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Homological full-and-faithfulness of comodule inclusion and contramodule forgetful functors

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Abstract
In this paper, we consider a conilpotent coalgebra C over a field k. Let $\Upsilon \colon C-\mathsf{Comod} \longrightarrow C^*-\mathsf{Mod}$ be the natural functor of inclusion of the category of C-comodules into the category of C^*-modules, and let $\Theta \colon C-\mathsf{Contra} \longrightarrow C^*-\mathsf{Mod}$ be the natural forgetful functor. We prove that the functor Υ induces a fully faithful triangulated functor on bounded (below) derived categories if and only if the functor Θ induces a fully faithful triangulated functor on bounded (above) derived categories, and if and only if the k-vector space $\mathrm{Ext}^n_C(k, k)$ is finite-dimensional for all $n \geq 0$. We call such coalgebras "weakly finitely Koszul".

1. Introduction

In this paper, we work with coassociative, counital coalgebras over a field k. For any such coalgebra C, the dual k-vector space C^* is naturally an associative, unital algebra over k. One has to choose between two opposite ways of defining the multiplication on C^*. We prefer the notation in which any left C-comodule becomes a left C^*-module, and any right C-comodule becomes a right C^*-module.

It is known, at least, since 1960s that the resulting exact functor $\Upsilon \colon C-\mathsf{Comod} \longrightarrow C^*-\mathsf{Mod}$ is fully faithful [44, Section 2.1]. Following the terminology of the book [44], C^*-modules coming from C-comodules are often called "rational" in the literature. The essential image of the functor Υ is a *hereditary pretorsion class* in $C^*-\mathsf{Mod}$: this means that the full subcategory $\Upsilon(C-\mathsf{Comod})$ is closed under subobjects, quotient objects, and infinite coproducts in $C^*-\mathsf{Mod}$. However, this full subcategory *need not* be closed under extensions. In other words, $\Upsilon(C-\mathsf{Comod})$ is *not* always a torsion class or a Serre subcategory in $C^*-\mathsf{Mod}$. A module extension of two comodules need not be a comodule.

When is the essential image of Υ closed under extensions in $C^*-\mathsf{Mod}$? There is a vast body of literature on this topic, including the papers [3, 4, 10, 15, 41, 43, 46]. In this paper, we discuss further questions going in this direction, under an additional assumption. The assumption is that the coalgebra C is *conilpotent*. The conilpotent coalgebras were called "pointed irreducible" in the terminology of [44]. For a conilpotent coalgebra C, the full subcategory $C-\mathsf{Comod}$ is closed under extensions in $C^*-\mathsf{Mod}$ if and only if the coalgebra C is *finitely cogenerated* [41, Corollary 2.4 and Section 2.5], [43, Theorem 4.6], [15, Corollary 21], [4, Theorem 2.8], [3, Proposition 3.13 and Corollary 3.14], [10, Lemma 1.2 and Theorem 4.8]. If C is not finitely cogenerated, then there is a two-dimensional C^*-module which is not a C-comodule, but an extension of two one-dimensional C-comodules.

Let A and B be two abelian categories, and $\Phi \colon \mathsf{B} \longrightarrow \mathsf{A}$ be a fully faithful exact functor. Then the essential image of Φ is a full subcategory closed under kernels and cokernels in A. The full subcategory $\Phi(\mathsf{B})$ is closed under extensions in A if and only if the functor Φ induces an isomorphism

$$\Phi \colon \mathrm{Ext}^1_\mathsf{B}(X, Y) \longrightarrow \mathrm{Ext}^1_\mathsf{A}(\Phi(X), \Phi(Y)) \tag{1}$$

for all objects X, $Y \in \mathsf{B}$. Generally speaking, for a fully faithful exact functor Φ, the map (1) is a monomorphism, but not necessarily an isomorphism.

Thus, the following question is a natural extension of the question about extension closedness of $\Upsilon(C-\mathsf{Comod})$ in $C^*-\mathsf{Mod}$. Put $\mathsf{B} = C-\mathsf{Comod}$, $\mathsf{A} = C^*-\mathsf{Mod}$, and $\Phi = \Upsilon$. Consider the induced maps on the Ext spaces

$$\Phi \colon \mathrm{Ext}^i_\mathsf{B}(X, Y) \longrightarrow \mathrm{Ext}^i_\mathsf{A}(\Phi(X), \Phi(Y)). \qquad (2)$$

When is the map (2) in isomorphism for all X, $Y \in \mathsf{B}$? Generally speaking, for an exact functor of abelian categories $\Phi \colon \mathsf{B} \longrightarrow \mathsf{A}$, all one can say is that the map (2) is a monomorphism for $i = n + 1$ and all X, $Y \in \mathsf{B}$ whenever it is an isomorphism for $i = n$ and all X, $Y \in \mathsf{B}$.

Let C be a conilpotent coalgebra over a field k. Then the one-dimensional k-vector space k has a unique left C-comodule structure (and a unique right C-comodule structure) provided by the unique coaugmentation of C. In this context, we show that the maps (2) are isomorphisms for $\Phi = \Upsilon$ and all $1 \le i \le n$ if and only if the k-vector spaces $\mathrm{Ext}^i_C(k, k)$ (computed in the abelian category of left or right C-comodules) are finite-dimensional for all $1 \le i \le n$. In particular, $\mathrm{Ext}^1_C(k, k)$ is the vector space of cogenerators of a conilpotent coalgebra C; so C is finitely cogenerated if and only if $\mathrm{Ext}^1_C(k, k)$ is finite-dimensional.

Alongside with the abelian categories of left and right *comodules* over a coalgebra C, there are much less familiar, but no less natural abelian categories of left and right *C-contramodules* [24]. Endowing the dual vector space C^* to a coalgebra C with the natural algebra structure in which any left C-comodule is a left C^*-module and any right C-comodule is a right C^*-module, one also obtains a natural left C^*-module structure on any left C-contramodule. So there is an exact forgetful functor $\Theta \colon C-\mathsf{Contra} \longrightarrow C^*-\mathsf{Mod}$.

The functor Θ is *not* in general fully faithful. It was shown in the paper [30, Theorem 2.1] that that the functor Θ is fully faithful for any finitely cogenerated conilpotent coalgebra C. In this paper, we demonstrate a counterexample proving the (much easier) converse implication: if a conilpotent coalgebra C is *not* finitely cogenerated, then the functor Θ is *not* fully faithful.

More generally, the maps (2) are isomorphisms for $\Phi = \Theta$ and all $0 \le i \le n - 1$ (for separated contramodules Y; and also for $0 \le i \le n - 2$ and arbitrary Y) if and only if the k-vector spaces $\mathrm{Ext}^i_C(k, k)$ are finite-dimensional for all $1 \le i \le n$. Summarizing the assertions for comodules and contramodules, let us state the following theorem.

Theorem 1.1. *Let C be a conilpotent coalgebra over a field k and $n \ge 1$ be an integer. Then the following five conditions are equivalent:*

(i) *the map*

$$\mathrm{Ext}^i_C(L, M) \longrightarrow \mathrm{Ext}^i_{C^*}(L, M)$$

induced by the inclusion functor $\Upsilon \colon C-\mathsf{Comod} \longrightarrow C^*-\mathsf{Mod}$ is an isomorphism for all left C-comodules L and M, and all $1 \le i \le n$;

(ii) *the map*

$$\mathrm{Ext}^i_{C^\mathrm{op}}(L, M) \longrightarrow \mathrm{Ext}^i_{C^{*\mathrm{op}}}(L, M)$$

induced by the inclusion functor $\mathsf{Comod}-C \longrightarrow \mathsf{Mod}-C^*$ is an isomorphism for all right C-comodules L and M, and all $1 \le i \le n$;

(iii) *the map*

$$\mathrm{Ext}^{C,i}(P, Q) \longrightarrow \mathrm{Ext}^i_{C^*}(P, Q) \qquad (3)$$

induced by the forgetful functor $\Theta \colon C-\mathsf{Contra} \longrightarrow C^*-\mathsf{Mod}$ is an isomorphism for all left C-contramodules P, all separated left C-contramodules Q, and all $0 \le i \le n - 1$;

(iv) *the k-vector space $\mathrm{Ext}^i_C(k, k)$ is finite-dimensional for all $1 \le i \le n$;*
(v) *the k-vector space $\mathrm{Ext}^{C,i}(k, k)$ is finite-dimensional for all $1 \le i \le n$.*

If any one of the equivalent conditions (i–v) holds, then the map (3) is an isomorphism for all left C-contramodules P and Q and all $0 \le i \le n - 2$.

Notice a curious cohomological dimension shift in comparison between the assertions about the comodule and contramodule inclusion/forgetful functors. Let $n \geq 1$ be the minimal integer for which the vector space $\mathrm{Ext}^n_C(k, k)$ is infinite-dimensional. Then the map $\mathrm{Ext}^n_C(k, k) \longrightarrow \mathrm{Ext}^n_{C*}(k, k)$ induced by the functor Υ is injective, but not surjective. In fact, the dimension cardinality of the vector space $\mathrm{Ext}^n_{C*}(k, k)$ is larger than that of $\mathrm{Ext}^n_C(k, k)$ in this case: $\mathrm{Ext}^n_{C*}(k, k)$ is as large as the double dual vector space $\mathrm{Ext}^n_C(k, k)^{**}$ to $\mathrm{Ext}^n_C(k, k)$.

At the same time, denoting by T an infinite-dimensional k-vector space endowed with the trivial C-contramodule structure, the map $\mathrm{Ext}^{C,n}(T, k) \longrightarrow \mathrm{Ext}^n_{C*}(T, k)$ induced by the functor Θ is *not injective* (for the integer n as in the previous paragraph). Consequently, there exists a projective C-contramodule P such that the map $\mathrm{Ext}^{C,n-1}(P, k) \longrightarrow \mathrm{Ext}^{n-1}_{C*}(P, k)$ is injective, but not surjective. For $n > 1$ this, of course, means that $\mathrm{Ext}^{n-1}_{C*}(P, k) \neq 0$, while $\mathrm{Ext}^{C,n-1}(P, k) = 0$ as it should be. Notice that both the contramodules P and k are separated.

Let us emphasize that *we do not know* whether the equivalent conditions (i–v) of Theorem 1.1 imply bijectivity of the maps (3) for arbitrary (nonseparated) contramodules Q and $i = n - 1$. This remains an open question.

Returning to the discussion of a fully faithful exact functor $\Phi \colon \mathsf{B} \longrightarrow \mathsf{A}$, it is clear that the maps of Ext groups (2) induced by Φ are isomorphisms if and only if the induced triangulated functor between the bounded derived categories $\Phi^{\mathrm{b}} \colon \mathsf{D}^{\mathrm{b}}(\mathsf{B}) \longrightarrow \mathsf{D}^{\mathrm{b}}(\mathsf{A})$ is fully faithful. Now assume that there are enough injective objects in the abelian category A and the functor $\Phi \colon \mathsf{B} \longrightarrow \mathsf{A}$ has a right adjoint. Then the functor Φ^{b} is fully faithful if and only if the similar functor between the bounded below derived categories $\Phi^+ \colon \mathsf{D}^+(\mathsf{B}) \longrightarrow \mathsf{D}^+(\mathsf{A})$ is fully faithful.

Dually, assume that there are enough projective objects in A and the functor $\Phi \colon \mathsf{B} \longrightarrow \mathsf{A}$ has a left adjoint. Then the functor Φ^{b} is fully faithful if and only if the similar functor between the bounded above derived categories $\Phi^- \colon \mathsf{D}^-(\mathsf{B}) \longrightarrow \mathsf{D}^-(\mathsf{A})$ is fully faithful [31, Proposition 6.5].

In the situation at hand, there are enough projective and injective objects in the categories of modules over associative rings. The comodule inclusion functor $\Upsilon \colon C-\mathsf{Comod} \longrightarrow C^*-\mathsf{Mod}$ has a right adjoint functor $\Gamma \colon C^*-\mathsf{Mod} \longrightarrow C-\mathsf{Comod}$, while the contramodule forgetful functor $\Theta \colon C-\mathsf{Contra} \longrightarrow C^*-\mathsf{Mod}$ has a left adjoint functor $\Delta \colon C^*-\mathsf{Mod} \longrightarrow C-\mathsf{Contra}$. Consequently, Theorem 1.1 implies the following result about full-and-faithfulness of induced triangulated functors.

Theorem 1.2. *For any conilpotent coalgebra C over a field k, the following eight conditions are equivalent:*

  (i) *the triangulated functor $\Upsilon^{\mathrm{b}} \colon \mathsf{D}^{\mathrm{b}}(C-\mathsf{Comod}) \longrightarrow \mathsf{D}^{\mathrm{b}}(C^*-\mathsf{Mod})$ induced by the comodule inclusion functor $\Upsilon \colon C-\mathsf{Comod} \longrightarrow C^*-\mathsf{Mod}$ is fully faithful;*

  (ii) *the triangulated functor $\Upsilon^+ \colon \mathsf{D}^+(C-\mathsf{Comod}) \longrightarrow \mathsf{D}^+(C^*-\mathsf{Mod})$ induced by the comodule inclusion functor Υ is fully faithful;*

  (iii) *the triangulated functor $\mathsf{D}^{\mathrm{b}}(\mathsf{Comod}-C) \longrightarrow \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}-C^*)$ induced by the comodule inclusion functor $\mathsf{Comod}-C \longrightarrow \mathsf{Mod}-C^*$ is fully faithful;*

  (iv) *the triangulated functor $\mathsf{D}^+(\mathsf{Comod}-C) \longrightarrow \mathsf{D}^+(\mathsf{Mod}-C^*)$ induced by the comodule inclusion functor is fully faithful;*

  (v) *the triangulated functor $\Theta^{\mathrm{b}} \colon \mathsf{D}^{\mathrm{b}}(C-\mathsf{Contra}) \longrightarrow \mathsf{D}^{\mathrm{b}}(C^*-\mathsf{Mod})$ induced by the contramodule forgetful functor $\Theta \colon C-\mathsf{Contra} \longrightarrow C^*-\mathsf{Mod}$ is fully faithful;*

  (vi) *the triangulated functor $\Theta^- \colon \mathsf{D}^-(C-\mathsf{Contra}) \longrightarrow \mathsf{D}^-(C^*-\mathsf{Mod})$ induced by the contramodule forgetful functor Θ is fully faithful;*

  (vii) *the k-vector space $\mathrm{Ext}^n_C(k, k)$ is finite-dimensional for all $n \geq 0$;*

  (viii) *the k-vector space $\mathrm{Ext}^{C,n}(k, k)$ is finite-dimensional for all $n \geq 0$.*

Let us discuss the condition of finite-dimensionality of the Ext spaces $\mathrm{Ext}^i_C(k, k)$ for all $i \geq 0$ in some detail. One can consider the special case when the coalgebra C is positively graded with finite-dimensional components; so C is the graded dual coalgebra to a positively graded algebra $A = \bigoplus_{m=0}^{\infty} A_m$

with $\dim A_m < \infty$ for all $m \geq 0$ and $A_0 = k$. Then one has $\mathrm{Ext}^i_C(k,k) \simeq \bigoplus_{j=i}^\infty \mathrm{Ext}^{i,j}_A(k,k)$ and $\mathrm{Ext}^i_A(k,k) \simeq \prod_{j=i}^\infty \mathrm{Ext}^{i,j}_A(k,k)$, where i is the usual cohomological grading on the Ext spaces, while the *internal grading* j is induced by the grading on A (cf. [17, Section 1 of Chapter 1], [35, Section 2.1], and [19, Section 2]).

Assume further that A is multiplicatively generated by A_1 with relations of degree 2; so A is a quadratic algebra with finite-dimensional components over k. Then the vector spaces $\mathrm{Ext}^i_C(k,k)$ or $\mathrm{Ext}^i_A(k,k)$ still need *not* be finite-dimensional; in fact, $\mathrm{Ext}^3_C(k,k)$ or (equivalently) $\mathrm{Ext}^3_A(k,k)$ can be infinite-dimensional already [1, Theorem 7.6], [7], [17, Section 6 of Chapter 6].

Nevertheless, for a (*homogeneous*) *Koszul* k-algebra A with finite-dimensional grading components [17, 40] and the graded dual coalgebra C, the vector spaces $\mathrm{Ext}^i_C(k,k)$ and $\mathrm{Ext}^i_A(k,k)$ are, of course, finite-dimensional for all $i \geq 0$. More generally, for a finitely cogenerated conilpotent coalgebra C, all the vector spaces $\mathrm{Ext}^i_C(k,k)$ are finite-dimensional whenever the Ext-algebra $\mathrm{Ext}^*_C(k,k)$ is generated by $\mathrm{Ext}^1_C(k,k)$. This includes the important particular case when the Ext-algebra $\mathrm{Ext}^*_C(k,k)$ is Koszul (but the coalgebra C need not be graded) described in [39, Main Theorem] and [26, Sections 5–6]. Here we keep assuming that the coalgebra C is finitely cogenerated.

