that the condition in 3.8 (iii) holds.

The identity arrow of $P = D_0$ to itself represents an element $d \in M(P)$ which is a generic element of $M(P)$: if $X \in \mathrm{Sub}(P)$, and $d \in M(X)$, then we have a factorization



with $D_n \twoheadrightarrow D_0$ coming from the diagram (3). But then, of course, X must be the identity subobject of P.

We say that a model M of a site $\mathscr{C}$ is *prime* if for any model M', any object A of $\mathscr{C}$ and any elements $a \in MA$ and $a' \in M'A$, if $\mathbf{t}(a) \subseteq \mathbf{t}(a')$, then there is a morphism $\alpha:M \to M'$ such that $\alpha A(a) = a'$. (This terminology derives from an analogous, although not identical, model theoretical terminology.)

PROPOSITION 3.11′. *Let $\mathscr{C}$ be a regular site, $\mathscr{E} = \mathbf{Sh}(\mathscr{C})$, and $\epsilon:\mathscr{C} \to \mathscr{E}$ the canonical functor. If the model M of $\mathscr{C}$ is principal, respectively principal prime, then so is the corresponding model N of $\mathscr{E}$ (for which $N\epsilon \cong M$).*

Proof. Suppose M is principal. Then M is represented by a cofiltered diagram of primes of $\mathscr{C}$ with transition morphisms covering epis, by 3.6 (i). Note that if P is a prime in $\mathscr{C}$, then $\epsilon\mathbf{P}$ is a prime in $\mathscr{E}$; and the left Kan extension of $\mathscr{C}(P, -)$ along ϵ is $\mathscr{E}(\epsilon P, -)$. Therefore, by the last assertion of 1.2, we obtain that N is represented by a cofiltered diagram of primes in $\mathscr{E}$ with transition morphisms epis and hence, by 3.6 (i), N is principal.

Now let M be a principal prime model of $\mathscr{C}$ and N be its extension to $\mathscr{E}$. Suppose M' is another model of $\mathscr{C}$ and N' its extension to $\mathscr{E}$. Suppose $e \in NE$ and $e' \in N'E$ with $\mathbf{t}(e) \subseteq \mathbf{t}(e')$. The type of e is a prime subobject P_0 of E which is the quotient of some prime P belonging to $\mathscr{C}$. There is an element $p \in MP$ that maps to e under the morphism $P \to E$. The type of p is P', a prime subobject of P. The image of P' under $P \to E$ is P_0, since e is in this image and the type of e is generated by P_0. Thus we have a covering epi $P' \twoheadrightarrow P_0$. Since $e' \in N'P_0$ and $P' \twoheadrightarrow P_0$, it follows that there is an element $p' \in M'P'$ that maps to e'. From the fact that M is prime, it follows that there is a morphism $\alpha:M \rightarrow M'$ such that $\alpha P'(p) = p'$. α corresponds to a morphism $\beta:N \rightarrow N'$. The fact that $\alpha P'(p) = p'$ and the evident commutativity imply that $\beta P_0(e) = e'$.

THEOREM 3.12. *Let $\mathscr{C}$ be a separable site and M be a countable principal model of $\mathscr{C}$. Then M is prime.*

In the next two lemmas, M is principal.

LEMMA 3.13. *Suppose $a \in MA$ is generic for A and $a' \in M'A$, $b \in MB$ are arbitrary. Then there is an element $b' \in M'B$ such that*

$$\mathbf{t}\langle a, b\rangle \subseteq \mathbf{t}\langle a', b'\rangle.$$

Proof. Let $C \subseteq A \times B$ be the generator of $\mathbf{t}\langle a, b\rangle$. The image of the composite

$$C \rightarrow A \times B \rightarrow A$$

belongs to the type of a which is A, i.e., the composite is an epi. Hence the composite

$$M'C \rightarrow M'A \times M'B \rightarrow M'A$$

is surjective, which means there is a $c' \in M'C$ that maps to a'. Let b' be its other coordinate. Clearly, $C \in \mathbf{t}\langle a', b'\rangle$.

COROLLARY 3.14. *Suppose*

$$\langle a_1, a_2, \dots, a_{n+1}\rangle \in MA_1 \times MA_2 \times \dots \times MA_{n+1}$$

and

$$\langle a'_1, a'_2, \dots, a'_n\rangle \in M'A_1 \times M'A_2 \times \dots \times M'A_n$$

are such that

$$\mathbf{t}\langle a_1, a_2, \dots, a_n\rangle \subseteq \mathbf{t}\langle a'_1, a'_2, \dots, a'_n\rangle,$$

then there is an $a'_{n+1} \in M'A_{n+1}$ such that

$$\mathbf{t}\langle a_1, a_2, \dots, a_{n+1}\rangle \subseteq \mathbf{t}\langle a'_1, a'_2, \dots, a'_{n+1}\rangle.$$

Proof. Take A to be the type of $\langle a_1, a_2, \dots, a_n\rangle$ and $B = A_{n+1}$ in the previous lemma.

PROPOSITION 3.15. *Let M and M' be two models of $\mathscr{C}$. If $\langle a, b\rangle \in M(A \times B)$ and $\langle a', b'\rangle \in M'(A \times B)$ are such that*

$$\mathbf{t}\langle a, b\rangle \subseteq \mathbf{t}\langle a', b'\rangle,$$

then $\mathbf{t}(a) \subseteq \mathbf{t}(a')$.

Proof. Let $A_0 \subseteq A$ such that $a \in MA_0$. Then $b \in MB$, so that

$$\langle a, b\rangle \in M(A_0 \times B),$$

whence

$$\langle a', b'\rangle \in M'(A_0 \times B),$$

whence $a' \in M'A_0$.