With these important special cases in mind, we call a conilpotent coalgebra C *weakly finitely Koszul* if the vector space $\mathrm{Ext}^n_C(k,k)$ is finite-dimensional for every $n \geq 0$. This terminology goes back to [35, Sections 5.4 and 5.7].

Let us emphasize that *we do not know* whether the induced triangulated functors between the unbounded derived categories

$$\Upsilon^\varnothing \colon \mathsf{D}(C\text{–}\mathsf{Comod}) \longrightarrow \mathsf{D}(C^*\text{–}\mathsf{Mod}) \quad \text{and} \quad \Theta^\varnothing \colon \mathsf{D}(C\text{–}\mathsf{Contra}) \longrightarrow \mathsf{D}(C^*\text{–}\mathsf{Mod})$$

are fully faithful under the equivalent conditions of Theorem 1.2 (or even for a graded coalgebra C graded dual to a Koszul algebra A with finite-dimensional components) in general. However, the triangulated functors $\Upsilon^\varnothing$ and $\Theta^\varnothing$ are known to be fully faithful for any finitely cogenerated conilpotent *cocommutative* coalgebra C. This is a particular case of [25, Theorems 1.3 and 2.9] (applied to the complete Noetherian commutative local k-algebra $R = C^*$ with its maximal ideal $I = \mathfrak{m}$).

In this connection, it should first of all be mentioned that all finitely cogenerated conilpotent cocommutative coalgebras C are weakly finitely Koszul since they are Artinian, and consequently, co-Noetherian [29, Section 2]. Quite generally, any left or right co-Noetherian conilpotent coalgebra is weakly finitely Koszul (and so is any finitely cogenerated left or right cocoherent coalgebra). But this does not seem to be enough. The proofs of [25, Theorems 1.3 and 2.9] are essentially based on the observation that *the right adjoint functor* Γ *to* Υ *and the left adjoint functor* Δ *to* Θ *have finite homological dimensions* (cf. [28, Theorem 6.4] and [31, Proposition 6.5]). We cannot think of any general noncommutative versions of these properties, proved in [18] and [25] using commutative Koszul complexes.

2. Preliminaries on coalgebras, comodules, and contramodules

Unless otherwise mentioned, all *coalgebras*, *comodules*, and *contramodules* in this paper are presumed to be coassociative and counital; all *coalgebra homomorphisms* are presumed to preserve the counit. Dually, all *rings*, *algebras*, and *modules* are presumed to be associative and unital.

A (coassociative, counital) *coalgebra* C over a field k is a k-vector space endowed with k-linear maps of *comultiplication* $\mu \colon C \longrightarrow C \otimes_k C$ and *counit* $\epsilon \colon C \longrightarrow k$ satisfying the usual coassociativity and counitality axioms (which can be obtained by writing down the definition of an associative, unital k-algebra in the tensor notation and inverting the arrows). We suggest the books [16, 44] as the standard reference sources on coalgebras over a field. The present author's surveys [24, Section 1], [36, Section 3] can be used as additional reference sources.

Let C be a coalgebra over k. A *right C-comodule* N is a k-vector space endowed with a k-linear map of *right coaction* $\nu \colon N \longrightarrow N \otimes_k C$ satisfying the usual coassociativity and counitality axioms (which can be obtained by inverting the arrows in the definition of a module over an associative, unital algebra).

Similarly, a *left C-comodule M* is a k-vector space endowed with a k-linear map of *left coaction* $v\colon M \longrightarrow C \otimes_k M$ satisfying the coassociativity and counitality axioms.

A *left C-contramodule* (see [5, Section III.5], [20, Sections 0.2.4 and 3.1.1, and Appendix A], [24, Sections 1.1–1.6], and [36, Section 8]) is a k-vector space P endowed with a k-linear map of *left contraaction* $\pi\colon \operatorname{Hom}_k(C, P) \longrightarrow P$ satisfying the following *contraassociativity* and *contraunitality* axioms. First, the two maps $\operatorname{Hom}(C, \pi)$ and $\operatorname{Hom}(\mu, P)\colon \operatorname{Hom}_k(C \otimes_k C,\ P) \simeq \operatorname{Hom}_k(C, \operatorname{Hom}_k(C, P)) \longrightarrow \operatorname{Hom}_k(C, P)$ must have equal compositions with the contraaction map $\pi\colon \operatorname{Hom}_k(C, P) \longrightarrow P$,

$$\operatorname{Hom}_k(C \otimes_k C,\ P) \simeq \operatorname{Hom}_k(C, \operatorname{Hom}_k(C, P)) \rightrightarrows \operatorname{Hom}_k(C, P) \longrightarrow P.$$

Second, the composition of the map $\operatorname{Hom}(\epsilon, P)\colon P \longrightarrow \operatorname{Hom}_k(C, P)$ with the map $\pi\colon \operatorname{Hom}_k(C, P) \longrightarrow P$ must be equal to the identity endomorphism of P,

$$P \longrightarrow \operatorname{Hom}_k(C, P) \longrightarrow P.$$

Here the k-vector space isomorphism $\operatorname{Hom}_k(C \otimes_k C,\ P) \simeq \operatorname{Hom}_k(C, \operatorname{Hom}_k(C, P))$ is obtained as a particular case of the adjunction isomorphism $\operatorname{Hom}_k(U \otimes_k V,\ W) \simeq \operatorname{Hom}_k(V, \operatorname{Hom}_k(U, W))$, which holds for any vector spaces U, V, and W.

The definition of a *right C-contramodule* is similar, with the only difference that the isomorphism $\operatorname{Hom}_k(C \otimes_k C,\ P) \simeq \operatorname{Hom}_k(C, \operatorname{Hom}_k(C, P))$ arising as a particular case of the identification $\operatorname{Hom}_k(V \otimes_k U,\ W) \simeq \operatorname{Hom}_k(V, \operatorname{Hom}_k(U, W))$ is used.

For any right C-comodule N and any k-vector space V, the vector space $\operatorname{Hom}_k(N, V)$ has a natural left C-contramodule structure. The left contraaction map

$$\pi\colon \operatorname{Hom}_k(C, \operatorname{Hom}_k(N, V)) \simeq \operatorname{Hom}_k(N \otimes_k C,\ V) \longrightarrow \operatorname{Hom}_k(N, V)$$

is constructed by applying the functor $\operatorname{Hom}_k(-, V)$ to the right coaction map $v\colon N \longrightarrow N \otimes_k C$, i. e., $\pi = \operatorname{Hom}_k(v, V)$.

Let C^{op} denote the opposite coalgebra to C (i. e., the same vector space endowed with the same counit and the left-right opposite comultiplication to that in C). Similarly, given a ring A, we denote by A^{op} the opposite ring. Then right C-comodules are the same things as left C^{op}-comodules, and vice versa.

We will use the notation $\operatorname{Hom}_C(-, -)$ for the vector spaces of morphisms in the category of left C-comodules $C-\mathsf{Comod}$ and the notation $\operatorname{Hom}_{C^{\mathrm{op}}}(-, -)$ for the vector spaces of morphisms in the category of right C-comodules $\mathsf{Comod}-C$. The vector spaces of morphisms in the left and right contramodule categories $C-\mathsf{Contra}$ and $\mathsf{Contra}\text{-}C$ will be denoted by $\operatorname{Hom}^C(-, -)$ and $\operatorname{Hom}^{C^{\mathrm{op}}}(-, -)$. Similarly, the Yoneda Ext spaces are denoted by $\operatorname{Ext}^*_C(-, -)$ in $C-\mathsf{Comod}$, by $\operatorname{Ext}^*_{C^{\mathrm{op}}}(-, -)$ in $\mathsf{Comod}-C$, and by $\operatorname{Ext}^{C,*}(-, -)$ in $C-\mathsf{Contra}$.

The category of left C-comodules $C-\mathsf{Comod}$ is a locally finite Grothendieck abelian category. Any C-comodule is the union of its finite-dimensional subcomodules [44, Propositions 2.1.1–2.1.2 and Corollary 2.1.4], [36, Lemma 3.1(b)]. The forgetful functor $C-\mathsf{Comod} \longrightarrow k-\mathsf{Vect}$ from the category of C-comodules to the category of k-vector spaces is exact and preserves the infinite coproducts (but *not* the infinite products). So the coproduct (and more generally, filtered direct limit) functors are exact in $C-\mathsf{Comod}$, while the functors of infinite product are usually *not* exact.

Left C-comodules of the form $C \otimes_k V$ and right C-comodules of the form $V \otimes_k C$, where V ranges over the k-vector spaces, are called the *cofree C-comodules*. For any left C-comodule L, the vector space of C-comodule morphisms $L \longrightarrow C \otimes_k V$ is naturally isomorphic to the vector space of k-linear maps $L \longrightarrow V$,

$$\operatorname{Hom}_C(L,\ C \otimes_k V) \simeq \operatorname{Hom}_k(L, V).$$

Hence, the cofree comodules are injective (as objects of $C-\mathsf{Comod}$ or $\mathsf{Comod}-C$). A C-comodule is injective if and only if it is a direct summand of a cofree one.

The category of left C-contramodules $C-\mathsf{Contra}$ is a locally presentable abelian category with enough projective objects. The forgetful functor $C-\mathsf{Contra} \longrightarrow k-\mathsf{Vect}$ is exact and preserves the infinite products (but *not* the infinite coproducts). Hence, the functors of infinite product are exact in $C-\mathsf{Contra}$, while the coproduct functors are usually *not* exact.

Left C-contramodules of the form $\mathrm{Hom}_k(C, V)$, where $V \in k-\mathsf{Vect}$, are called the *free* C-contramodules. For any left C-contramodule Q, the vector space of C-contramodule morphisms $\mathrm{Hom}_k(C, V) \longrightarrow Q$ is naturally isomorphic to the vector space of k-linear maps $V \longrightarrow Q$,

$$\mathrm{Hom}^C(\mathrm{Hom}_k(C, V), Q) \simeq \mathrm{Hom}_k(V, Q).$$

Hence, the free contramodules are projective (as objects of $C-\mathsf{Contra}$). A C-contramodule is projective if and only if it is a direct summand of a free one.

Let us introduce a simplified version of the *Sweedler notation* [44, Section 1.2], [16, Notation 1.4.2] for the comultiplication in C. Given an element $c \in C$, we write

$$\mu(c) = c_{(1)} \otimes c_{(2)} \in C \otimes_k C.$$

Following the convention in [20, 24, 36] (which is opposite to the convention in [16, 44]), we define the associative algebra structure on the dual vector space $C^* = \mathrm{Hom}_k(C, k)$ to a coalgebra C by the formula

$$(fg)(c) = f(c_{(2)})g(c_{(1)}) \qquad \text{for all } f, \ g \in C^* \text{ and } c \in C.$$

The counit on C induces a unit in C^* in the obvious way.

Then, for any left C-comodule M, the composition

$$C^* \otimes_k M \longrightarrow C^* \otimes_k C \otimes_k M \longrightarrow M$$

of the map induced by the coaction map $\nu \colon M \longrightarrow C \otimes_k M$ and the map induced by the pairing map $C^* \otimes_k C \longrightarrow k$ endows M with a left C^*-module structure. Similarly, for any right C-comodule N, the composition

$$N \otimes_k C^* \longrightarrow N \otimes_k C \otimes_k C^* \longrightarrow N$$

endows N with a right C^*-module structure. Finally, for any left C-contramodule P, the composition

$$C^* \otimes_k P \longrightarrow \mathrm{Hom}_k(C, P) \longrightarrow P$$

of the natural embedding of vector spaces $C^* \otimes_k P \rightarrowtail \mathrm{Hom}_k(C, P)$ and the contraaction map $\pi \colon \mathrm{Hom}_k(C, P) \longrightarrow P$ endows P with a left C^*-module structure.

We have constructed the *comodule inclusion functors*

$$\Upsilon \colon C-\mathsf{Comod} \longrightarrow C^*-\mathsf{Mod},$$
$$\mathsf{Comod}-C \longrightarrow \mathsf{Mod}-C^*$$

and the *contramodule forgetful functor*

$$\Theta \colon C-\mathsf{Contra} \longrightarrow C^*-\mathsf{Mod}.$$

The comodule inclusion functors (for a coalgebra C over a field k) are always fully faithful (see [44, Propositions 2.1.1–2.1.2 and Theorem 2.1.3(e)] for a discussion). The contramodule forgetful functor is *not* fully faithful in general, as we will see in Example 8.2 below.

3. Conilpotent coalgebras and minimal resolutions

What we call *conilpotent* coalgebras (in the terminology going back to [39, Section 3.1], [19, Section 4.1]) were called "pointed irreducible" coalgebras in [44, Section 8.0]. We refer to [36, Sections 3.3–3.4] for an introductory discussion.

Let D be a coalgebra without counit over a field k. For every $n \geq 1$, there is the uniquely defined *iterated comultiplication map* $\mu^{(n)} \colon D \longrightarrow D^{\otimes n+1}$. The coalgebra D is said to be *conilpotent* if for every $d \in D$ there exists $n \geq 1$ such that $\mu^{(n)}(d) = 0$ in $D^{\otimes n+1}$. Clearly, one then also has $\mu^{(m)}(d) = 0$ for all $m \geq n$.

A *coaugmentation* γ of a coalgebra C is a homomorphism of (counital) coalgebras $\gamma\colon k \longrightarrow C$. So the composition $\epsilon\gamma\colon k \longrightarrow C \longrightarrow k$ must be the identity map.

Given a coaugmented coalgebra (C, γ), the cokernel $D = C/\gamma(k)$ of the map γ has a unique coalgebra structure for which the natural surjection $C \longrightarrow D$ is a homomorphism of noncounital coalgebras. A coaugmented coalgebra C us called *conilpotent* if the noncounital coalgebra D is conilpotent (in the sense of the definition above).

Obviously, no nonzero noncounital coalgebra homomorphisms $k \longrightarrow D$ exist for a conilpotent noncounital coalgebra D. Consequently, a conilpotent coaugmented coalgebra (C, γ) admits no other coaugmentation but γ.

The following result is a version of Nakayama lemma for conilpotent noncounital coalgebras.

Lemma 3.1. *(a) Let D be a conilpotent noncounital coalgebra and $M \neq 0$ be a noncounital left D-comodule. Then the coaction map $M \longrightarrow D \otimes_k M$ is not injective.*

(b) Let D be a conilpotent noncounital coalgebra and $P \neq 0$ be a noncounital left D-contramodule. Then the contraaction map $\mathrm{Hom}_k(D, P) \longrightarrow P$ is not surjective.

Proof. Part (a): for the sake of contradiction, assume that the coaction map $\nu\colon M \longrightarrow D \otimes_k M$ is injective. Then the iterated coaction map $\nu^{(n)}\colon M \longrightarrow D^{\otimes n} \otimes_k M$ is also injective for every $n \geq 1$.

Pick a nonzero element $x \in M$, and write $\nu(x) = \sum_{i=1}^r d_r \otimes y_r$ for some $d_i \in D$ and $y_i \in M$. Choose $n \geq 1$ such that $\mu^{(n)}(d_i) = 0$ in $D^{\otimes n+1}$ for every $1 \leq i \leq r$. Then $\nu^{(n+1)}(x) = \sum_{i=1}^r \mu^{(n)}(d_i) \otimes y_i = 0$ in $D^{\otimes n+1} \otimes_k M$, a contradiction.

The proof of part (b) is a bit more involved; it can be found in [20, Lemma A.2.1]. For a discussion of generalizations and other versions of the comodule and contramodule Nakayama lemmas, see [24, Lemma 2.1]. □

Let E be a subcoalgebra in a coalgebra C. Then in any left C-comodule M there exists a unique maximal subcomodule whose C-comodule structure arises from an E-comodule structure. We denote this subcomodule, which can be computed as the kernel of the composition of maps $M \longrightarrow C \otimes_k M \longrightarrow C/E \otimes_k M$, by $_E M \subset M$. The similar subcomodule of a right C-comodule N will be denoted by $N_E \subset N$.