Proof of the theorem. Enumerate the elements of M and find a sequence $\langle a_1', a_2', \ldots, a_n', \ldots\rangle$ of elements of M' such that for all n,

$$\mathbf{t}\langle a_1, a_2, \ldots, a_n\rangle \subseteq \mathbf{t}\langle a_1', a_2', \ldots, a_n'\rangle.$$

We define the function $f:M \to M'$ by letting $\varphi(a_n) = a_n'$. To see that φ is a morphism, we must show that if $f:A \to B$ is a morphism of $\mathscr{C}$, then for any $a \in MA$,

$$\varphi B \circ Mf(a) = M'f \circ \varphi A(a).$$

Let $b = Mf(a)$. Then by applying 3.15 to a string $\langle a_1, a_2, \ldots, a_n\rangle$ long enough to include both a and b, we may conclude that

$$\mathbf{t}\langle a, b\rangle \subseteq \mathbf{t}\langle \varphi(a), \varphi(b)\rangle.$$

But the type of the latter is evidently included in the graph of f which is the image of

$$(\mathrm{id}, f):A \to A \times B.$$

But then the graph of f is also contained in the type of $\langle a', b'\rangle$ which implies that $M'f(a') = b'$.

Recall the terminology "sufficient class of models", "enough models" from Section 1. A sufficient category of models is one whose class of objects is sufficient.

COROLLARY 3.16. *A separable, prime-generated site has enough principal, prime models. As a consequence, a prime-generated separable* GT *has enough principal prime models.*

Proof. This follows readily by 3.11, 3.12 and 3.3.

A principal model of a site is called *atomic* if the type of every element is generated by an atom.

THEOREM 3.17. *Let $\mathscr{C}$ be a separable site and M and M' any two countable atomic models of $\mathscr{C}$. Then for any elements $a \in MA$ and $a' \in M'A$ with* $\mathbf{t}(a) \subseteq \mathbf{t}(a')$, *there is an isomorphism $f:M \to M'$ with $f(a) = a'$.*

Proof. We begin by observing that for any two elements $a \in MA$ and $a' \in M'A$, if $\mathbf{t}(a) \subseteq \mathbf{t}(a')$, then they are actually equal for there can be no proper inclusions among atoms. The argument proceeds much like that of 3.12 above except that the construction goes back and forth to insure that all elements are captured. To be precise, begin by enumerating the elements of the union of the models. Suppose that at stage n, we have sequences $a_1, a_2, \ldots, a_n$ of elements of M and $a'_1, a'_2, \ldots, a'_n$ of elements of M' so that when $a_i \in MA_i$, then $a'_i \in M'A_i$ and furthermore that

$$\mathbf{t}\langle a_1, a_2, \ldots, a_n\rangle = \mathbf{t}\langle a'_1, a'_2, \ldots, a'_n\rangle.$$

The $n+1$st element in the enumeration is either an element of M or of M'. Suppose it belongs to MA_{n+1}. Then call it a_{n+1}. We know from 3.14 and the remark at the beginning of the proof that there is an element $a'_{n+1} \in M'A_{n+1}$ such that

$$\mathbf{t}\langle a_1, a_2, \ldots, a_{n+1}\rangle = \mathbf{t}\langle a'_1, a'_2, \ldots, a'_{n+1}\rangle.$$

It is possible that a'_{n+1} appeared earlier on the list, say $a'_{n+1} = a'_m$. We have, by Proposition 3.15,

$$\mathbf{t}\langle a'_m, a'_{n+1}\rangle = \mathbf{t}\langle a_m, a_{n+1}\rangle.$$

But then $A_m = A_{n+1}$ and the type of $\langle a'_m, a'_{n+1}\rangle$ is included in the diagonal of $A_m \times A_m$, whence so is the type of $\langle a_m, a_{n+1}\rangle$. But this means that $a_{n+1} = a_m$. Then the function $f:M \to M'$ given by $f(a_n) = a'_n$ is well defined. It is evidently a bijection and the proof that it is a morphism of models is the same as that of 3.12.

Call a site a *regular epimorphism site* if it has finite limits and has a basis of covers consisting of regular epimorphisms. We do not require the site to be regular.

THEOREM 3.18. *The category of sheaves for a regular epimorphism site has enough principal prime models.*

Proof. The conclusion follows from the results of [**3**]. See Theorem 17 of that paper for the proof that there are enough principal models and Proposition 8 for their primeness. Also see [**16**], Corollary 4.3.

The same result is true for presheaf categories. We could actually infer it from results we already have by observing that our hypothesis that a site have finite limits is actually too strong. Only pullbacks along covers need exist and in a presheaf topos that is no restriction. Thus a presheaf topos is a regular epimorphism site. Nonetheless, it seems useful to actually describe the principal prime models; they are the evaluation functors. They are principal from Proposition 3.6 (iii) since the category has, and they preserve, arbitrary intersections (indeed all limits). To see that the evaluation functors are prime, let $\mathscr{E} = \mathbf{Set}^{\mathbf{K}}$ and for K an object of $\mathbf{K}$, let K^* denote the model of $\mathscr{E}$ which is evaluation at K.