Dually, any left C-contramodule P admits a unique maximal quotient contramodule whose C-contramodule structure arises from an E-contramodule structure. We denote this quotient contramodule, which can be computed as the cokernel of the composition of maps $\mathrm{Hom}_k(C/E, P) \longrightarrow \mathrm{Hom}_k(C, P) \longrightarrow P$, by $^E P \twoheadleftarrow P$. A further discussion of the functors $M \longmapsto {}_E M$ and $P \longmapsto {}^E P$ can be found in [29, Section 2] or [36, Section 8.4].

For any right C-comodule N, any subcoalgebra $E \subset C$, and any k-vector space V, there is a natural isomorphism of left E-contramodules

$$^E\mathrm{Hom}_k(N, V) \simeq \mathrm{Hom}_k(N_E, V), \tag{4}$$

where the left contramodule structure on the space of linear maps from a right comodule to a vector space is constructed as explained in Section 2. The natural isomorphism (4) follows immediately from the constructions of the functors $M \longmapsto {}_E M$ and $P \longmapsto {}^E P$ above.

In this section, we will be interested in the particular case when (C, γ) is a coaugmented (eventually, conilpotent) coalgebra and $E = \gamma(k) \subset C$. In this case, we put $_\gamma M = {}_E M$ and $^\gamma P = {}^E P$. The similar notation for a right C-comodule N is $N_\gamma = N_E$, and for a right C-contramodule Q it is $Q^\gamma = Q^E$. Endowing the one-dimensional k-vector space k with the (left and right) C-comodule and C-contramodule structures defined in terms of γ, one has natural isomorphisms of k-vector spaces

$$_\gamma M \simeq \mathrm{Hom}_C(k, M) \quad \text{and} \quad \mathrm{Hom}_k(^\gamma P, k) \simeq \mathrm{Hom}^C(P, k).$$

One can further compute the vector space $^\gamma P$ as the *contratensor product* $k \odot_C P$ (see [24, Section 3.1] for the definition), but we will not need to use this fact.

Lemma 3.2. *(a) Let (C, γ) be a conilpotent coaugmented coalgebra and $M \neq 0$ be a left C-comodule. Then $_\gamma M \neq 0$.*

(b) Let (C, γ) be a conilpotent coaugmented coalgebra and $P \neq 0$ be a left C-contramodule. Then $^\gamma P \neq 0$.

Proof. This is an equivalent restatement of Lemma 3.1 for $D = C/\gamma(k)$. □

For a conilpotent coalgebra C, the subcomodule $_\gamma M$ of a C-comodule M can be also described as the socle (i.e., the maximal semisimple subcomodule) of M, and the quotient contramodule $^\gamma P$ of a C-contramodule P can be described as the cosocle (i.e., the maximal semisimple quotient contramodule) of P.

Lemma 3.3. *Let (C, γ) be a conilpotent coaugmented coalgebra. Then*

(a) a morphism of left C-comodules $f: L \longrightarrow M$ is injective if and only if the induced map of vector spaces $_\gamma f: {_\gamma L} \longrightarrow {_\gamma M}$ is injective;

(b) a morphism of left C-contramodules $f: P \longrightarrow Q$ is surjective if and only if the induced map of vector spaces $^\gamma f: {^\gamma P} \longrightarrow {^\gamma Q}$ is surjective.

Proof. Part (a): if the map f is injective, then so is the map $_\gamma f$ since $_\gamma N$ is a vector subspace in N for every left C-comodule N. Conversely, the functor $N \longmapsto {_\gamma N}$ is left exact; so if $K = \ker(f) \in C-\mathsf{Comod}$, then $_\gamma K = \ker(_\gamma f)$. Now if $_\gamma K = 0$, then $K = 0$ by Lemma 3.2(a). The proof of part (b) is similar (or rather, dual-analogous in the sense of [36, Section 8.2]). The functor $P \longmapsto {^\gamma P}$ is right exact, etc. □

Lemma 3.4. *Let (C, γ) be a conilpotent coaugmented coalgebra. Then*

(a) a morphism of injective left C-comodules $f: I \longrightarrow J$ is an isomorphism if and only if the induced map of vector spaces $_\gamma f: {_\gamma I} \longrightarrow {_\gamma J}$ is an isomorphism;

(b) a morphism of projective left C-contramodules $f: P \longrightarrow Q$ is an isomorphism if and only if the induced map of vector spaces $^\gamma f: {^\gamma P} \longrightarrow {^\gamma Q}$ is an isomorphism.

Proof. Part (a): if the map $_\gamma f$ is injective, then the morphism f is injective by Lemma 3.3(a). Now if the C-comodule I is injective, then f is a split monomorphism in $C-\mathsf{Comod}$. Therefore, $_\gamma \mathrm{coker}(f) = \mathrm{coker}(_\gamma f)$ and Lemma 3.2(a) tell us that f is an isomorphism whenever $_\gamma f$ is. The proof of part (b) is dual-analogous. □

Lemma 3.5. *Let (C, γ) be a conilpotent coaugmented coalgebra. Then*

(a) for any left C-comodule M there exists a cofree left C-comodule J together with an injective C-comodule morphism $f: M \longrightarrow J$ such that the induced map $_\gamma f: {_\gamma M} \longrightarrow {_\gamma J}$ is an isomorphism of k-vector spaces;

(b) for any left C-contramodule Q there exists a free left C-contramodule P together with a surjective C-contramodule morphism $f: P \longrightarrow Q$ such that the induced map $^\gamma f: {^\gamma P} \longrightarrow {^\gamma Q}$ is an isomorphism of k-vector spaces.

Proof. Part (a): put $V = {_\gamma M}$ and $J = C \otimes_k V$. Then the injective map $V \longrightarrow J$ induced by the map $\gamma: k \longrightarrow C$ is a C-comodule morphism $M \supset {_\gamma M} \longrightarrow J$. Since the cofree comodules are injective, this morphism can be extended to a C-comodule morphism $f: M \longrightarrow J$. The morphism f is injective by Lemma 3.3(a). The comodule J can be also described as an injective envelope of the comodule M in the comodule category $C-\mathsf{Comod}$. The proof of part (b) is dual analogous; the contramodule P can be also described as a projective cover of the contramodule Q in contramodule category $C-\mathsf{Contra}$ (cf. [32, Example 12.3]). □

Corollary 3.6. *Let C be a conilpotent (coaugmented) coalgebra. Then*
  (a) a C-comodule is injective if and only if is cofree;
  (b) a C-contramodule is projective if and only if it is free.

Proof. Generally speaking, over a coalgebra C over a field k, the injective comodules are the direct summands of the cofree ones, and the projective contramodules are the direct summands of the free ones (see Section 2). In the situation at hand with a conilpotent coalgebra C, the assersion of part (a) follows straighforwardly from Lemmas 3.4(a) and 3.5(a). Part (b) follows from Lemmas 3.4(b) and 3.5(b). $\square$

Let $K^\bullet$ be a complex of left C-comodules. We will say that the complex $K^\bullet$ is *minimal* if the differential in the complex of vector spaces ${}_\gamma K^\bullet$ vanishes. Similarly, a complex of left C-contramodules $Q_\bullet$ is said to be *minimal* if the differential in the complex of vector spaces ${}^\gamma Q_\bullet$ vanishes.

Let M be a C-comodule. An injective coresolution $M \longrightarrow J^\bullet$ of the comodule M is said to be *minimal* if the complex of comodules $J^\bullet$ is minimal. Dually, a projective resolution $Q_\bullet \longrightarrow P$ of a C-contramodule P is said to be *minimal* if the complex of contramodules $Q_\bullet$ is minimal.

Proposition 3.7. *The following assertions hold for any conilpotent coalgebra C.*
  (a) Any C-comodule M admits a minimal injective coresolution. A minimal injective coresolution of M is unique up to a (nonunique) isomorphism.
  (b) Any C-contramodule P admits a minimal projective resolution. A minimal projective resolution of P is unique up to a (nonunique) isomorphism.

Proof. Part (a): to construct a minimal injective coresolution of M, all one needs to do is to iterate the construction of Lemma 3.5(a). Pick an injective C-comodule J^0 together with an injective morphism of C-comodules $M \longrightarrow J^0$ such that the induced map ${}_\gamma M \longrightarrow {}_\gamma J^0$ is an isomorphism. Then the induced map ${}_\gamma J^0 \longrightarrow {}_\gamma(J^0/M)$ is zero. Pick an injective C-comodule J^1 together with an injective morphism of C-comodules $J^0/M \longrightarrow J^1$ such that the induced map ${}_\gamma(J^0/M) \longrightarrow {}_\gamma J^1$ is an isomorphism, etc.

To prove uniqueness, let $M \longrightarrow I^\bullet$ and $M \longrightarrow J^\bullet$ be two minimal injective coresolutions of M. Then there exists a morphism of complexes of C-comodules $f\colon I^\bullet \longrightarrow J^\bullet$ making the triangular diagram $M \longrightarrow I^\bullet \longrightarrow J^\bullet$ commutative. Now we have ${}_\gamma I^\bullet = \mathrm{Hom}_C(k, I^\bullet)$ and ${}_\gamma J^\bullet = \mathrm{Hom}_C(k, I^\bullet)$. The map of complexes $\mathrm{Hom}_C(k, f)\colon \mathrm{Hom}_C(k, I^\bullet) \longrightarrow \mathrm{Hom}_C(k, J^\bullet)$ induces an isomorphism of the cohomology spaces, as both the complexes compute $\mathrm{Ext}^*_C(k, M)$. Since both the complexes ${}_\gamma I^\bullet$ and ${}_\gamma J^\bullet$ have vanishing differentials by assumption, it follows that $\mathrm{Hom}_C(k, f) = {}_\gamma f$ is a termwise isomorphism of complexes of vector spaces. Applying Lemma 3.4(a), we conclude that f is a termwise isomorphism of complexes of C-comodules.

The proof of part (b) is dual-analogous. $\square$

4. The left and right comodule and contramodule Ext comparison

Let (C, γ) be a coaugmented coalgebra over a field k (as defined in Section 3). Then the one-dimensional k-vector space k can be endowed with left and right C-comodule and C-contramodule structures defined in terms of γ. We recall the notation $\mathrm{Ext}^*_C(-, -)$ for the Ext spaces in the category of left C-comodules $C-\mathsf{Comod}$ and $\mathrm{Ext}^*_{C^{\mathrm{op}}}(-, -)$ for the Ext spaces in the category of right C-comodules $\mathsf{Comod}-C$, as well as the notation $\mathrm{Ext}^{C,*}(-, -)$ for the Ext spaces in the category of left C-contramodules $C-\mathsf{Contra}$.

The condition that the vector spaces $\mathrm{Ext}^n_C(k, k)$ be finite-dimensional plays a key role in this paper. In this section, we explain that this condition is left-right symmetric: it holds for a coalgebra C if and only if it holds for C^{op}. Furthermore, the vector space $\mathrm{Ext}^n_C(k, k)$ is finite-dimensional if and only if the vector space $\mathrm{Ext}^{C,n}(k, k)$ is. The following proposition is certainly essentially well known; we spell out the details here for the sake of completeness of the exposition.

Proposition 4.1. *Let (C, γ) be a coaugmented coalgebra over k. Then there are natural isomorphisms of Ext spaces*

$$\mathrm{Ext}^n_C(k, k) \simeq \mathrm{Ext}^n_{C^{\mathrm{op}}}(k, k) \qquad \textit{for all } n \geq 0.$$

Moreover, in fact, the Ext-algebras $\mathrm{Ext}^*_C(k, k)$ and $\mathrm{Ext}^*_{C^{\mathrm{op}}}(k, k)$ are naturally opposite to each other (as graded algebras),

$$\mathrm{Ext}^*_{C^{\mathrm{op}}}(k, k) \simeq \mathrm{Ext}^*_C(k, k)^{\mathrm{op}}.$$

The module analogue of Proposition 4.1 is well known: for any augmented associative algebra A over a field k, there are natural isomorphisms of k-vector spaces $\mathrm{Ext}^n_A(k, k) \simeq \mathrm{Tor}^A_n(k, k)^* \simeq \mathrm{Ext}^n_{A^{\mathrm{op}}}(k, k)$, making $\mathrm{Ext}^*_{A^{\mathrm{op}}}(k, k)$ the graded algebra with the opposite multiplication to $\mathrm{Ext}^*_A(k, k)$.

First proof. Put $C_+ = C/\gamma(k)$. For any left C-comodule M, the reduced cobar coresolution

$$0 \longrightarrow M \longrightarrow C \otimes_k M \longrightarrow C \otimes_k C_+ \otimes_k M \longrightarrow C \otimes_k C_+ \otimes_k C_+ \otimes_k M \longrightarrow \cdots$$

is an injective coresolution of M in the abelian category $C-\mathsf{Comod}$. Applying the functor $\mathrm{Hom}_C(k, -)$, we obtain a cobar complex

$$M \longrightarrow C_+ \otimes_k M \longrightarrow C_+ \otimes_k C_+ \otimes_k M \longrightarrow \cdots \tag{5}$$

computing the vector spaces $\mathrm{Ext}^*_C(k, M)$. In particular, the vector spaces $\mathrm{Ext}^*_C(k, k)$ are computed by the cobar complex

$$k \longrightarrow C_+ \longrightarrow C_+ \otimes_k C_+ \longrightarrow C_+ \otimes_k C_+ \otimes_k C_+ \longrightarrow \cdots \tag{6}$$

[39, Section 1.1], [19, Section 2.1].

Reordering the tensor factors inside every term of the complex in the opposite way identifies the complex (6) for a coalgebra C with the similar complex for the opposite coalgebra C^{op}. This suffices to prove the first assertion of the proposition.

To prove the second assertion, one needs to observe that the obvious (free associative) multiplication on the cobar complex (6) induces a multiplication on its cohomology spaces that is precisely opposite to the composition multiplication on $\mathrm{Ext}^*_C(k, k)$. Moreover, the obvious left action of the DG-algebra (6) on the complex (5) induces a left action of the cohomology algebra of (6) on the cohomology of (5) corresponding to the natural graded right module structure over $\mathrm{Ext}^*_C(k, k)$ on $\mathrm{Ext}^*_C(k, M)$. Thus, the isomorphism $\mathrm{Ext}^*_C(k, k) \simeq \mathrm{Ext}^*_{C^{\mathrm{op}}}(k, k)$ obtained by comparing the cobar complexes (6) for C and C^{op} makes the multiplication on the latter Ext-algebra opposite to the one on the former one.

This argument was sketched in [19, Section 2.4]. An alternative proof of the first assertion of the proposition was suggested in [19, Section 2.5]. □

Second proof. Let $C-\mathsf{comod}$ and $\mathsf{comod}-C$ denote the abelian categories of *finite-dimensional* left and right C-comodules, respectively. One makes two observations, which, taken together, imply both the assertions of the proposition.

First, the fully faithful, exact inclusions of abelian categories $C-\mathsf{comod} \longrightarrow C-\mathsf{Comod}$ and $\mathsf{comod}-C \longrightarrow \mathsf{Comod}-C$ induce isomorphisms on the Ext spaces. Moreover, the triangulated functors between bounded above derived categories $\mathsf{D}^-(C-\mathsf{comod}) \longrightarrow \mathsf{D}^-(C-\mathsf{Comod})$ and $\mathsf{D}^-(\mathsf{comod}-C) \longrightarrow \mathsf{D}^-(\mathsf{Comod}-C)$ induced by these inclusions of abelian categories are fully faithful. This is a particular case of a well-known general result for abelian or even exact categories (see [12, opposite version of Proposition 1.7.11], [13, opposite version of Theorem 13.2.8], [14, opposite version of Theorem 12.1(b)], or [37, Proposition 2.1]). The point is that for any finite-dimensional C-comodule L, any C-comodule M, and any surjective C-comodule morphism $M \longrightarrow L$, there exists a finite-dimensional C-subcomodule $M' \subset M$ such that the composition $M' \longrightarrow M \longrightarrow L$ is surjective.