PROPOSITION 3.19. *Let E be an object of $\mathscr{E}$, K be an object of $\mathbf{K}$ and $k \in E(K)$. Let $E_{(k)}$ denote the intersection of all subfunctors of $E_0 \subseteq E$ for which $k \in E_0(K)$. Then $E_{(k)}$ is prime; conversely, every prime of E is of the form $E_{(k)}$ for some K in $\mathbf{K}$ and $k \in E(K)$.*

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Proof. To see that $E_{(k)}$ is prime, it is sufficient to observe that limits and colimits in a functor category are computed "pointwise". Hence if $E_{(k)} = \bigvee E_i$, then we must have $k \in E_i(K)$ for at least one i, whence $E_i = E_0$ by minimality of the latter. Conversely, if E is any functor, it is evident that if $E = \bigvee E_{(k)}$ as k ranges over all elements of E and if E is prime, then E must be one of the $E_{(k)}$.

PROPOSITION 3.20. *The evaluation functors are prime.*

Proof. To show that the model K^* is prime, we must show that for any prime E, any generator $k_0 \in K^*(E) = E(K)$, any model M and any $x \in M(E)$, there is an $\alpha:K^* \to M$ such that

$$\alpha E(k_0) = x.$$

To say that k_0 generates E means that the corresponding morphism

$$k_0:\mathrm{Hom}(K, -) \to E$$

is epi, which implies that

$$M(k_0):M(\mathrm{Hom}(K, -)) \to M(E)$$

is surjective. This means that there is an element $y \in M(\mathrm{Hom}(K, -))$ such that

$$M(k_0)(y) = x.$$

Then define a function

$$\alpha F:F(K) \to M(F),$$

for each object F of $\mathscr{E}$ by letting

$$\alpha F(k) = M(k)(y).$$

Here we identify elements of $F(K)$ with natural transformations of $\mathrm{Hom}(K, -)$ to F. To show that this is natural, let $\beta:F \to G$ be a natural transformation of functors. We must show that

$$\begin{array}{ccc} F(K) & \xrightarrow{\beta K} & G(K) \\ {\scriptstyle \alpha F}\downarrow & & \downarrow{\scriptstyle \alpha G} \\ M(F) & \xrightarrow[M\beta]{} & M(G) \end{array}$$

commutes. Let

$$k:\mathrm{Hom}(K, -) \to F.$$

Then

$$\alpha G \circ \beta K(k) = \alpha G(\beta K(k)) = M(\beta K(k))(y)$$

$$= M(\beta \circ k)(y) = M\beta \circ Mk(y) = M\beta \circ \alpha F(k).$$

The fact that $\beta K(k) = \beta \circ k$ is just part of the identification of elements of a functor with a natural transformation. Thus αF is the component at F of a natural transformation $\alpha : K^* \to M$. Since

$$\alpha E(k_0) = M(k_0)(y) = x,$$

we have verified the necessary condition.

COROLLARY 3.21. *A presheaf category has enough principal prime models.*

Proof. This is immediate from 3.19 and 3.20.

It is easy to see that a GT with enough principal models is prime-generated. Below, we will show (Corollary 3.25) that a GT with enough principal prime models has a continuous powerful embedding into a presheaf category. In fact, we formulate a condition, weaker than having enough principal prime models, which is necessary and sufficient for such an embedding to exist.

Let $\mathbf{M}$ be a subcategory of Mod($\mathscr{C}$). We say that the model M in $\mathbf{M}$ is *prime relative to* $\mathbf{M}$ if the condition defining "M is prime" above holds with M' and α constrained to lie in $\mathbf{M}$.

THEOREM 3.22. *A Grothendieck topos has a continuous powerful embedding into a presheaf category if and only if it has a sufficient category of principal models which are all prime relative to that category.*

Proof. Let $\mathscr{E}$ have a powerful embedding into a functor category $\mathbf{Set}^{\mathbf{K}}$. The evaluation functors K^* are principal models of $\mathbf{Set}^{\mathbf{K}}$ and are prime as models of that category. They are prime as models of $\mathscr{E}$ relative only to models of that category that extend to models of $\mathbf{Set}^{\mathbf{K}}$. In particular, they are prime relative to the class of all models of the form K^* and hence form a sufficient class of principal models prime relative to that class. This proves one direction.

Now suppose that $\mathscr{E}$ is a Grothendieck topos with a sufficient category $\mathbf{M}$ of principal models prime relative to that category. We have the evaluation functor

$$\mathrm{Ev}:\mathscr{E} \to \mathbf{Set}^{\mathbf{M}}.$$

It is continuous since all objects of $\mathbf{M}$ are continuous functors. Since the class Ob($\mathbf{M}$) of models is sufficient, Ev is conservative, hence also faithful. Next we wish to show that Ev is surjective on subobject lattices. Let $\Gamma \subseteq \mathrm{Ev}(A)$. This means that for each model M of $\mathbf{M}$, we have the subset $\Gamma(M) \subseteq MA$ and that when $\alpha : M \to N$ is a natural transformation in $\mathbf{M}$,

$$\alpha A(\Gamma(M)) \subseteq \Gamma(N).$$

Form the subobject of A,

$$C = \bigvee\{B \in \mathrm{Sub}(A) \mid M(B) \subseteq \Gamma(M) \text{ for all } M \text{ in } \mathbf{M}\}.$$

It is clear from the fact that models preserve arbitrary sups that $M(C) \subseteq \Gamma(M)$ for all M in $\mathbf{M}$. So we want to show the converse. We require:

PROPOSITION 3.23. *Let $P \subseteq A$ be a prime subobject and M a model in* $\mathbf{M}$ *that contains a generic element $p \in M(P)$ for P. Then $P \subseteq C$ if and only if $p \in \Gamma(M)$.*

Proof. The necessity is obvious. Suppose $p \in \Gamma(M)$; we wish to show that $P \leqq C$ in $\mathrm{Sub}(A)$. By the definition of C, it suffices to show that $N(P) \subseteq \Gamma(N)$ for all N in $\mathbf{M}$. So let N be a model in $\mathbf{M}$ and $a \in N(P)$. Since M is prime relative to $\mathbf{M}$ and p is generic for P, there is a natural transformation $\alpha:M \to N$ with $\alpha P(p) = a$. By the naturality of the inclusion Γ in $\mathrm{Ev}(A)$ and since $p \in \Gamma(M)$, we have $a \in \Gamma(N)$ as desired.