Second, for any finite-dimensional right C-comodule N, the dual vector space N^* is naturally a finite-dimensional left C-comodule. Here, it helps to notice that for any such N there exists a finite-dimensional

subcoalgebra $E \subset C$ such that the C-comodule structure on N arises from an E-comodule structure [36, Lemma 3.1(c)]. Consequently, there is a natural anti-equivalence of abelian categories

$$N \longmapsto N^* : (\mathsf{comod}-C)^{\mathsf{op}} \longrightarrow C-\mathsf{comod}$$

taking the right C-comodule k to the left C-comodule k. This anti-equivalence induces the desired anti-isomorphism of the Ext algebras. $\square$

Lemma 4.2. *Let C be a coalgebra over a field k. Let $M \in \mathsf{Comod}-C$ be a right C-comodule and $L \in \mathsf{comod}-C$ be a finite-dimensional right C-comodule. Then the vector space of left C-contramodule homomorpisms $M^* \longrightarrow L^*$ is naturally isomorphic to the double dual vector space to the vector space of right C-comodule homomorphisms $L \longrightarrow M$,*

$$\mathrm{Hom}^C(M^*, L^*) \simeq \mathrm{Hom}_{C^{\mathsf{op}}}(L, M)^{**}. \tag{7}$$

This isomorphism identifies the map $\mathrm{Hom}_{C^{\mathsf{op}}}(L, M) \longrightarrow \mathrm{Hom}^C(M^*, L^*)$ induced by the contravariant functor of vector space dualization $N \longmapsto N^* : \mathsf{Comod}-C \longrightarrow C-\mathsf{Contra}$ with the natural inclusion $\mathrm{Hom}_{C^{\mathsf{op}}}(L, M) \longrightarrow \mathrm{Hom}_{C^{\mathsf{op}}}(L, M)^{**}$ of a vector space $\mathrm{Hom}_{C^{\mathsf{op}}}(L, M)$ into its double dual vector space.

Proof. Notice that for any k-vector space V and any finite-dimensional k-vector space W there is a natural isomorphism of k-vector spaces

$$\mathrm{Hom}_k(V^*, W^*) \simeq \mathrm{Hom}_k(W, V)^{**}, \tag{8}$$

where $V^* = \mathrm{Hom}_k(V, k)$ and $W^* = \mathrm{Hom}_k(W, k)$. The isomorphism (8) identifies the map $\mathrm{Hom}_k(W, V) \longrightarrow \mathrm{Hom}_k(V^*, W^*)$ induced by the contravariant functor $U \longmapsto U^* : k-\mathsf{Vect} \longrightarrow k-\mathsf{Vect}$ with the natural inclusion $\mathrm{Hom}_k(W, V) \longrightarrow \mathrm{Hom}_k(W, V)^{**}$ of a vector space $\mathrm{Hom}_k(W, V)$ into its double dual vector space.

Now, for any two right C-comodules L and M, the vector space $\mathrm{Hom}_{C^{\mathsf{op}}}(L, M)$ is the kernel of (the difference of) a natural pair of maps $\mathrm{Hom}_k(L, M) \rightrightarrows \mathrm{Hom}_k(L, M \otimes_k C)$,

$$0 \longrightarrow \mathrm{Hom}_{C^{\mathsf{op}}}(L, M) \longrightarrow \mathrm{Hom}_k(L, M) \rightrightarrows \mathrm{Hom}_k(L, M \otimes_k C). \tag{9}$$

For any two left C-contramodules P and Q, the vector space $\mathrm{Hom}^C(P, Q)$ is the kernel of (the difference of) a natural pair of maps $\mathrm{Hom}_k(P, Q) \rightrightarrows \mathrm{Hom}_k(\mathrm{Hom}_k(C, P), Q)$,

$$0 \longrightarrow \mathrm{Hom}^C(P, Q) \longrightarrow \mathrm{Hom}_k(P, Q) \rightrightarrows \mathrm{Hom}_k(\mathrm{Hom}_k(C, P), Q). \tag{10}$$

In particular, for $P = M^*$ and $Q = L^*$, we have

$$0 \longrightarrow \mathrm{Hom}^C(M^*, L^*) \longrightarrow \mathrm{Hom}_k(M^*, L^*) \rightrightarrows \mathrm{Hom}_k(\mathrm{Hom}_k(C, M^*), L^*). \tag{11}$$

It remains to use the natural isomorphism (8) in order to show that the double dual vector space functor $U \longmapsto U^{**}$ takes the rightmost pair of parallel morphisms in the sequence (9) to the rightmost pair of parallel morphisms in the sequence (11) for a finite-dimensional C-comodule L. $\square$

Proposition 4.3. *Let (C, γ) be a coaugmented coalgebra over k. Then the vector space $\mathrm{Ext}^{C,n}(k, k)$ is naturally isomorphic to the double dual vector space to the vector space $\mathrm{Ext}^n_{C^{\mathsf{op}}}(k, k)$,*

$$\mathrm{Ext}^{C,n}(k, k) \simeq \mathrm{Ext}^n_{C^{\mathsf{op}}}(k, k)^{**} \qquad \text{for all } n \geq 0.$$

This isomorphism identifies the map $\mathrm{Ext}^n_{C^{\mathsf{op}}}(k, k) \longrightarrow \mathrm{Ext}^{C,n}(k, k)$ induced by the exact contravariant functor of vector space dualization $N \longmapsto N^* : \mathsf{Comod}-C \longrightarrow C-\mathsf{Contra}$ with the natural inclusion $\mathrm{Ext}^n_{C^{\mathsf{op}}}(k, k) \longrightarrow \mathrm{Ext}^n_{C^{\mathsf{op}}}(k, k)^{**}$ of a vector space $\mathrm{Ext}^n_{C^{\mathsf{op}}}(k, k)$ into its double dual vector space.

Proof. This observation goes back to the discussion in [20, Section A.1.2]. The point is that the functor $N \longmapsto N^* : \mathsf{Comod}-C \longrightarrow C-\mathsf{Contra}$ takes injective right C-comodules to projective left C-contramodules, which makes it easy to compute the induced map of the Ext spaces.

For any coalgebra C and any right C-comodule M, pick an injective coresolution

$$0 \longrightarrow M \longrightarrow J^0 \longrightarrow J^1 \longrightarrow J^2 \longrightarrow \cdots \tag{12}$$

of M in the abelian category $\mathsf{Comod}{-}C$. Applying the dual vector space functor $\mathrm{Hom}_k(-, k)$ to the complex (12), we obtain the complex

$$0 \longleftarrow \mathrm{Hom}_k(M, k) \longleftarrow \mathrm{Hom}_k(J^0, k) \longleftarrow \mathrm{Hom}_k(J^1, k) \longleftarrow \cdots, \tag{13}$$

which is a projective resolution of the left C-contramodule $M^* = \mathrm{Hom}_k(M, k)$.

Now let L be a another right C-comodule. Then one can compute the vector spaces $\mathrm{Ext}^n_{C^{\mathrm{op}}}(L, M)$ as the cohomology spaces of the complex obtained by applying the functor $\mathrm{Hom}_{C^{\mathrm{op}}}(L, -)$ to the coresolution (12). Similarly, one can compute the vector spaces $\mathrm{Ext}^{C,n}(M^*, L^*)$ as the cohomology spaces of the complex obtained by applying the functor $\mathrm{Hom}^C(-, L^*)$ to the resolution (13). Taking into account Lemma 4.2, we obtain a natural isomorphism of complexes

$$\mathrm{Hom}^C((J^\bullet)^*, L^*) \simeq \mathrm{Hom}_{C^{\mathrm{op}}}(L, J^\bullet)^{**}$$

inducing a natural isomorphism of the cohomology spaces

$$\mathrm{Ext}^{C,n}(M^*, L^*) \simeq \mathrm{Ext}^n_{C^{\mathrm{op}}}(L, M)^{**}$$

for all right C-comodules M, all finite-dimensional right C-comodules L, and all integers $n \geq 0$. Moreover, it follows from the second assertion of the lemma that the map $\mathrm{Ext}^n_{C^{\mathrm{op}}}(L, M) \longrightarrow \mathrm{Ext}^{C,n}(M^*, L^*)$ induced by the functor $N \longmapsto N^* \colon \mathsf{Comod}{-}C \longrightarrow C{-}\mathsf{Contra}$ agrees with the obvious map $\mathrm{Ext}^n_{C^{\mathrm{op}}}(L, M) \longrightarrow \mathrm{Ext}^n_{C^{\mathrm{op}}}(L, M)^{**}$. $\qquad\square$

5. Weakly finite Koszulity implies comodule Ext isomorphism

The aim of this section is to prove the following theorem.

Theorem 5.1. *Let C be a conilpotent coalgebra over a field k and $n \geq 0$ be an integer such that the vector space $\mathrm{Ext}^i_C(k, k)$ is finite-dimensional for all $1 \leq i \leq n$. Then the map of Ext spaces*

$$\mathrm{Ext}^i_C(L, M) \longrightarrow \mathrm{Ext}^i_{C^*}(L, M)$$

induced by the inclusion functor $\Upsilon \colon C{-}\mathsf{Comod} \longrightarrow C^*{-}\mathsf{Mod}$ is an isomorphism for all left C-comodules L and M and all $0 \leq i \leq n$.

We start with a lemma describing some vector spaces of morphisms in the category of left C^*-modules for a coalgebra C. A right C-comodule N is said to be *finitely cogenerated* if it is a subcomodule of a cofree C-comodule $V \otimes_k C$ with a finite-dimensional vector space of cogenerators V.

Lemma 5.2. *Let C be a coalgebra over k, let N be a right C-comodule, and let U be a k-vector space. Then there is a natural monomorphism of k-vector spaces*

$$\eta_{N,U} \colon N \otimes_k U \longrightarrow \mathrm{Hom}_{C^*}(N^*, C \otimes_k U),$$

which is an isomorphism whenever N is a finitely cogenerated injective C-comodule. Here the left C^*-module structure on $N^* = \mathrm{Hom}_k(N, k)$ is obtained by dualizing the right C^*-module structure on N (or equivalently, by applying the forgetful functor to the natural left C-contramodule structure on N^*), while the left C^*-module structure on $C \otimes_k U$ comes from the cofree left C-comodule structure.

Proof. The map $\eta_{N,U}$ is defined by the formula

$$\eta_{N,U}(x \otimes u)(f) = f(x_{(0)})x_{(1)} \otimes u,$$

for all $x \in N$, $u \in U$, and $f \in N^*$. Here, $v(x) = x_{(0)} \otimes x_{(1)} \in N \otimes_k C$ is a notation for the right C-coaction map $v \colon N \longrightarrow N \otimes_k C$.

To prove the isomorphism assertion for a finitely cogenerated injective C-comodule N, it suffices to consider the case $N = V \otimes_k C$, where $\dim_k V < \infty$. In this case, $N^* = C^* \otimes_k V^*$ is a free left C^*-module, so $\operatorname{Hom}_{C^*}(N^*, \, C \otimes_k U) = \operatorname{Hom}_k(V^*, \, C \otimes_k U) = V \otimes_k C \otimes_k U = N \otimes_k U$.

To prove that $\eta_{N,U}$ is an injective map for any C-comodule N, consider the k-linear map

$$\operatorname{Hom}_{C^*}(N^*, \, C \otimes_k U) \longrightarrow \operatorname{Hom}_k(N^*, U)$$

induced by the counit map $\epsilon \colon C \longrightarrow k$. Notice that the counit is *not* a C-comodule morphism, and consequently not a C^*-module morphism, but only a linear map of k-vector spaces. The composition

$$N \otimes_k U \xrightarrow{\ \eta\ } \operatorname{Hom}_{C^*}(N^*, \, C \otimes_k U) \longrightarrow \operatorname{Hom}_k(N^*, U)$$

is the natural injective map of k-vector spaces $N \otimes_k U \longrightarrow \operatorname{Hom}_k(N^*, U)$ (defined for any two vector spaces U and $W = N$). Since the composition is a monomorphism, so is the map $\eta = \eta_{N,U}$. □

Let $C = (C, \gamma)$ be a coaugmented coalgebra. For every $m \geq 0$, denote by $F_m C \subset C$ the kernel of the composition $C \longrightarrow C^{\otimes m+1} \longrightarrow (C/\gamma(k))^{\otimes m+1}$, where $\mu^{(m)} \colon C \longrightarrow C^{\otimes m+1}$ is the iterated comultiplication map and $C^{\otimes m+1} \longrightarrow (C/\gamma(k))^{\otimes m+1}$ is the natural surjection. So one has $F_{-1} C = 0$ and $F_0 C = \gamma(k) \subset C$. One can check that F is a comultiplicative filtration on C, that is,

$$\mu(F_m C) \subset \sum\nolimits_{p+q=m} F_p C \otimes_k F_q C \subset C \otimes_k C$$

for all $m \geq 0$ [39, Section 3.1].

Let L be a left C-comodule. Denote by $F_m L \subset L$ the full preimage of the subspace $F_m C \otimes_k L \subset C \otimes_k L$ under the coaction map $\nu \colon L \longrightarrow C \otimes_k L$. So one has $F_{-1} L = 0$ and $F_0 L = {}_\gamma L$ (in the notation of Section 3). One can check that F is a comultiplicative filtration on L compatible with the filtration F on C, that is,

$$\nu(F_m L) \subset \sum\nolimits_{p+q=m} F_p C \otimes_k F_q L \subset C \otimes_k L$$

for all $m \geq 0$ [19, Section 4.1].

In other words, this means that $F_m L$ is a C-subcomodule of L for every $m \geq 0$ *and* the successive quotient C-comodules $F_m L / F_{m-1} L$ are trivial (i.e., the coaction of C in them is induced by the coaugmentation γ). In fact, one has $F_m L / F_{m-1} L = {}_\gamma(L/F_{m-1} L) \subset L/F_{m-1} L$ for every $m \geq 1$.

By the definition, a coaugmented coalgebra C is conilpotent if and only if $C = \bigcup_{m=0}^{\infty} F_m C$ (i.e., the increasing filtration F on C is exhaustive). In this case, it follows that $L = \bigcup_{m=0}^{\infty} F_m L$, that is the increasing filtration F on any C-comodule is exhaustive as well.

Proof of Theorem 5.1. The argument resembles the proofs of the comodule Ext comparison theorems in [35, Sections 5.4 and 5.7] (see specifically [35, Theorem 5.21]). Notice first of all that the functor Υ is always fully faithful by [44, Propositions 2.1.1–2.1.2 and Theorem 2.1.3(e)].

According to Lemma A.5 from the appendix, in order to prove the theorem it suffices to show that $\operatorname{Ext}_{C^*}^i(L, J) = 0$ for all left C-comodules L, all injective left C-comodules J, and all integers $1 \leq i \leq n$. Equivalently, this means that the space $\operatorname{Ext}_{C^*}^i(L, \, C \otimes_k U)$ should vanish for all k-vector spaces U and $1 \leq i \leq n$.

According to the discussion preceding this proof, the C-comodule L has a natural increasing filtration with (direct sums of) the trivial C-comodule k as the successive quotients. Using the Eklof lemma [6, Lemma 1], the problem reduces to showing that $\operatorname{Ext}_{C^*}^i(k, \, C \otimes_k U) = 0$ for all k-vector spaces U.

Let

$$0 \longrightarrow k \longrightarrow V_0 \otimes_k C \longrightarrow V_1 \otimes_k C \longrightarrow V_2 \otimes_k C \longrightarrow \cdots \qquad (14)$$
$$\longrightarrow V_{n-1} \otimes_k C \longrightarrow V_n \otimes_k C \xrightarrow{\ \tau_n\ } V_{n+1} \otimes_k C \longrightarrow \cdots$$

be a minimal injective/cofree coresolution of the right C-comodule k, as per Corollary 3.6(a) and Proposition 3.7(a). Here V_i, $i \geq 0$, are some k-vector spaces. Computing the Ext spaces $\operatorname{Ext}_{C^{\mathrm{op}}}^*(k, k)$

using the injective coresolution (14), we obtain isomorphisms $V_i \simeq \mathrm{Ext}^i_{C^{\mathrm{op}}}(k, k)$ (in particular, $V_0 \simeq k$). Furthermore, Proposition 4.1 provides isomorphisms $\mathrm{Ext}^i_{C^{\mathrm{op}}}(k, k) \simeq \mathrm{Ext}^i_C(k, k)$. So the vector space V_i is finite-dimensional for $0 \le i \le n$ by assumption.

Applying the dual vector space functor $\mathrm{Hom}_k(-, k)$ to the complex of right C-comodules (14), we obtain a resolution of the left C^*-module k,

$$0 \longleftarrow k \longleftarrow C^* \otimes_k V_0^* \longleftarrow C^* \otimes_k V_1^* \longleftarrow C^* \otimes_k V_2^* \longleftarrow \cdots \qquad (15)$$
$$\longleftarrow C^* \otimes_k V_{n-1}^* \longleftarrow C^* \otimes_k V_n^* \longleftarrow (V_{n+1} \otimes_k C)^* \longleftarrow \cdots$$

The exact complex of left C^*-modules (15) starts and proceeds up to the homological degree n as a resolution by finitely generated projective C^*-modules before turning into a resolution by some arbitrary C^*-modules (in fact, projective left C-contramodules) from the homological degree $n + 1$ on.

For every $0 \le i \le n + 1$ and any left C^*-module J, the space $\mathrm{Ext}^i_{C^*}(k, J)$ can be computed as the degree i cohomology space of the complex obtained by applying the functor $\mathrm{Hom}_{C^*}(-, J)$ to the resolution (15) of the left C^*-module k (see Lemma A.1). We are only interested in $0 \le i \le n$ now, so let us write down the resulting complex in the cohomological degrees from 0 to $n + 1$. It has the form

$$0 \longrightarrow V_0 \otimes_k J \longrightarrow V_1 \otimes_k J \longrightarrow \cdots \longrightarrow V_n \otimes_k J \longrightarrow \mathrm{Hom}_{C^*}((V_{n+1} \otimes_k C)^*, \ J),$$

which for $J = C \otimes_k U$ turns into

$$0 \longrightarrow V_0 \otimes_k C \otimes_k U \longrightarrow V_1 \otimes_k C \otimes_k U \longrightarrow \cdots \qquad (16)$$
$$\longrightarrow V_n \otimes_k C \otimes_k U \xrightarrow{\ \theta\ } \mathrm{Hom}_{C^*}((V_{n+1} \otimes_k C)^*, \ C \otimes_k U).$$

In the cohomological degrees $\le n$, the complex (16) can be simply obtained by applying the vector space tensor product functor $- \otimes_k U$ to the coresolution (14). This follows from Lemma 5.2. Consequently, the complex (16) is exact in the cohomological degrees $0 < i < n$. It is only the complicated rightmost term of (16) that remains to be dealt with.

Finally, the same Lemma 5.2 provides a commutative square diagram of k-linear maps

$$(17)$$

where the leftmost vertical isomorphism is $\eta = \eta_{V_n \otimes_k C, \ U}$, the rightmost vertical monomorphism is $\eta = \eta_{V_{n+1} \otimes_k C, \ U}$, the horizontal arrows are induced by the differential τ_n in the coresolution (14), and the diagonal composition θ is the rightmost differential in the complex (16). Now it is clear from the diagram (17) that the kernel of the map $\theta \colon V_n \otimes_k C \otimes_k U \longrightarrow \mathrm{Hom}_{C^*}((V_{n+1} \otimes_k C)^*, \ C \otimes_k U)$ coincides with the kernel of the map $\tau_n \otimes_k U \colon V_n \otimes_k C \otimes_k U \longrightarrow V_{n+1} \otimes_k C \otimes_k U$ (because the rightmost vertical map η is injective). Thus, exactness of the coresolution (14) implies exactness of the complex (16) in the cohomological degree n as well. □

Corollary 5.3. *Let C be a conilpotent coalgebra over a field k and $n \ge 0$ be an integer such that the vector space $\mathrm{Ext}^i_C(k, k)$ is finite-dimensional for all $1 \le i \le n$. Then the map of Ext spaces*

$$\mathrm{Ext}^i_C(L, M) \longrightarrow \mathrm{Ext}^i_{C^*}(L, M)$$

induced by the comodule inclusion functor $\Upsilon \colon C\text{-}\mathsf{Comod} \longrightarrow C^*\text{-}\mathsf{Mod}$ is injective for all left C-comodules L and M and all $0 \le i \le n + 1$.

Proof. This is a purely formal consequence of Theorem 5.1. See Lemma A.3 in the appendix for the case $n = 0$ and Lemma A.4(a) for the general case. □

6. Comodule Ext isomorphism implies weakly finite Koszulity

The aim of this section is to prove the following theorem.

Theorem 6.1. *Let C be a conilpotent coalgebra over a field k and $n \geq 1$ be the minimal integer for which the vector space $\mathrm{Ext}^n_C(k, k)$ is infinite-dimensional. Then the map $\mathrm{Ext}^n_C(k, k) \longrightarrow \mathrm{Ext}^n_{C^*}(k, k)$ induced by the comodule inclusion functor $\Upsilon: C-\mathsf{Comod} \longrightarrow C^*-\mathsf{Mod}$ is injective, but not surjective. In fact, if λ is the dimension cardinality of the k-vector space $\mathrm{Ext}^n_C(k, k)$ and κ is the cardinality of the field k, then the dimension cardinality of $\mathrm{Ext}^n_{C^*}(k, k)$ is equal to κ^{κ^λ}.*

The following lemma computing the dimension cardinality of the dual vector space is a classical result.

Lemma 6.2. *Let V be an infinite-dimensional vector space of dimension cardinality λ over a field k of cardinality κ. Then the dimension cardinality of the dual k-vector space $V^* = \mathrm{Hom}_k(V, k)$ is equal to κ^λ.*

Proof. This is [11, Section IX.5]. □

Let C be a conilpotent coalgebra. Then the vector space $\mathrm{Ext}^1_C(k, k)$ can be interpreted as the *space of cogenerators* of the coalgebra C (we refer to [26, Lemma 5.2] for a discussion). So we will say that a conilpotent coalgebra C is *finitely cogenerated* if the vector space $\mathrm{Ext}^1_C(k, k)$ is finite-dimensional.

For any coaugmented coalgebra (C, γ), the dual linear map $\gamma^*: C^* \longrightarrow k$ to the coaugmentation $\gamma: k \longrightarrow C$ defines an augmentation on the algebra C^*. Accordingly, the one-dimensional vector space k can be endowed with left and right C^*-module structures provided by the augmentation γ^*. These C^*-module structures on k can be also viewed as coming from the left and right C-comodule structures on k induced by the coaugmentation γ. The same module structures also come from the left and right C-contramodule structures on k induced by γ.

We start with a discussion of the case $n = 1$ in Theorem 6.1.

Proposition 6.3. *Let C be a conilpotent coalgebra over a field k. Then there is a natural k-vector space monomorphism*

$$\mathrm{Ext}^1_C(k, k)^{**} \rightarrowtail \mathrm{Ext}^1_{C^*}(k, k). \tag{18}$$

The dimension cardinality of the vector space $\mathrm{Ext}^1_{C^*}(k, k)$ is equal to that of the vector space $\mathrm{Ext}^1_C(k, k)^{**}$.

Proof. Notice first of all the isomorphism $\mathrm{Ext}^*_{C^{\mathrm{op}}}(k, k) \simeq \mathrm{Ext}^*_C(k, k)$ provided by Proposition 4.1. Furthermore, let A be an augmented k-algebra with an augmentation $\alpha: A \longrightarrow k$ and the augmentation ideal $A_+ = \ker(\alpha) \subset A$. Then the vector space $\mathrm{Ext}^1_A(k, k) \simeq \mathrm{Tor}^A_1(k, k)^* \simeq \mathrm{Ext}^1_{A^{\mathrm{op}}}(k, k)$ is computed as the kernel of the k-linear map

$$A_+^* \longrightarrow (A_+ \otimes_k A_+)^*$$

dual to the multiplication map $A_+ \otimes_k A_+ \longrightarrow A_+$. In particular, the vector space $\mathrm{Ext}^1_{C^*}(k, k)$ is naturally isomorphic to the kernel of the map

$$C_+^{**} \longrightarrow (C_+^* \otimes_k C_+^*)^* \tag{19}$$

dual to the multiplication map $C_+^* \otimes_k C_+^* \longrightarrow C_+^*$ (where $C_+ = C/\gamma(k)$).

On the other hand, from the cobar complex (6) one can immediately see that the vector space $\mathrm{Ext}^1_C(k, k)$ is naturally isomorphic to the kernel of the comultiplication map

$$C_+ \longrightarrow C_+ \otimes_k C_+.$$

Hence, the double dual vector space $\operatorname{Ext}^1_C(k,k)^{**}$ is naturally isomorphic to the kernel of the map

$$C_+^{**} \longrightarrow (C_+ \otimes_k C_+)^{**}. \tag{20}$$

Notice that $(C_+^* \otimes_k C_+^*)$ is naturally a subspace in $(C_+ \otimes_k C_+)^*$; hence $(C_+^* \otimes_k C_+^*)^*$ is a quotient space of $(C_+ \otimes_k C_+)^{**}$. One can also observe that the cokernel of the map $(C_+ \otimes_k C_+)^* \longrightarrow C_+^*$ is naturally a quotient space of the cokernel of the map $C_+^* \otimes_k C_+^* \longrightarrow C_+^*$. Comparing the maps (19) and (20), one immediately obtains the desired monomorphism (18).

To compute the dimension cardinality, put $V = \operatorname{Ext}^1_C(k,k)$, and notice that C is a subcoalgebra of the cofree conilpotent (tensor) coalgebra $\operatorname{Ten}(V) = \bigoplus_{n=0}^\infty V^{\otimes n}$ cospanned by V [26, Lemma 5.2] (see [36, Sections 2.3 and 3.3] for an introductory discussion). Hence, if V is infinite-dimensional, then the dimension cardinalities of V and C_+ are equal to each other. Now V^{**} is a subspace in $\operatorname{Ext}^1_{C^*}(k,k)$, which is in turn a subspace in C_+^{**}. Thus, the dimension cardinalities of all the three vector spaces V^{**}, $\operatorname{Ext}^1_{C^*}(k,k)$, and C_+^{**} are equal to each other. If V is finite-dimensional, then the map (18) is an isomorphism by Theorem 5.1 (for $n=1$). □

Example 6.4. *Let C be an infinitely cogenerated conilpotent coalgebra. Interpreted in the light of Lemma A.3 from the appendix, Proposition 6.3 tells us that there exists a short exact sequence of left C^*-modules*

$$0 \longrightarrow k \longrightarrow M \longrightarrow k \longrightarrow 0 \tag{21}$$

with the two-dimensional C^*-module M not coming from any C-comodule (while the one-dimensional C^*-module k, of course, comes from the one-dimensional C-comodule k) via the comodule inclusion functor Υ. Let us explain how to construct a short exact sequence (21) explicitly.

Let C be a conilpotent coalgebra. Recall the notation $F_0C = \gamma(k)$ and $F_1C = \ker(C \to C_+ \otimes_k C_+)$ from Section 5. Then we have a natural direct sum decomposition $F_1C = F_0C \oplus V$, where $V = \ker(C_+ \to C_+ \otimes_k C_+) = \operatorname{Ext}^1_C(k,k)$. Choose a linear function $f\colon V^* \longrightarrow k$. The composition of linear maps $C^* \twoheadrightarrow (F_1C)^* \twoheadrightarrow V^* \xrightarrow{f} k$ defines a linear function $\tilde{f}\colon C^* \longrightarrow k$.

Let M be the two-dimensional k-vector space with the basis vectors e_1 and e_2. Define the left action of C^* in M by the formulas $ae_1 = \gamma^*(a)e_1$ and $ae_2 = \gamma^*(a)e_2 + \tilde{f}(a)e_1$ for all $a \in C^*$ (where $\gamma^*\colon C^* \longrightarrow k$ is the dual map to the coaugmentation γ). Then $ke_1 \subset M$ is a C^*-submodule of M isomorphic to $\Upsilon(k)$ and M/ke_1 is a quotient module of M also isomorphic to $\Upsilon(k)$. The C^*-module M itself belongs to the essential image of the functor Υ if and only if the linear function $f\colon V^* \longrightarrow k$ comes from a vector in V.

Before passing to the case $n > 1$ in Theorem 6.1, we need a preparatory lemma.

Lemma 6.5. *Let (C, γ) be a coaugmented coalgebra over k and N be a right C-comodule. Then there is a natural monomorphism of k-vector spaces*

$$\xi_N\colon (N_\gamma)^{**} \longrightarrow \operatorname{Hom}_{C^*}(N^*, k)$$

from the double dual vector space to the vector subspace $N_\gamma \subset N$ to the vector space of left C^*-module morphisms $N^* \longrightarrow k$. The map ξ_N is an isomorphism whenever the coalgebra C is conilpotent and finitely cogenerated.

Proof. The inclusion $N_\gamma \longrightarrow N$ is a morphism of right C-comodules. Consequently, the k-vector space dual map $N^* \longrightarrow (N_\gamma)^*$ is an epimorphism of left C^*-modules (and in fact, of left C-contramodules), where the C^*-module (or C-contramodule) structure on $(N_\gamma)^*$ is induced by the (co)augmentation. Applying the functor $\operatorname{Hom}_{C^*}(-, k)$, we obtain the desired injective map of k-vector spaces

$$\xi_N : (N_\gamma)^{**} = \operatorname{Hom}_{C^*}((N_\gamma)^*, k) \longrightarrow \operatorname{Hom}_{C^*}(N^*, k).$$

If the coalgebra C is conilpotent and finitely cogenerated, then the contramodule forgetful functor $C-\mathsf{Contra} \longrightarrow C^*-\mathsf{Mod}$ from the category of left C-contramodules to the category of left C^*-modules is fully faithful [30, Theorem 2.1] (see Theorem 7.4 below). Consequently, we have $\mathrm{Hom}_{C^*}(N^*, k) = \mathrm{Hom}^C(N^*, k)$.

It remains to observe that the natural map

$$(N_\gamma)^{**} \longrightarrow \mathrm{Hom}^C(N^*, k) \tag{22}$$

is an isomorphism for any coaugmented coalgebra (C, γ). Indeed, we have

$$\mathrm{Hom}^C(N^*, k) \simeq \mathrm{Hom}_k(^\gamma(N^*), k) \simeq \mathrm{Hom}_k((N_\gamma)^*, k) = (N_\gamma)^{**},$$

because $^\gamma(N^*) \simeq (N_\gamma)^*$. The latter isomorphism is a particular case of the formula (4) in Section 3. Alternatively, one can obtain the isomorphism (22) as a particular case of the isomorphism (7) from Lemma 4.2 for $L = k$ and $M = N$. □

Proposition 6.6. *Let C be a finitely cogenerated conilpotent coalgebra over a field k and $n \geq 2$ an integer such that the vector space $\mathrm{Ext}^i_C(k, k)$ is finite-dimensional for all $1 \leq i \leq n-1$. Then there is a natural k-vector space isomorphism*

$$\mathrm{Ext}^n_C(k, k)^{**} \simeq \mathrm{Ext}^n_{C^*}(k, k). \tag{23}$$

Proof. Similarly to the proof of Theorem 5.1, we consider a minimal injective/cofree coresolution (14) of the right C-comodule k. The assumption of the proposition implies that the vector spaces $V_i \simeq \mathrm{Ext}^i_C(k, k)$ are finite-dimensional for all $0 \leq i \leq n-1$. Applying the dual vector space functor $\mathrm{Hom}_k(-, k)$ to the complex (14), we obtain a resolution of the left C^*-module k,

$$0 \longleftarrow k \longleftarrow C^* \otimes_k V_0^* \longleftarrow C^* \otimes_k V_1^* \longleftarrow C^* \otimes_k V_2^* \longleftarrow \cdots \tag{24}$$
$$\longleftarrow C^* \otimes_k V_{n-1}^* \longleftarrow (V_n \otimes_k C)^* \longleftarrow (V_{n+1} \otimes_k C)^* \longleftarrow \cdots$$

(cf. formula (15), which holds under the stricter assumptions of Theorem 5.1).

Lemma A.1 from the appendix tells us that, for every $0 \leq i \leq n$ and any left C^*-module M, the space $\mathrm{Ext}^i_{C^*}(k, M)$ can be computed as the degree i cohomology space of the complex obtained by applying the functor $\mathrm{Hom}_{C^*}(-, M)$ to the resolution (24) of the left C^*-module k. For $M = k$, the resulting complex turns into

$$0 \longrightarrow V_0 \longrightarrow V_1 \longrightarrow \cdots \longrightarrow V_{n-1} \tag{25}$$
$$\longrightarrow \mathrm{Hom}_{C^*}((V_n \otimes_k C)^*, \ k) \longrightarrow \mathrm{Hom}_{C^*}((V_{n+1} \otimes_k C)^*, k).$$

By Lemma 6.5, the complex (25) is naturally isomorphic to the complex

$$0 \longrightarrow V_0 \longrightarrow V_1 \longrightarrow \cdots \longrightarrow V_{n-1} \longrightarrow V_n^{**} \longrightarrow V_{n+1}^{**} \tag{26}$$

obtained by applying the functor $((-)_\gamma)^{**}$ to the coresolution (14). As the latter coresolution was chosen to be minimal, the differential in the complex (26) (or, which is the same, in the complex (25)) vanishes. Thus, $\mathrm{Ext}^n_{C^*}(k, k) \simeq V_n^{**} \simeq \mathrm{Ext}^n_C(k, k)^{**}$.