We now return to the proof of the theorem. Let p be an arbitrary element of $\Gamma(M)$ and let P generate $\mathbf{t}(p)$. Then clearly p is generic for P and it follows from the preceding that $P \leqq C$, whence $p \in M(C)$.

We have shown that $\mathrm{Ev}(C) = \Gamma$, proving the surjectivity of Ev on subobjects.

The proof of 3.22 gives

PROPOSITION 3.24. *If* $\mathbf{K}$ *is a sufficient category of principal models of a* GT $\mathscr{E}$, *all prime relative to* $\mathbf{K}$, *then the evaluation functor*

$$\mathrm{Ev}:\mathscr{E} \to \mathbf{Set}^{\mathbf{K}}$$

is a powerful embedding.

COROLLARY 3.25. *Any Grothendieck topos with enough principal prime models has a powerful continuous embedding into a presheaf category. In particular, any separable prime-generated Grothendieck topos, and the topos of sheaves over any regular epimorphism site have such embeddings. In the separable prime-generated case, the presheaf category can be chosen to be* $\mathbf{Set}^{\mathbf{K}}$ *with* $\mathbf{K}$ *a countable category.*

Proof. If $\mathscr{E}$ has enough principal prime models, then the full subcategory $\mathbf{K}$ of $\mathrm{Mod}(\mathscr{E})$ with objects the principal prime models will be sufficient, and every model in $\mathbf{K}$ will be prime relative to $\mathbf{K}$. Therefore, $\mathscr{E}$ has a powerful continuous embedding into $\mathbf{Set}^{\mathbf{K}}$ by 3.24. The particular cases follow from 3.16 and 3.18. It is clear that, in the separable prime-generated case, we may choose $\mathbf{K}$ (a non full subcategory of $\mathrm{Mod}(\mathscr{E})$) to be countable.

Question. Does every prime-generated coherent GT have a powerful continuous embedding into a presheaf category?

In **[14]** and **[16]**, the weaker result is shown that such a GT has continuous full embedding into a presheaf category.

PROPOSITION 3.26. *Suppose the* GT $\mathscr{E}$ *has enough principal prime models. Then the evaluation functor*

$$e:\mathscr{E} \to (\text{Mod } \mathscr{E}, \mathbf{Set})$$

is continuous, full and faithful.

Proof. Only the fullness requires proof. Suppose

$$g:eA \to eB$$

is a morphism in (Mod $\mathscr{E}$, **Set**). Consider the graph of $g:\Phi \in \text{Sub}(e(A \times B))$. With **K** the full subcategory of Mod $\mathscr{E}$ consisting of the principal prime models, we have that the composite

$$e':\mathscr{E} \to (\text{Mod } \mathscr{E}, \mathbf{Set}) \to (\mathbf{K}, \mathbf{Set}),$$

with the second arrow induced by the inclusion of **K** into mod $\mathscr{E}$, is powerful; namely, e' is the same as the evaluation functor Ev considered in Proposition 3.24. Therefore, there is $X \in \text{Sub}(A \times B)$ such that

$$M(X) = (e'X)M = \Phi(M) \quad \text{for all } M \in \mathbf{K}.$$

Also, since e' is conservative, X is the graph of some $f:A \to B$ and $M(f) = g_M$ for all $M \in \mathbf{K}$.

To see that $N(f) = g_N$ for all $N \in \text{Mod}(\mathscr{E})$ (which will complete the proof), it suffices to see that if $g, g':eA \to eB$ coincide on all M in $\mathbf{K}:g_M = g'_M$, then they coincide (on all N in Mod $\mathscr{E}$). But this follows from the fact that "maps from elements of **K** to any given N in Mod($\mathscr{E}$) are jointly surjective". More precisely, given any N in Mod ($\mathscr{E}$) and $a \in N(A)$, there are $M \in \mathbf{K}$, $a \in M(A)$, and an $h:M \to N$ such that

$$h_A(a') = a.$$

(Namely, A is the **J**-union of some prime subobjects; so there is $X \in \text{Sub}(A)$, X prime, such that $a \in N(X)$; let $M \in \mathbf{K}$ be such that there is a generic element $a' \in M(X)$; then since M is prime, there is h as required.) To show that $g_N = g'_N$, consider the diagram of sets

$$\begin{array}{ccc} M(A) & \xrightarrow{h_A} & N(A) \\ g_M \downarrow\downarrow g'_M & & g_N \downarrow\downarrow g'_N \\ M(B) & \xrightarrow[h_B]{} & N(B) \end{array}$$

and the corresponding diagram of elements of those respective sets

$$\begin{array}{ccc} a' & \longmapsto & a = h_A(a') \\ \Big\downarrow\Big\downarrow & & \Big\downarrow\Big\downarrow \\ g_M(a') = g'_M(a') & \longmapsto & g_N(a) = g'_N(a) \end{array}$$

from which we conclude that $g_N(a) = g'_N(a)$ for all $a \in N(F)$.

Remark. It is easy to see by an example that e is not powerful in general.