This proves existence of *an* isomorphism (23). To show that this isomorphism is *natural*, consider an arbitrary injective coresolution

$$0 \longrightarrow k \longrightarrow J^0 \longrightarrow J^1 \longrightarrow J^2 \longrightarrow \cdots \tag{27}$$

of the right C-comodule k. Then

$$0 \longleftarrow k \longleftarrow (J^0)^* \longleftarrow (J^1)^* \longleftarrow (J^2)^* \longleftarrow \cdots \tag{28}$$

is a (nonprojective) resolution of the left C^*-module k. However, the coresolution (27) is homotopy equivalent (as a complex of injective right C-comodules) to the minimal coresolution (14). Hence, the resolution (28) is homotopy equivalent (as a complex of left C^*-modules) to the initially projective

resolution (24). Consequently, the natural map (47) from Lemma A.1 for the nonprojective resolution (28) is naturally isomorphic to the similar map for the initially projective resolution (24).

Thus, one can compute $\mathrm{Ext}^i_{C^*}(k, k)$ for $0 \leq i \leq n$ using the nonprojective resolution (28), obtaining natural isomorphisms

$$\mathrm{Ext}^i_{C^*}(k, k) \simeq H^i \mathrm{Hom}_{C^*}((J^\bullet)^*, k) \simeq H^i((J^\bullet_\gamma)^{**}) \simeq (H^i(J^\bullet_\gamma))^{**} \simeq \mathrm{Ext}^i_C(k, k)^{**}$$

(with the second isomorphism provided by Lemma 6.5) for all $0 \leq i \leq n$. $\qquad\square$

Proof of Theorem 6.1. The map $\mathrm{Ext}^n_C(L, M) \longrightarrow \mathrm{Ext}^n_{C^*}(L, M)$ is injective for all left C-comodules L and M in our assumptions by Corollary 5.3. The dimension cardinality assertion of the theorem is provided by Proposition 6.3 (for $n = 1$) or Proposition 6.6 (for $n \geq 2$) together with Lemma 6.2. The nonsurjectivity assertion follows from the dimension inequality. $\qquad\square$

Remark 6.7. *Arguing more carefully, one can show that, in the assumptions of Proposition 6.6, the isomorphism (23) identifies the map $\mathrm{Ext}^n_C(k, k) \longrightarrow \mathrm{Ext}^n_{C^*}(k, k)$ induced by the comodule inclusion functor $\Upsilon\colon C-\mathsf{Comod} \longrightarrow C^*-\mathsf{Mod}$ with the natural inclusion $\mathrm{Ext}^n_C(k, k) \longrightarrow \mathrm{Ext}^n_{C^*}(k, k)^{**}$ of a vector space $\mathrm{Ext}^n_C(k, k)$ into its double dual vector space. For this purpose, one can start with comparing the proofs of Propositions 4.3 and 6.6 in order to observe that the forgetful functor $\Theta\colon C-\mathsf{Contra} \longrightarrow C^*-\mathsf{Mod}$ induces an isomorphism $\mathrm{Ext}^{C,n}(k, k) \simeq \mathrm{Ext}^n_{C^*}(k, k)$ in the assumptions of Proposition 6.6. Then the result of Proposition 4.3 identifies the map $\mathrm{Ext}^n_{C^{op}}(k, k) \longrightarrow \mathrm{Ext}^n_{C^*}(k, k)$ induced by the dual vector space functor $N \longmapsto N^*\colon \mathsf{Comod}-C \longrightarrow C^*-\mathsf{Mod}$ with the natural inclusion $\mathrm{Ext}^n_{C^{op}}(k, k) \longrightarrow \mathrm{Ext}^n_{C^{op}}(k, k)^{**}$. It remains to restrict the consideration to finite-dimensional comodules and use the observations from the second proof of Proposition 4.1.*

7. Weakly finite Koszulity implies contramodule Ext isomorphism

Before stating the main theorem of this section, let us have a discussion of *separated* and *nonseparated* contramodules.

Let $C = (C, \gamma)$ be a coaugmented coalgebra and P be a left C-contramodule. For every $m \geq 0$, denote by $F^m P \subset P$ the image of the subspace $\mathrm{Hom}_k(C/F_{m-1}C, P) \subset \mathrm{Hom}_k(C, P)$ under the contraaction map $\pi\colon \mathrm{Hom}_k(C, P) \longrightarrow P$. So one has $F^0 P = P$ and $F^1 P = \ker(P \twoheadrightarrow {}^\gamma P)$ (in the notation of Sections 3 and 5). One can check that F is a contramultiplicative decreasing filtration on P compatible with the increasing filtration F on C, that is

$$\pi(\mathrm{Hom}_k(C/F_{q-1}C, F^p P)) \subset F^{p+q} P$$

for all $p, q \geq 0$.

In other words, this means that $F^m P$ is a C-subcontramodule of P for every $m \geq 0$ *and* the successive quotient C-contramodules $F^m P/F^{m+1} P$ are trivial (i.e., the contraaction of C in them is induced by the coaugmentation γ). In fact, one has $F^m P/F^{m+1} P = {}^\gamma(F^m P)$ for every $m \geq 0$.

The problem with the decreasing filtration F on P is that it is *not* in general separated (not even when C is a conilpotent coalgebra); see the counterexample in [24, Section 1.5] (going back to [42, Example 2.5], [48, Example 3.20], and [20, Section A.1.1]). The next lemma explains how this problem can be partially rectified (for a conilpotent coalgebra C).

We say that a vector space P endowed with a decreasing filtration F is *separated* if the natural map to the projective limit

$$\lambda_{P,F}\colon P \longrightarrow \varprojlim_{m \geq 1} P/F^m P$$

is injective, and that P is *complete* if the map $\lambda_{P,F}$ is surjective. A contramodule P over a conilpotent coalgebra C is said to be *separated* (respectively, *complete*) if it is separated (resp., complete) with respect to its natural decreasing filtration F constructed above.

Notice that, for any subcontramodule Q in a contramodule P, one has $F^m Q \subset F^m P$ for all $m \geq 0$. Thus, any subcontramodule of a separated contramodule is separated. For a free contramodule $P = \mathrm{Hom}_k(C, V)$, one has $F^m P = \mathrm{Hom}_k(C/F_{m-1}C, V)$. Hence, all free (and therefore, all projective) contramodules over a conilpotent coalgebra are separated and complete.

Lemma 7.1. *Let C be a conilpotent coalgebra. Then*

(a) any C-contramodule is complete;

(b) any C-contramodule can be presented as the quotient contramodule of a separated contramodule by a separated subcontramodule.

Proof. Part (b) holds because any contramodule is a quotient of a free contramodule, which is separated, and any subcontramodule of a separated contramodule is separated. For part (a), see [20, Lemma A.2.3]. $\qquad\square$

The aim of this section is to prove the following theorem.

Theorem 7.2. *Let C be a conilpotent coalgebra over a field k and $n \geq 1$ be an integer such that the vector space $\mathrm{Ext}^i_C(k, k)$ is finite-dimensional for all $1 \leq i \leq n$. Then the map of the Ext spaces*

$$\mathrm{Ext}^{C,i}(P, Q) \longrightarrow \mathrm{Ext}^i_{C^*}(P, Q) \tag{29}$$

induced by the forgetful functor $\Theta \colon C-\mathsf{Contra} \longrightarrow C^*-\mathsf{Mod}$ is an isomorphism for all left C-contramodules P, all separated left C-contramodules Q, and all $0 \leq i \leq n - 1$. The map (29) is also an isomorphism for all left C-contramodules P and Q and all $0 \leq i \leq n - 2$.

The following lemma, describing some tensor products of C^*-modules for a coalgebra C, is a partial dual version of Lemma 5.2.

Lemma 7.3. *Let C be a coalgebra over k, let N be a left C-comodule, and let U be a k-vector space. Then there is a natural map of k-vector spaces*

$$\zeta_{N,U} \colon N^* \otimes_{C^*} \mathrm{Hom}_k(C, U) \longrightarrow \mathrm{Hom}_k(N, U)$$

which is an isomorphism whenever N is a finitely cogenerated injective C-comodule. Here, the left C^*-module structure on $\mathrm{Hom}_k(C, U)$ comes from the free left C-contramodule structure, or equivalently, is induced by the right C^*-module structure on C (coming from the right C-comodule structure on C).

Proof. The map $\zeta_{N,U}$ is defined by the formula

$$\zeta_{N,U}(f \otimes g)(x) = f(x_{(0)}) g(x_{(-1)})$$

for all $f \in N^*$, $g \in \mathrm{Hom}_k(C, U)$, and $x \in N$. Here, $\nu(x) = x_{(-1)} \otimes x_{(0)} \in C \otimes_k N$ is a notation for the left C-coaction map $\nu \colon N \longrightarrow C \otimes_k N$. So $f(x_{(0)}) \in k$ and $g(x_{(-1)}) \in U$.

To prove the isomorphism assertion for a finitely cogenerated injective C-comodule N, it suffices to consider the case $N = C \otimes_k V$, where $\dim_k V < \infty$. In this case, $N^* = V^* \otimes_k C^*$ is a free right C^*-module, so $N^* \otimes_{C^*} \mathrm{Hom}_k(C, U) = V^* \otimes_k \mathrm{Hom}_k(C, U) = \mathrm{Hom}_k(V, \mathrm{Hom}_k(C, U)) = \mathrm{Hom}_k(C \otimes_k V, U)$. $\qquad\square$

The most difficult part of the proof of Theorem 7.2 is the case of $n = 1$. This is a previously known result [30]. Notice that, according to the following theorem, no separatedness assumptions are needed in Theorem 7.2 for $n = 1$.

Theorem 7.4. *Let C be a finitely cogenerated conilpotent coalgebra over a field k. Then the contramodule forgetful functor $\Theta \colon C-\mathsf{Contra} \longrightarrow C^*-\mathsf{Mod}$ is fully faithful.*

Proof. Moreover, the forgetful functor $C-\mathsf{Contra} \longrightarrow R-\mathsf{Mod}$ (defined as the composition $C-\mathsf{Contra} \longrightarrow C^*-\mathsf{Mod} \longrightarrow R-\mathsf{Mod}$) is fully faithful for any dense subring $R \subset C^*$ in the natural pro-finite-dimensional (otherwise known as linearly compact or pseudocompact) topology of the k-algebra C^*. See [30, Theorem 2.1]. For a discussion of related results with further references, see [24, Section 3.8]. □

Proof of Theorem 7.2. The argument resembles the proofs of the contramodule Ext comparison theorems in [35, Sections 5.4 and 5.7] (see specifically [35, Theorem 5.20]). Notice first of all that the coalgebra C is finitely cogenerated in the assumptions of Theorem 7.2, so the functor Θ is fully faithful by Theorem 7.4.

To prove the first assertion of the theorem, let Q be a separated left C-contramodule. According to Lemma A.6, it suffices to show that $\mathrm{Ext}^i_{C^*}(P, Q) = 0$ for all projective left C-contramodules P and all integers $1 \le i \le n-1$.

Lemma 7.1(a) says that the C-contramodule Q is complete with respect to its natural decreasing filtration F; so it is separated and complete under our assumptions. According to the discussion preceding the lemma, the successive quotient contramodules $F^m Q / F^{m-1} Q$ have trivial C-contramodule structures (i.e., their C-contramodule structures are induced by the coaugmentation γ). Using the dual Eklof lemma [6, Proposition 18], the question reduces to showing that $\mathrm{Ext}^i_{C^*}(P, T) = 0$ for all k-vector spaces T with trivial C-contramodule structures.

Any vector space T is a direct summand of the dual vector space W^* to some vector space W. Thus it suffices to show that the vector space $\mathrm{Ext}^i_{C^*}(P, W^*) \simeq \mathrm{Tor}^{C^*}_i(W, P)^*$ vanishes. Finally, W is a direct sum of copies of the one-dimensional vector space k, and Tor commutes with the direct sums. So we need to show that $\mathrm{Tor}^{C^*}_i(k, P) = 0$ for any projective left C-contramodule P and $1 \le i \le n-1$ (and the trivial right C^*-module structure on k). Equivalently, this means that the space $\mathrm{Tor}^{C^*}_i(k, \mathrm{Hom}_k(C, U))$ should vanish for all k-vector spaces U and $1 \le i \le n-1$.

This time, we consider a minimal injective/cofree coresolution of the left C-comodule k,

$$0 \longrightarrow k \longrightarrow C \otimes_k V_0 \longrightarrow C \otimes_k V_1 \longrightarrow C \otimes_k V_2 \longrightarrow \cdots \tag{30}$$
$$\longrightarrow C \otimes_k V_{n-1} \longrightarrow C \otimes_k V_n \longrightarrow C \otimes_k V_{n+1} \longrightarrow \cdots,$$

and apply the dual vector space functor $\mathrm{Hom}_k(-, k)$ to it in order to obtain an initially projective resolution of the right C^*-module k,

$$0 \longleftarrow k \longleftarrow V_0^* \otimes_k C^* \longleftarrow V_1^* \otimes_k C^* \longleftarrow V_2^* \otimes_k C^* \longleftarrow \cdots \tag{31}$$
$$\longleftarrow V_{n-1}^* \otimes_k C^* \longleftarrow V_n^* \otimes_k C^* \longleftarrow (C \otimes_k V_{n+1})^* \longleftarrow \cdots$$

Here the vector space $V_i \simeq \mathrm{Ext}^i_C(k, k)$ is finite-dimensional for $0 \le i \le n$ by assumption. For every $0 \le i \le n+1$ and any left C^*-module P, the space $\mathrm{Tor}^{C^*}_i(k, P)$ can be computed as the degree i homology space of the complex obtained by applying the functor $- \otimes_{C^*} P$ to the resolution (31) of the right C^*-module k (see Lemma A.2). We are only interested in $0 \le i \le n-1$ now, so we only write down the resulting complex in the cohomological degrees from 0 to n. It has the form

$$0 \longleftarrow V_0^* \otimes_k P \longleftarrow V_1^* \otimes_k P \longleftarrow \cdots \longleftarrow V_{n-1}^* \otimes_k P \longleftarrow V_n^* \otimes_k P,$$

which for $P = \mathrm{Hom}_k(C, U)$ turns into

$$0 \longleftarrow \mathrm{Hom}_k(V_0, \mathrm{Hom}_k(C, U)) \longleftarrow \mathrm{Hom}_k(V_1, \mathrm{Hom}_k(C, U)) \longleftarrow \cdots \tag{32}$$
$$\longleftarrow \mathrm{Hom}_k(V_{n-1}, \mathrm{Hom}_k(C, U)) \longleftarrow \mathrm{Hom}_k(V_n, \mathrm{Hom}_k(C, U)).$$

By Lemma 7.3, the complex (32) (in the cohomological degrees $\le n$) can be simply obtained by applying the contravariant vector space Hom functor $\mathrm{Hom}_k(-, U)$ to the coresolution (30). Consequently, the complex (32) is exact in the cohomological degrees $0 < i \le n-1$. Therefore, $\mathrm{Tor}^{C^*}_i(k, \mathrm{Hom}_k(C, U)) = 0$ for $0 < i \le n-1$, as desired.

Having proved the first assertion of the theorem, we can now easily deduce the second one. For a nonseparated left C-contramodule Y, by Lemma A.6 we only need to show that $\mathrm{Ext}^i_{C^*}(P, Y) = 0$

for all projective left C-contramodules P and $1 \leq i \leq n-2$. Lemma 7.1(b) provides a short exact sequence $0 \longrightarrow Q' \longrightarrow Q'' \longrightarrow Y \longrightarrow 0$ with separated C-contramodules Q' and Q''. In the long exact sequence $\cdots \longrightarrow \mathrm{Ext}^i_{C^*}(P, Q'') \longrightarrow \mathrm{Ext}^i_{C^*}(P, Y) \longrightarrow \mathrm{Ext}^{i+1}_{C^*}(P, Q') \longrightarrow \cdots$, we have $\mathrm{Ext}^i_{C^*}(P, Q'') = 0 = \mathrm{Ext}^{i+1}_{C^*}(P, Q')$, hence $\mathrm{Ext}^i_{C^*}(P, Y) = 0$. $\qquad\square$

Corollary 7.5. *Let C be a conilpotent coalgebra over a field k and $n \geq 0$ be an integer such that the vector space $\mathrm{Ext}^i_C(k, k)$ is finite-dimensional for all $1 \leq i \leq n$. Then the map of the Ext spaces*

$$\mathrm{Ext}^{C,i}(P, Q) \longrightarrow \mathrm{Ext}^i_{C^*}(P, Q) \tag{33}$$

induced by the forgetful functor $\Theta\colon C{-}\mathsf{Contra} \longrightarrow C^*{-}\mathsf{Mod}$ is injective for all left C-contramodules P, all separated left C-contramodules Q, and all $0 \leq i \leq n$. The map (33) *is also injective for all left C-contramodules P and Q and all $0 \leq i \leq n-1$.*

Proof. No separatedness assumption is needed in the obvious case $n = 0$. For $n \geq 1$, this is a purely formal consequence of Theorem 7.2. See Lemma A.3 for the case $n = 1$ (when the separatedness assumption is not needed, either, by virtue of Theorem 7.4) and Lemma A.4(a) for the general case. $\qquad\square$

8. Contramodule Ext isomorphism implies weakly finite Koszulity

The aim of this section is to prove the following theorem. Together with the results of the previous Sections 5–7, this will allow us to finish the proof of Theorem 1.1 from the introduction.