We now turn to a discussion of atomic toposes.

Let $\mathscr{C} = (\mathbf{C}, \mathbf{J})$ be a regular atomic site. Then every model of $\mathscr{C}$ is clearly principal; every object is a $\mathbf{J}$-union of atoms and the model preserves $\mathbf{J}$-unions so the type of each element is generated by a unique atom in the respective subobject lattice. In particular, the type of the unique element of $M(1)$, for a model M is a unique atom of Sub(1), which we will call the *type of M* and denote $\mathbf{t}(M)$. Clearly, if $\mathbf{t}(M) \neq \mathbf{t}(M')$, then there is no morphism $M \rightarrow M'$.

Let us now suppose, in addition, that $\mathscr{C}$ (and hence $\mathscr{E}$) is separable. It follows from Theorem 3.17 that two countable models are isomorphic if and only if they have the same type. Moreover, the category of countable models and the isomorphisms between them satisfies the hypotheses of Proposition 3.24 and we have that $\mathscr{E}$ has a powerful embedding into a category of G-**Set** for a countable groupoid G. In case $\mathscr{E}$ is connected, we can take G to be a group. Note the easily verified fact that a GT having a powerful continuous embedding into G-**Set** is equivalent to $\hat{G}$-**Set**, the category of continuous $\hat{G}$-actions (on discrete sets) for a topological group $\hat{G}$ with underlying discrete group G.

COROLLARY 3.27. *A separable atomic* GT *has a continuous powerful embedding into G-***Set** *for a countable groupoid G. A connected separable atomic* GT *is equivalent to $\hat{G}$-***Set** *for a countable topological group $\hat{G}$.*

This conclusion is well known; it was stated explicitly in [**16**] as Theorem 3.8. Let us note that a similar characterization holds for coherent atomic toposes; see Corollary 3.3 in [**16**]. A related theorem, actually a generalization, appears in [**12**], referring to an arbitrary connected topos with a point and characterizing such a topos as the category of "discrete G-spaces for an (open) spatial group G" (here, however, the "spatial group" means something more general than a topological group).

THEOREM 3.28. *A Boolean Grothendieck topos with enough points is atomic.*

Proof. Let $\mathscr{E}$ be such a topos. Then we have a set $\{M_i\}$ of models of $\mathscr{E}$ which are collectively faithful. Each M_i induces a morphism

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$$\text{Hom}(E, 2) \rightarrow \text{Hom}(M_i E, M_i 2) = \text{Hom}(M_i E, 2)$$

and these morphisms preserve union since M preserves colimits. But they are also Boolean homomorphisms and hence preserve all the infinite Boolean operations. They are also collectively faithful, so that we have an embedding

$$\text{Hom}(E, 2) \rightarrow \prod \text{Hom}(M_i E, 2).$$

But the latter is a complete atomic Boolean algebra and it is standard (and easy) to see that a complete subalgebra of a complete atomic Boolean algebra is again one.

THEOREM 3.29. *Let $\mathcal{E}$ be a separable Grothendieck topos. The following are equivalent*:

(i) *$\mathcal{E}$ is connected atomic.*

(ii) *$\mathcal{E}$ is connected and Boolean.*

(iii) *$\mathcal{E}$ has exactly two distinct subtoposes.*

(iv) *$\mathcal{E}$ has exactly one countable model up to isomorphism.*

In fact, the implications (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) *are true without the separability assumption.*

Proof. (i) $\Rightarrow$ (ii) is clear. To see (ii) $\Rightarrow$ (iii), note the following consequence of Giraud's theorem for Grothendieck toposes. Any subtopos is the category of sheaves over $\mathcal{E}$ with a topology extending the canonical topology, and any such topology is generated by the canonical covers plus covers each consisting of monomorphisms; if $\mathcal{E}$ is Boolean, then the latter monomorphisms can all be taken to have codomain 1. So, if $\mathcal{E}$ is connected, the subtopos is determined by whether 1 is covered by 0 or not.

Conversely, for every non-degenerate GT $\mathcal{E}$, $\mathcal{E}_{\neg\neg}$ is a non-degenerate subtopos of $\mathcal{E}$. So, if (iii) holds, we must have $\mathcal{E} = \mathcal{E}_{\neg\neg}$, hence $\mathcal{E}$ is Boolean.

In the case $\mathcal{E}$ is separable, it has enough points, and therefore (ii) implies (i) by 3.28.

Assume (iv). Let $(\mathbf{C}, \mathbf{J})$ be a separable regular defining site for $\mathcal{E}$, and let $\epsilon:(\mathbf{C}, \mathbf{J}) \rightarrow \mathcal{E}$ be the canonical functor. Suppose $\{X_s\}$, $s \in S$ is a $\mathbf{J}$-dense family of subobjects of A in $\mathbf{C}$; i.e., if $Y \in \text{Sub}(A)$, and $Y \cap S$ is empty for all $s \in S$, then Y is empty. We claim that

$$\bigvee^{(\mathbf{J})}_{s \in S} X_S = A.$$

If not, there is a model M of $(\mathbf{C}, \mathbf{J})$ in which $\bigvee_{s \in S} M(X_S)$ is a proper subobject of $M(A)$. On the other hand, consider the topology $\mathbf{J}'$ on $\mathbf{C}$ generated by $\mathbf{J}$ together with the single family

$$\{X_s \rightarrowtail A\}, \quad s \in S.$$

The site $(\mathbf{C}, \mathbf{J}')$ is non-degenerate (1 is not empty) since the composite

$$\mathbf{C} \xrightarrow{\epsilon} \mathscr{E} \to \mathscr{E}_{\neg\neg}$$

is $\mathbf{J}'$-continuous, and $\mathscr{E}_{\neg\neg}$ is non-degenerate. Also, $(\mathbf{C}, \mathbf{J}')$ is separable, so there is a countable model M' of $(\mathbf{C}, \mathbf{J}')$; this is a model of $(\mathbf{C}, \mathbf{J})$ for which

$$\vee_{s \in S} M'(X_s) = M'(A),$$

hence M and M' are not isomorphic, contrary to (iv).

What we have shown is that for any object A of $\mathbf{C}$, the subobject lattice of ϵA is a Boolean algebra. Since every object of $\mathscr{E}$ has a cover by objects of the form ϵA, it is easy to see that it follows that $\mathrm{Sub}(E)$ is a Boolean algebra for every object E of $\mathscr{E}$, i.e., $\mathscr{E}$ is Boolean.

It is easy to see that it also follows that $\mathscr{E}$ is connected. We have shown that (iv) implies (ii).

Peter Freyd has pointed out an example of a connected Boolean topos which is not atomic (equivalently, it does not have a point). Let C be the monoid freely generated by generators x and y, and let $\mathscr{E}$ be $(\mathbf{Set}^C)_{\neg\neg}$. $\mathscr{E}$ is connected Boolean. Consider the geometric morphism $p:\mathscr{E} \to \mathbf{Set}^C$:

$$\mathscr{E} \underset{p_*}{\overset{p^*}{\rightleftarrows}} \mathbf{Set}^C$$

in which p_* is the inclusion. For the unit

$$\eta:\mathrm{id}(\mathbf{Set}^C) \to p_* \circ p^*$$

of the adjunction $p^* \dashv p_*$ and for a C-set X,

$$\eta X: X \to p_* p^* X$$

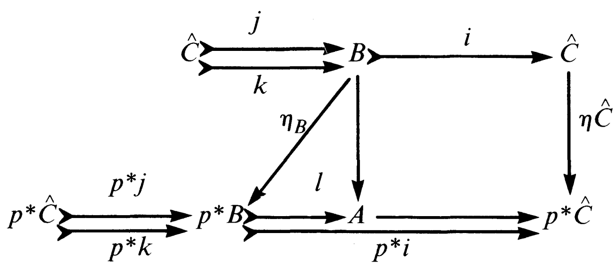
is a dense map (a covering in the $\neg\neg$-topology).

Let $\hat{C}$ be the C-set whose underlying set is C, and in which the action of an element c of C is multiplication by c on the left. $p^*\hat{C}$ is a non-zero object of $\mathscr{E}$; we claim then there is no subobject of $p^*\hat{C}$ which is an atom.

Note that for any $w \in C$, the sub-C-set of $\hat{C}$ generated by w:$\{w'w | w' \in C\}$, is isomorphic to $\hat{C}$ itself, by the map

$$w' \mapsto w'w.$$

It follows that any non-empty sub-C-set of $\hat{C}$ contains a sub-C-set which, as a C-set, is isomorphic to $\hat{C}$ itself. Also, the sub-C-sets of $\hat{C}$ generated by x and y are disjoint and isomorphic to $\hat{C}$ itself; so, for any non-empty sub-C-set B of $\hat{C}$, we can find two monomorphisms $\hat{C} \rightrightarrows B$ with an empty equalizer. Let A be any subobject of $p^*\hat{C}$, and consider the following diagram in $\mathbf{Set}^C$.



in which B is defined so that the right hand square is a pullback. B is non-empty since $\eta_{\hat{C}}$ is dense. $l:p^*B \to A$ exists because of the adjunction $p^* \dashv p_*$, and it makes the two triangles in which it takes part commutative for the same reason. It follows that l is a monomorphism. The left exact functor p^* produces two disjoint non-empty subobjects (represented by $l \circ (p^*j)$ and $l \circ (p^*k)$) of A, hence A is not an atom, as claimed.

4. A conceptual completeness result. In this section, $\mathrm{Mod}_\sigma(\mathscr{C})$ denotes the category of countable models of $\mathscr{C}$ and $\mathrm{Mod}_\sigma(\mathscr{C})^{\cong}$ denotes the category of countable models with isomorphisms as morphisms. Any continuous $I:\mathscr{B} \to \mathscr{C}$ induces a functor

$$I^{\cong}:\mathrm{Mod}_\sigma(\mathscr{C})^{\cong} \to \mathrm{Mod}_\sigma(\mathscr{B})^{\cong}.$$

It also induces a geometric morphism

$$\mathbf{Sh}(I):\mathbf{Sh}(\mathscr{B}) \to \mathbf{Sh}(\mathscr{C})$$

that has a right-adjoint part $(\mathbf{Sh}(I))_*$ as well as a left-adjoint part.

THEOREM 4.1. *Suppose $I:\mathscr{B} \to \mathscr{C}$ is a continuous functor between separable Boolean sites such that the induced functor*

$$I^{\cong}:\mathrm{Mod}_\sigma(\mathscr{C})^{\cong} \to \mathrm{Mod}_\sigma(\mathscr{B})^{\cong}$$

on countable models and isomorphisms is full and faithful. Then $\mathbf{Sh}(I)$ *is an inclusion of Grothendieck toposes, i.e.,* $(\mathbf{Sh}(I))_*$ *is full and faithful.*

COROLLARY 4.2. *Suppose $p:\mathscr{F} \to \mathscr{E}$ is a separable geometric morphism between separable Boolean (that is atomic) toposes which induces a full inclusion between the categories of points. Then the right adjoint p_* is full and faithful.*

Proof of 4.2 *from* 4.1. We remarked in Section 1 that countable models of $\mathscr{C}$ and those of $\mathbf{Sh}(\mathscr{C})$ are essentially the same, for a separable site $\mathscr{C}$. Therefore, the corollary follows from the theorem and Proposition 1.10.