Theorem 8.1. *Let C be a conilpotent coalgebra over a field k and $n \geq 1$ be the minimal integer for which the vector space $\mathrm{Ext}^n_C(k, k)$ is infinite-dimensional. Let T be an infinite-dimensional k-vector space endowed with the trivial left C-contramodule structure (provided by the coaugmentation $\gamma\colon C \longrightarrow k$). Then the map $\mathrm{Ext}^{C,n}(T, k) \longrightarrow \mathrm{Ext}^n_{C^*}(T, k)$ induced by the contramodule forgetful functor $\Theta\colon C{-}\mathsf{Contra} \longrightarrow C^*{-}\mathsf{Mod}$ is not injective. Consequently, there exists a projective left C-contramodule P such that the map $\mathrm{Ext}^{C,n-1}(P, k) \longrightarrow \mathrm{Ext}^{n-1}_{C^*}(P, k)$ induced by the functor Θ is (injective, but) not surjective.*

We start with a discussion of the case $n = 1$ in Theorem 8.1. This is the only case when $\mathrm{Ext}^{C,n-1}(P, k) \neq 0$.

Example 8.2. *Let C be an infinitely cogenerated conilpotent coalgebra over k, and let T be an infinite-dimensional vector space endowed with the trivial left C-contramodule structure. Then we claim that there exists a nonsplit short exact sequence of left C-contramodules*

$$0 \longrightarrow k \longrightarrow Q \longrightarrow T \longrightarrow 0$$

that splits as a short exact sequence of left C^*-modules. Consequently, the splitting map $q\colon Q \longrightarrow k$ is a morphism of C^*-modules but not a morphism of C-contramodules. So the forgetful functor Θ is not full.

To construct the desired short exact sequence, put $C_+ = C/\gamma(k)$ and $V = \ker(C_+ \to C_+ \otimes_k C_+) = \mathrm{Ext}^1_C(k, k)$, as in Example 6.4. So we have $F_0 C = \gamma(k)$ and $F_1 C = F_0 C \oplus V$. Notice that $V^* \otimes_k T$ is naturally a proper vector subspace in $\mathrm{Hom}_k(V, T)$, as both the vector spaces V and T are infinite-dimensional. Essentially, $V^* \otimes_k T \subset \mathrm{Hom}_k(V, T)$ is the subspace of all linear maps $V \longrightarrow T$ of finite rank. Choose a linear function $\phi\colon \mathrm{Hom}_k(V, T) \longrightarrow k$ vanishing on $V^* \otimes_k T$.

Let the underlying vector space of Q be simply the direct sum $k \oplus T$. Define the contraaction map $\pi\colon \mathrm{Hom}_k(C, Q) \longrightarrow Q$ as follows. The component $\pi_{k,T}\colon \mathrm{Hom}_k(C, k) \longrightarrow T$ of the map π is zero. The component $\pi_{k,k}\colon \mathrm{Hom}_k(C, k) \longrightarrow k$ is induced by the coaugmentation γ. The component $\pi_{T,T}\colon \mathrm{Hom}_k(C, T) \longrightarrow T$ is also induced by γ; specifically, it is defined as the composition

$\mathrm{Hom}_k(C,T) \longrightarrow \mathrm{Hom}_k(k,T) \longrightarrow T$ *of the surjective map* $\mathrm{Hom}_k(\gamma,T)\colon \mathrm{Hom}_k(C,T) \longrightarrow \mathrm{Hom}_k(k,T)$ *and the identity isomorphism* $\mathrm{Hom}_k(k,T) \simeq T$.

Finally, the component $\pi_{T,k}\colon \mathrm{Hom}_k(C,T) \longrightarrow k$ *of the map* π *is defined as the composition* $\mathrm{Hom}_k(C,T) \longrightarrow \mathrm{Hom}_k(F_1 C,T) \longrightarrow \mathrm{Hom}_k(V,T) \longrightarrow k$ *of the surjective map* $\mathrm{Hom}_k(C,T) \longrightarrow \mathrm{Hom}_k(F_1 C,T)$ *induced by the inclusion* $F_1 C \longrightarrow C$, *the direct summand projection* $\mathrm{Hom}_k(F_1 C,T) = \mathrm{Hom}_k(F_0 C \oplus V,\ T) \longrightarrow \mathrm{Hom}_k(V,T)$, *and the chosen linear function* $\phi\colon \mathrm{Hom}_k(V,T) \longrightarrow k$.

Then the direct sum decomposition of vector spaces $Q = k \oplus T$ *still holds as a direct sum decomposition in the module category* $C^*-\mathsf{Mod}$, *since the composition of the embedding* $C^* \otimes_k T \longrightarrow \mathrm{Hom}_k(C,T)$ *with the map* $\pi_{T,k}\colon \mathrm{Hom}_k(C,T) \longrightarrow k$ *is the zero map. But the* C-*contramodule structure on* Q *is nontrivial (not induced by the coaugmentation* γ *), so* Q *is not isomorphic to* $k \oplus T$ *in* $C-\mathsf{Contra}$.

In order to pass from some object Q in Example 8.2 to a projective object P in the context Theorem 8.1 for $n = 1$, we will use Lemma A.7 from the appendix.

In the case $n > 1$, we need the following simple coalgebra-theoretic lemma.

Lemma 8.3. *Let* (C,γ) *be a coaugmented coalgebra, N be a right C-comodule, and T be a k-vector space. Then there is a natural isomorphism of k-vector spaces*

$$\mathrm{Hom}^C(\mathrm{Hom}_k(N,T),k) \simeq \mathrm{Hom}_k(N_\gamma,T)^*.$$

Proof. One has $^\gamma\mathrm{Hom}_k(N,T) \simeq \mathrm{Hom}_k(N_\gamma,T)$ by formula (4) from Section 3. Hence

$$\mathrm{Hom}^C(\mathrm{Hom}_k(N,T),k) \simeq \mathrm{Hom}_k(^\gamma\mathrm{Hom}_k(N,T),k) \simeq \mathrm{Hom}_k(\mathrm{Hom}_k(N_\gamma,T),k)$$

as desired (cf. Lemma 6.5). □

Proposition 8.4. *Let C be a finitely cogenerated conilpotent coalgebra over a field k and $n \geq 2$ be the minimal integer for which the vector space* $\mathrm{Ext}_C^n(k,k)$ *is infinite-dimensional. Let T be an infinite-dimensional k-vector space endowed with the trivial left C-contramodule structure (provided by the coaugmentation γ). Then the map* $\mathrm{Ext}^{C,n}(T,k) \longrightarrow \mathrm{Ext}_{C^*}^n(T,k)$ *induced by the contramodule forgetful functor* $\Theta\colon C-\mathsf{Contra} \longrightarrow C^*-\mathsf{Mod}$ *is surjective, but not injective. Consequently, one has* $\mathrm{Ext}_{C^*}^{n-1}(P,k) \neq 0$ *for the projective/free left C-contramodule* $P = \mathrm{Hom}_k(C,T)$.

Proof. Our strategy of proving the first assertion of the proposition is to explicitly compute the map of Ext spaces in question, and see that it is surjective but not injective. As in the proof of Theorem 5.1, we choose a minimal injective/cofree coresolution (14) of the right C-comodule k,

$$0 \longrightarrow k \longrightarrow V_0 \otimes_k C \longrightarrow V_1 \otimes_k C \longrightarrow V_2 \otimes_k C \longrightarrow \cdots \tag{34}$$
$$\longrightarrow V_{n-1} \otimes_k C \longrightarrow V_n \otimes_k C \longrightarrow V_{n+1} \otimes_k C \longrightarrow \cdots$$

The vector space $V_i \simeq \mathrm{Ext}_{C^{op}}^i(k,k) \simeq \mathrm{Ext}_C^i(k,k)$ is finite-dimensional for $0 \leq i \leq n-1$ and infinite-dimensional for $i = n$ by assumption.

Applying the contravariant vector space Hom functor $\mathrm{Hom}_k(-,T)$ to the complex (34), we obtain a minimal projective resolution of the left C-contramodule T,

$$0 \longleftarrow T \longleftarrow \mathrm{Hom}_k(C,\mathrm{Hom}_k(V_0,T)) \longleftarrow \mathrm{Hom}_k(C,\mathrm{Hom}_k(V_1,T)) \longleftarrow \cdots \tag{35}$$
$$\longleftarrow \mathrm{Hom}_k(C,\mathrm{Hom}_k(V_{n-1},T)) \longleftarrow \mathrm{Hom}_k(C,\mathrm{Hom}_k(V_n,T))$$
$$\longleftarrow \mathrm{Hom}_k(C,\mathrm{Hom}_k(V_{n+1},T)) \longleftarrow \cdots$$

On the other hand, applying the dual vector space functor $\mathrm{Hom}_k(-,k)$ to the complex (34), we obtain a resolution of the left C^*-module k

$$0 \longleftarrow k \longleftarrow C^* \otimes_k V_0^* \longleftarrow C^* \otimes_k V_1^* \longleftarrow C^* \otimes_k V_2^* \longleftarrow \cdots \tag{36}$$
$$\longleftarrow C^* \otimes_k V_{n-1}^* \longleftarrow (V_n \otimes_k C)^* \longleftarrow (V_{n+1} \otimes_k C)^* \longleftarrow \cdots$$

as in formula (24) from the proof of Proposition 6.6. Applying the vector space tensor product functor $-\otimes_k T$ to the complex (36), we get an initially projective resolution of the left C^*-module T,

$$0 \longleftarrow T \longleftarrow C^* \otimes_k V_0^* \otimes_k T \longleftarrow C^* \otimes_k V_1^* \otimes_k T \longleftarrow \cdots \tag{37}$$
$$\longleftarrow C^* \otimes_k V_{n-1}^* \otimes_k T \longleftarrow (V_n \otimes_k C)^* \otimes_k T \longleftarrow (V_{n+1} \otimes_k C)^* \otimes_k T.$$

For any left C-contramodule Y, the projective resolution (35) of the trivial left C-contramodule T can be used in order to compute the vector spaces $\mathrm{Ext}^{C,i}(T, Y)$ in all cohomological degrees $i \geq 0$. On the other hand, the resolution (37) of the trivial left C^*-module T is only initially projective. By Lemma A.1, for any left C^*-module Y, the resolution (37) can be used in order to compute the vector spaces $\mathrm{Ext}_{C^*}^i(T, Y)$ in the cohomological degrees $0 \leq i \leq n$.

Now we have a natural injective morphism of complexes of left C^*-modules from the complex (37) to the complex (35), acting by the identity map on the leftmost terms T. This morphism of resolutions can be used in order to compute the maps $\mathrm{Ext}^{C,i}(T, Y) \longrightarrow \mathrm{Ext}_{C^*}^i(T, Y)$ in the cohomological degrees $0 \leq i \leq n$ for any left C-contramodule Y.

Put $Y = k$. Applying the contramodule Hom functor $\mathrm{Hom}^C(-, k)$ to the projective resolution (35) of the trivial left C-contramodule T, we obtain a complex

$$0 \longrightarrow \mathrm{Hom}_k(V_0, T)^* \longrightarrow \mathrm{Hom}_k(V_1, T)^* \longrightarrow \cdots \tag{38}$$
$$\longrightarrow \mathrm{Hom}_k(V_{n-1}, T)^* \longrightarrow \mathrm{Hom}_k(V_n, T)^* \longrightarrow \mathrm{Hom}_k(V_{n+1}, T)^* \longrightarrow \cdots$$

By Lemma 8.3, the complex (38) is naturally isomorphic to the complex obtained by applying the functor $\mathrm{Hom}_k((-)_\gamma, T)^*$ to the coresolution (34). As the latter coresolution was chosen to be minimal, the differential in the complex (38) vanishes.

Applying the module Hom functor $\mathrm{Hom}_{C^*}(-, k)$ to the initially projective resolution (37) of the trivial left C^*-module T, we obtain a complex

$$0 \longrightarrow T^* \otimes_k V_0 \longrightarrow T^* \otimes_k V_1 \longrightarrow \cdots \longrightarrow T^* \otimes_k V_{n-1} \tag{39}$$
$$\longrightarrow \mathrm{Hom}_k(T, \mathrm{Hom}_{C^*}((V_n \otimes_k C)^*, k)) \longrightarrow \mathrm{Hom}_k(T, \mathrm{Hom}_{C^*}((V_{n+1} \otimes_k C)^*, k)).$$

The complex (39) can be also obtained by applying the covariant vector space Hom functor $\mathrm{Hom}_k(T, -)$ to the complex

$$0 \longrightarrow V_0 \longrightarrow V_1 \longrightarrow \cdots \longrightarrow V_{n-1} \tag{40}$$
$$\longrightarrow \mathrm{Hom}_{C^*}((V_n \otimes_k C)^*, k) \longrightarrow \mathrm{Hom}_{C^*}((V_{n+1} \otimes_k C)^*, k)$$

produced by applying the module Hom functor $\mathrm{Hom}_{C^*}(-, k)$ to the resolution (36) of the trivial C^*-module k (see formula (25) in the proof of Proposition 6.6).

By Lemma 6.5, the complex (40) is naturally isomorphic to the complex

$$0 \longrightarrow V_0 \longrightarrow V_1 \longrightarrow \cdots \longrightarrow V_{n-1} \longrightarrow V_n^{**} \longrightarrow V_{n+1}^{**} \tag{41}$$

obtained by applying the functor $((-)_\gamma)^{**}$ to the coresolution (34). Once again, as the latter coresolution was chosen to be minimal, the differential in the complex (41) (or, which is the same, in the complex (40)) vanishes. We have computed the complex (39) as having the form

$$0 \longrightarrow \mathrm{Hom}_k(T, V_0) \longrightarrow \mathrm{Hom}_k(T, V_1) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_k(T, V_{n-1}) \tag{42}$$
$$\longrightarrow \mathrm{Hom}_k(T, V_n^{**}) \longrightarrow \mathrm{Hom}_k(T, V_{n+1}^{**}),$$

or, which is the same,

$$0 \longrightarrow T^* \otimes_k V_0 \longrightarrow T^* \otimes_k V_1 \longrightarrow \cdots \longrightarrow T^* \otimes_k V_{n-1} \tag{43}$$
$$\longrightarrow (V_n^* \otimes_k T)^* \longrightarrow (V_{n+1}^* \otimes_k T)^*$$

and zero differential.

Now it is finally clear from the formulas (38) and (43) that the map

$$\mathrm{Ext}^{C,n}(T, k) \longrightarrow \mathrm{Ext}_{C^*}^n(T, k) \tag{44}$$

is isomorphic to the natural map

$$\mathrm{Hom}(V_n, T)^* \longrightarrow (V_n^* \otimes_k T)^*, \tag{45}$$

which can be obtained by applying the dual vector space functor $\mathrm{Hom}_k(-, k)$ to the natural embedding

$$V_n^* \otimes_k T \longrightarrow \mathrm{Hom}_k(V_n, T). \tag{46}$$

The map (46) is always injective, but for infinite-dimensional vector spaces V_n and T it is not surjective. Thus, the map (45), and consequently the desired map (44), is surjective but not injective.