COROLLARY 4.3. *In both* 4.1 *and* 4.2, *if the functor assumed to be full and faithful is actually an equivalence, then the geometric morphism asserted to be an inclusion is in fact an equivalence.*

Proof. The conservativeness of $(\mathbf{Sh}(I))^*$ follows from $I^{\cong}$ being essentially surjective on objects; if p is an inclusion such that p^* is conservative then p is an equivalence.

The theorem and the corollaries are conceptual completeness results. The first conceptual completeness result was proved in [**18**]; see Theorems 7.1.9, 9.2.9 and 9.2.10 for various formulations for the same result concerning coherent toposes. (Theorem 4.1 in the present paper neither implies, nor is implied by the result just mentioned.)

Theorem 4.1 with the added hypothesis that the sites involved should also be coherent was proved and used in [**15**]. It was pointed out in the same paper that the "coherent" special case of Theorem 4.1 is essentially equivalent to an unpublished theorem of Haim Gaifman (stated without category theory). The proof of Theorem 4.1 given below is, despite appearances, quite similar to the one given for the coherent case in [**15**].

The proof of 4.1 will be accomplished by the following two propositions.

PROPOSITION 4.4. *Under the hypothesis of* 4.1, *in fact with* $I^{\cong}$ *assumed only to be full,* $(\mathbf{Sh}(I))^*$ *is full on subobjects* (*see* 1.9).

The proposition is contained in Theorem 7.3.3 in [**18**]. The proof of 7.3.3, using the Craig interpolation theorem for $L_{\omega_1\omega}$, is only sketched there, but similar proofs are given in more detail.

PROPOSITION 4.5. *Under the hypotheses of* 4.1, $\mathbf{Sh}(I)$ *is localic* (*see* 1.9).

Before we turn to the proof of 4.5, let us point out that 4.1 follows from 4.4 and 4.5 by 1.9.

Proof of Proposition 4.5. Assume the hypotheses of 4.5. Let us fix a countable model M of $\mathscr{C}$. We have the "diagram" $\mathscr{D}$ of M as constructed in Section 2, especially Propositions 2.4 and 2.5, with the accompanying data

$$\mathscr{C} \xleftarrow{F} \mathscr{D} \xrightarrow{\hat{M}} \mathbf{Set}$$

Henceforth, we write N for $\hat{M}$.

PROPOSITION 4.6. *N is a principal model of $\mathscr{D}$ (see Section 3 for the notion of "principal model").*

Proof. Since $\mathscr{D} = (\mathbf{D}, \mathbf{L})$ is Boolean, N being principal is the same as its being atomic (see before 3.17). Suppose the conclusion fails, and D is an object of $\mathscr{D}$, and $d \in \hat{M}(D)$ is such that $\mathbf{t}(d)$ is not principal. $\mathbf{t}(d)$ is an $\mathbf{L}$-prime filter, hence, in particular, an ultrafilter in the Boolean algebra $\mathrm{Sub}(D)$. The complement $\neg\mathbf{t}(d)$ of $\mathbf{t}(d)$ in $\mathrm{Sub}(D)$ is ($\mathbf{L}$)-dense: if $X \in \mathrm{Sub}(D)$, and $X \wedge Y = 0$ for all $Y \in \neg\mathbf{t}(d)$, then $X \leqq Z$ for all

$Z \in \mathbf{t}(d)$; since $\mathbf{t}(d)$ is not principal, $X \notin \mathbf{t}(d)$; hence $\neg X$, the complement is in $\mathbf{t}(d)$, and so $X \leqq \neg X$, hence $X = 0$ as desired.

Next we claim that for any object B of $\mathscr{B}$, the set of global sections ΦB of FIB is dense. For if D is a subobject of FIB that misses every global section, then, since every element of $NFIB$ is determined by a global section (see 2.1 (ii)), it follows that $ND = 0$. But in a 2-valued category (see 2.4 (ix)), either $D \twoheadrightarrow 1$ in which case $ND \neq 0$ or $D = 0$.

By Proposition 1.8, there is a model P of $\mathscr{D}$ such that P takes $\neg\mathbf{t}(d)$ and each ΦB into surjective families. Since P takes the complement of $\mathbf{t}(d)$ to a cover, there can be no element of PD whose type is $\mathbf{t}(d)$, so that P cannot be isomorphic to N.

Using the hypotheses of the proposition, and the other properties of P, we will deduce that P is isomorphic to N; the resulting contradiction will establish the claim.

Consider the functor G defined in the proof of Proposition 2.5, and let us consider $G(P) = \langle PF, \alpha \rangle$ with

$$\alpha : NFI \to PFI$$

defined there. Because of the definition of α and because P takes each ΦB into a surjective family, each component of α is a surjection; by the two-valuedness of $\mathscr{D}$, it must be injective. We have that α is an isomorphism. Since $I^{\cong}$ is full, there is an isomorphism

$$\beta : NF \to PF$$

such that $\beta_I = \alpha$. Notice that the last fact means that β represents a morphism from the object

$$GN = \langle NF, \mathrm{id}: NFI \to NFI)$$

of the category $\mathbf{A}$ of 2.5 to the object GP. Thus, since G is full and faithful (see 2.5), there is a (unique) isomorphism $g:N \to P$ (for which $G(g) = p$) as promised.

Proposition 4.7. *The only automorphism of N is the identity.*

Proof. Let α be an automorphism of N. For an object B of $\mathscr{B}$, it is evident from 2.1 (ii) that αFIB is the identity. Thus αFI is the identity automorphism. The faithfulness of $\mathbf{Sh}(I)_*$ guarantees that αF is the identity. Finally, we conclude from the diagram (see 2.1 (iii))

$$\begin{array}{ccc} ND & \rightarrowtail & NFC \\ {\scriptstyle \alpha D}\downarrow & & \downarrow{\scriptstyle \alpha FC} \\ ND & \rightarrowtail & NFC \end{array}$$

that α is the identity on all of $\mathscr{D}$.

https://doi.org/10.4153/CJM-1987-009-x Published online by Cambridge University Press

PROPOSITION 4.8. *The type of every element of ND has as underlying object the object* 1.

Proof. Let $d \in ND$. Its type is principal, hence generated by an atom $A \subseteq D$. If NA has another element d', there is, by 3.17, an automorphism of N that maps d on d'. Thus the map $A \twoheadrightarrow 1$ induces an isomorphism $NA \to N1$. But this means that if B is the complement of the diagonal of $A \to A \times A$, then $NB = 0$. But the site is 2-valued, so either $B = 0$, whence $A = 1$, or B has global support, a contradiction.

PROPOSITION 4.9. *N is representable by* 1.

Given any morphism $f:X \to Y$ in $\mathscr{C}$, there is a largest subobject of $\Phi f \subseteq X$ with the following property: if in the diagram

$$Z \underset{h}{\overset{g}{\rightrightarrows}} X \xrightarrow{\ f\ } Y$$

g factors through Φf and $f \circ g = f \circ h$, then $g = h$.

Φf exists because **C** is Boolean. Using the "internal logic" of **C**, we can express Φ as follows:

$$\Phi f = [x \in X | \forall x' \in X, fx = fx' \Rightarrow x = x'].$$

A more direct description of Φf is as the universal image under the first projection of $\mathrm{kerp}(f) \Rightarrow \mathrm{diag}(X)$.

PROPOSITION 4.10. *For each object C of $\mathscr{C}$, the set of all $C_i \to C$, for which C_i can be embedded into an object of the form IB, covers C.*

Proof. If not, there will be a model M for which $\{MC_i \to MC\}$ fails to cover. Now let D be the "diagram" of M; let $F:\mathscr{C} \to \mathscr{D}$ and $N:\mathscr{D} \to$ **Set** have the properties described in Section 2 and 4.6-4.9. We may replace M by the isomorphic model NF (see 2.1 (i)); that is, we may suppose that $M = NF$. Let $c \in MC$ an element not in the image of the $\{MC_i\}$. For the morphism $c^{\#}:1 \rightarrowtail FC$ picking out c (see 4.8), there are objects C' of $\mathscr{C}$ and B of $\mathscr{B}$, morphism

$$\langle f, g\rangle : C' \to C \times IB$$

and elements $c' \in MC'$ and $b \in MIB$ such that

$$\begin{array}{ccc} 1 & \xrightarrow{\ c^{\#}\ } & FC \\ {\scriptstyle c'}\downarrow & & \downarrow{\scriptstyle id \times b^{\#}} \\ FC' & \xrightarrow{\ F\langle f, g\rangle\ } & FC \times FIB \end{array}$$

is a pullback. It follows that

$$\begin{array}{ccc} 1 & \rightarrowtail & 1 \\ {\scriptstyle c'}\downarrow & & \downarrow{\scriptstyle b^{\#}} \\ FC' & \xrightarrow[Fg]{} & FIB \end{array}$$

is also a pullback. But then it follows that for any arrow $h:D \to FC'$, if

$$Fg \circ h = Fg \circ c' \circ \langle\ \ \rangle$$

($\langle\ \ \rangle$ represents the terminal map of any object; it is the empty tuple), then

$$Fg \circ h = Fg \circ c' \circ \langle\ \ \rangle = b^{\#} \circ \langle\ \ \rangle,$$

and the universal mapping property of the pullback gives a unique map, necessarily $\langle\ \ \rangle:D \to 1$ such that $h = c' \circ \langle\ \ \rangle$. This shows that

$$c' \subseteq \Phi(Fg) = F(\Phi g),$$

the latter equality because a regular functor between Boolean regular categories preserves the construction of Φ. Thus we see that the composite arrow

$$\Phi g \to C' \to C$$

is a map from a subobject of IB to C whose image under M contains c, a contradiction.

This completes the proofs of Proposition 4.5 and of Theorem 4.1.

The following will be stated without proof.

THEOREM 4.11. *Suppose* $I:\mathscr{B} \to \mathscr{C}$ *is a continuous functor between Boolean* (*not necessarily separable*) *sites. Suppose that for all complete Boolean algebras* **B**, *the induced functor*

$$I^*:\mathrm{Con}(\mathscr{C}, \mathbf{Sh}(\mathbf{B})) \to \mathrm{Con}(\mathscr{B}, \mathbf{Sh}(\mathbf{B}))$$

is full and faithful. Then $(\mathbf{Sh}(I))_*$ *is full and faithful.*

COROLLARY 4.12. *Suppose* $p:\mathscr{F} \to \mathscr{E}$ *is a geometric morphism between Boolean* GT*s which induces a full inclusion between the categories of* **B**-*valued points, for all complete Boolean algebras* **B**. *Then* p_* *is full and faithful.*

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