To deduce the second assertion of the proposition, we apply Lemma A.4(b) to the contramodule forgetful functor $\Phi = \Theta$, the left C-contramodule $Y = k$, and the natural epimorphism of left C-contramodules $P = \mathrm{Hom}_k(C, T) \longrightarrow T$ with a kernel X. By the first assertion of the proposition, there exists a nonzero extension class $\beta \in \mathrm{Ext}^{C,n}(T, Y)$ annihilated by the map (44). The class β is also annihilated by the map $\mathrm{Ext}^{C,n}(T, Y) \longrightarrow \mathrm{Ext}^{C,n}(P, Y) = 0$ (as $n \geq 2$ and the C-contramodule P is projective). Finally, the map $\mathrm{Ext}^{C,n-1}(X, Y) \longrightarrow \mathrm{Ext}^{n-1}_{C^*}(X, Y)$ is injective by Corollary 7.5 (since the left C-contramodule $Y = k$ is separated). So Lemma A.4(b) tells us that the map $0 = \mathrm{Ext}^{C,n-1}(P, Y) \longrightarrow \mathrm{Ext}^{n-1}_{C^*}(P, Y)$ cannot be surjective. □

Proof of Theorem 8.1. The case $n = 1$ is covered by Example 8.2 with Lemma A.7. The case $n \geq 2$ is treated in Proposition 8.4. □

Proof of Theorem 1.1. Follows from Theorems 5.1, 6.1, 7.2, 8.1 (applied to the coalgebras C and C^{op}) and Propositions 4.1, 4.3. Specifically:

(iv) $\Longrightarrow$ (i) is Theorem 5.1;
(i) $\Longrightarrow$ (iv) is the first assertion of Theorem 6.1;
(iv) $\Longrightarrow$ (ii) is Theorem 5.1 for the coalgebra C^{op} together with Proposition 4.1;
(ii) $\Longrightarrow$ (iv) is the first assertion of Theorem 6.1 for the coalgebra C^{op} together with Proposition 4.1;
(iv) $\Longrightarrow$ (iii) is the first assertion of Theorem 7.2;
(iii) $\Longrightarrow$ (iv) is the second assertion of Theorem 8.1;
(iv) $\Longleftrightarrow$ (v) is Proposition 4.3.
The last assertion of Theorem 1.1 is provided by the second assertion of Theorem 7.2. □

9. Half-bounded derived full-and-faithfulness

The aim of this short section is to finish the proof of Theorem 1.2 from the introduction. The argument is based on the following result from the preprint [31].

Proposition 9.1. *(a) Let* A *and* B *be abelian categories, and* $\Upsilon \colon \mathsf{B} \longrightarrow \mathsf{A}$ *be a fully faithful exact functor. Assume that there are enough injective objects in the abelian category* A, *and the functor* Υ *has a right adjoint functor* $\Gamma \colon \mathsf{A} \longrightarrow \mathsf{B}$. *Then the induced triangulated functor between the bounded derived categories* $\Upsilon^{\mathrm{b}} \colon \mathsf{D}^{\mathrm{b}}(\mathsf{B}) \longrightarrow \mathsf{D}^{\mathrm{b}}(\mathsf{A})$ *is fully faithful if and only if the induced triangulated functor between the bounded below derived categories* $\Upsilon^+ \colon \mathsf{D}^+(\mathsf{B}) \longrightarrow \mathsf{D}^+(\mathsf{A})$ *is fully faithful.*

(b) Let A *and* B *be abelian categories, and* $\Theta \colon \mathsf{B} \longrightarrow \mathsf{A}$ *be a fully faithful exact functor. Assume that there are enough projective objects in the abelian category* A, *and the functor* Θ *has a left adjoint functor* $\Delta \colon \mathsf{A} \longrightarrow \mathsf{B}$. *Then the induced triangulated functor between the bounded derived categories* $\Theta^{\mathrm{b}} \colon \mathsf{D}^{\mathrm{b}}(\mathsf{B}) \longrightarrow \mathsf{D}^{\mathrm{b}}(\mathsf{A})$ *is fully faithful if and only if the induced triangulated functor between the bounded above derived categories* $\Theta^- \colon \mathsf{D}^-(\mathsf{B}) \longrightarrow \mathsf{D}^-(\mathsf{A})$ *is fully faithful.*

Proof. Part (b) is [31, Proposition 6.5 (c) $\Leftrightarrow$ (d)]. The point is that any one of the functors Θ^{b} and Θ^- is fully faithful if and only if the higher left derived functors $\mathbb{L}_n \Delta \colon \mathsf{A} \longrightarrow \mathsf{B}$, $n \geq 1$ of the functor Δ vanish on the essential image of the functor Θ. Part (a) is the dual assertion. □

Proof of Theorem 1.2. Follows from Theorem 1.1 and Proposition 9.1. Specifically:

(i) $\Longleftrightarrow$ (vii) is Theorem 1.1(i) $\Leftrightarrow$ (iv);

(iii) $\Longleftrightarrow$ (vii) is Theorem 1.1(ii) $\Leftrightarrow$ (iv);

(v) $\Longleftrightarrow$ (vii) is Theorem 1.1(iii) $\Leftrightarrow$ (iv) and the last assertion of Theorem 1.1;

(vii) $\Longleftrightarrow$ (viii) is Theorem 1.1(iv) $\Leftrightarrow$ (v);

(i) $\Longleftrightarrow$ (ii) is Proposition 9.1(a);

(iii) $\Longleftrightarrow$ (iv) is Proposition 9.1(a);

(v) $\Longleftrightarrow$ (vi) is Proposition 9.1(b).

In order to apply Proposition 9.1, it only needs to be explained why the functor $\Upsilon\colon C-\mathsf{Comod} \longrightarrow C^*-\mathsf{Mod}$ has a right adjoint and why the functor $\Theta\colon C-\mathsf{Contra} \longrightarrow C^*-\mathsf{Mod}$ has a left adjoint. In fact, these are quite general properties of coalgebras, as the assumption that C is conilpotent is not needed here.

For any coalgebra C over a field k, the comodule inclusion functor $\Upsilon\colon C-\mathsf{Comod} \longrightarrow C^*-\mathsf{Mod}$ has a right adjoint functor $\Gamma\colon C^*-\mathsf{Mod} \longrightarrow C-\mathsf{Comod}$. The functor Γ assigns to every left C^*-module N its maximal submodule belonging to the essential image of the functor Υ. In other words, $\Gamma(N)$ is the sum of all submodules of N whose C^*-module structure comes from a C-comodule structure. Equivalently, $\Gamma(N) \subset N$ is the submodule consisting of all the elements $x \in N$ whose annihilator ideals in C^* (with respect to the action map $C^* \times N \longrightarrow N$) contain the annihilator of some finite-dimensional vector subspace of C (with respect to the pairing map $C^* \times C \longrightarrow k$) [44, Theorem 2.1.3(d)].

For any coalgebra C over a field k, the contramodule forgetful functor $\Theta\colon C-\mathsf{Contra} \longrightarrow C^*-\mathsf{Mod}$ has a left adjoint functor $\Delta\colon C^*-\mathsf{Mod} \longrightarrow C-\mathsf{Contra}$. Given a left C^*-module M, the adjunction morphism $M \longrightarrow \Delta(M)$ is *not* surjective in general; so $\Delta(M)$ *cannot* be constructed as a quotient (contra)module of M.

To construct the functor Δ, one defines it on *free C^*-modules* by the rule $\Delta(C^* \otimes_k V) = \mathrm{Hom}_k(C, V)$ for every k-vector space V. Functoriality of the category object corepresenting a (given corepresentable) functor provides a natural way to define the action of Δ on morphisms of free C^*-modules. Then there is always a unique way to extend an abelian category-valued covariant additive functor defined on the full subcategory of free modules over some ring R to a right exact functor on the category of all R-modules. Simply put, to compute the contramodule $\Delta(M)$, one needs to represent M as the cokernel of a morphism of free C^*-modules $f\colon F' \longrightarrow F''$; then, $\Delta(M)$ is the cokernel of the contramodule map $\Delta(f)\colon\Delta(F') \longrightarrow \Delta(F'')$. $\qquad\square$

Remark 9.2. *Following the philosophy of the book [20] and the memoir [21] (see also the discussion in the survey [36, Section 7]), one is generally supposed to consider the coderived category of C-comodules* $\mathsf{D}^{\mathrm{co}}(C-\mathsf{Comod})$ *and the contraderived category of C-contramodules* $\mathsf{D}^{\mathrm{ctr}}(C-\mathsf{Contra})$, *rather than the conventional derived categories* $\mathsf{D}(C-\mathsf{Comod})$ *and* $\mathsf{D}(C-\mathsf{Contra})$. *Let us point out, in this connection, that the difference between the derived and the co/contraderived categories mostly does not manifest itself in the context of the present paper; certainly not in the context of Theorem 1.2. The point is that the distinction between the conventional derived and the co/contraderived categories of abelian categories is only relevant for unbounded complexes, while the results of the present paper mostly concern the* Ext *spaces, which can be computed in the bounded derived categories.*

Specifically, let A *be an abelian category with exact functors of infinite coproduct. Then the natural triangulated functor from the coderived to the derived category* $\mathsf{D}^{\mathrm{co}}(\mathsf{A}) \longrightarrow \mathsf{D}(\mathsf{A})$ *induces an equivalence of the full subcategories of bounded below complexes,* $\mathsf{D}^{\mathrm{co},+}(\mathsf{A}) \simeq \mathsf{D}^+(\mathsf{A})$. *Dually, if* B *is an abelian category with exact functors of infinite product, then the natural triangulated functor from the contraderived to the derived category* $\mathsf{D}^{\mathrm{ctr}}(\mathsf{B}) \longrightarrow \mathsf{D}(\mathsf{B})$ *induces an equivalence of the full subcategories of bounded above complexes,* $\mathsf{D}^{\mathrm{ctr},-}(\mathsf{B}) \simeq \mathsf{D}^-(\mathsf{B})$ *[21, Theorems 3.4.1 and 4.3.1], [23, Lemma A.1.3]. These references cover the case of co/contraderived categories in the sense of Positselski (see [36, Section 7] for the terminology); for a similar result for coderived categories in the sense of Becker, assume that there are enough injective objects in* A, *denote the full subcategory of injective objects by* $\mathsf{E} = \mathsf{A}_{\mathrm{inj}} \subset \mathsf{A}$, *and refer to [38, Lemma 5.4]. See also [20, Remark 4.1].*

10. Co-Noetherian and cocoherent conilpotent coalgebras

The aim of this section is to explain that certain Noetherianity or coherence-type conditions on a conilpotent coalgebra C imply weakly finite Koszulity. In particular, all *cocommutative* conilpotent coalgebras are weakly finitely Koszul; this fact will be relevant for the discussion in the next Section 11.

Let C be a coalgebra over a field k. A left C-comodule is said to be *finitely cogenerated* if it can be embedded as a subcomodule into a cofree C-comodule $C \otimes_k V$ with a finite-dimensional space of cogenerators V. A coalgebra C is said to be *left co-Noetherian* if every quotient comodule of a finitely cogenerated left C-comodule is finitely cogenerated [9, 29, 47]. A coalgebra C is said to be *left Artinian* if it is Artinian as an object of the category of left C-comodules $C-\mathsf{Comod}$, that is, any descending chain of left coideals in C terminates [9], [29, Section 2]. (Here by a *left coideal* in C one means a left subcomodule in the left C-comodule C.)

Lemma 10.1. *(a) A coalgebra C is left Artinian if and only if it is left co-Noetherian and the maximal cosemisimple subcoalgebra of C is finite-dimensional. In particular, a conilpotent coalgebra is left Artinian if and only if it is left co-Noetherian.*

(b) A coalgebra C is left Artinian if and only if its dual algebra C^* is right Noetherian.

Proof. See [9, Proposition 1.6 of the published version, or Proposition 2.5 of the `arXiv` version] or [29, Lemmas 2.6 and 2.10(b), Example 2.7, and the general discussion in Section 3]. $\square$

Proposition 10.2. *Any left or right co-Noetherian conilpotent coalgebra is weakly finitely Koszul.*

Proof. The weak finite Koszulity property of a conilpotent coalgebra is left-right symmetric by Proposition 4.1; so it suffices to consider the case of a left co-Noetherian conilpotent coalgebra C. Then it follows easily from the co-Noetherianity that every finitely cogenerated left C-comodule L has a coresolution $J^\bullet$ by finitely cogenerated cofree C-comodules J^n, $n \geq 0$. In particular, this applies to all finite-dimensional left C-comodules L; so the left C-comodule k has such a coresolution $J^\bullet$. It remains to compute the spaces $\operatorname{Ext}_C^n(k, k)$ as $H^n(\operatorname{Hom}_C(k, J^\bullet))$ and notice that the vector space $\operatorname{Hom}_C(k, J^n) = {}_\gamma J^n$ is finite-dimensional for every $n \geq 0$. $\square$

A left C-comodule is said to be *finitely copresented* if it can be obtained as the kernel of a morphism of cofree C-comodules $C \otimes_k V \longrightarrow C \otimes_k U$ with finite-dimensional vector spaces of cogenerators V and U. A coalgebra C is said to be *left cocoherent* if every finitely cogenerated quotient comodule of a finitely copresented left C-comodule is finitely copresented [29, Section 2].

Lemma 10.3. *Let C be a conilpotent coalgebra over k. Then*

(a) a left C-comodule L is finitely cogenerated if and only if the vector space $\operatorname{Hom}_k(k, L) = {}_\gamma L$ is finite-dimensional;

(b) a finitely cogenerated left C-comodule L is finitely copresented if and only if the vector space $\operatorname{Ext}_C^1(k, L)$ is finite-dimensional;

(c) in particular, the left C-comodule k is finitely copresented if and only if the coalgebra C is finitely cogenerated.

Proof. Part (a) follows from Lemma 3.5(a). To prove the "if" assertion in part (b), one can use Proposition 3.7(a) together with Corollary 3.6(a) to the effect that for any left C-comodule L there exists a left exact sequence of left C-comodules $0 \longrightarrow L \longrightarrow C \otimes V_0 \longrightarrow C \otimes_k V_1$ with $V_0 \simeq \operatorname{Hom}_C(k, L)$ and $V_1 \simeq \operatorname{Ext}_C^1(k, L)$. The "only if" assertion in part (b) is deduced from the fact that the quotient comodule of a finitely cogenerated comodule by a finitely copresented subcomodule is finitely cogenerated; see [47, Theorem 6] or [29, Lemma 2.8(a)]. In particular, the left C-comodule k is finitely copresented

if and only if the vector space $\operatorname{Hom}_k(k, C/\gamma(k)) = {}_\gamma(C/{}_\gamma C) = \operatorname{Ext}^1_C(k, k)$ is finite-dimensional. This is part (c). $\qquad\square$

Proposition 10.4. *Any finitely cogenerated left or right cocoherent conilpotent coalgebra is weakly finitely Koszul.*

Proof. It suffices to consider the case when C is left cocoherent. Then it follows from the cocoherence and the same fact that the quotient comodule of a finitely cogenerated comodule by a finitely copresented subcomodule is finitely cogenerated (mentioned in the proof of Lemma 10.3) that any finitely copresented left C-comodule L has a coresolution $J^\bullet$ by finitely cogenerated cofree C-comodules J^n, $n \geq 0$. If C is finitely cogenerated, then this applies to $L = k$ (by Lemma 10.3(c)); so the left C-comodule k has such a coresolution $J^\bullet$. The argument finishes exactly in the same way as the proof of Proposition 10.2. $\qquad\square$

Remark 10.5. *A coalgebra version of the Morita equivalence theory was developed by Takeuchi [45]. The notions of co-Noetherianity and co-coherence for coalgebras are not Morita–Takeuchi invariant (neither is conilpotence, of course; while Artinianity of coalgebras is invariant under the Morita–Takeuchi equivalence).*

The co-Noetherianity and cocoherence properties have Morita–Takeuchi invariant versions, which are called quasi-co-Noetherianity and quasi-cocoherence in [33, Section 3], [30, Sections 5.1–5.4]. (The former property was called "strict quasi-finiteness" in [9].) The key definition, going back to Takeuchi [45], is that of a 'quasi-finite comodule', called quasi-finitely cogenerated in the terminology of [33, 30]. A left C-comodule L is called quasi-finitely cogenerated if the vector space $\operatorname{Hom}_C(K, L)$ is finite-dimensional for any finite-dimensional left C-comodule K.

One can see from [29, Lemma 2.2(e)] that over a coalgebra C with finite-dimensional maximal cosemisimple subcoalgebra (in particular, over a conilpotent coalgebra C) the classes of finitely cogenerated and quasi-finitely cogenerated comodules coincide. Consequently, a conilpotent coalgebra is quasi-co-Noetherian in the sense of [33, 30] if and only if it is co-Noetherian, and a conilpotent coalgebra is quasi-cocoherent in the the sense of [33, 30] if and only if it is cocoherent. That is why we were not concerned with the quasi-co-Noetherianity and quasi-cocoherence properties in this section, but only with the co-Noetherianity and cocoherence properties.

11. Cocommutative conilpotent coalgebras

Let V be a k-vector space. We refer to [36, Sections 2.3 and 3.3] for an introductory discussion of the *cofree conilpotent* (*tensor*) coalgebra

$$\operatorname{Ten}(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n}$$

cospanned by a vector space V. The *cofree conilpotent cocommutative* (*symmetric*) coalgebra $\operatorname{Sym}(V)$ is defined as the subcoalgebra

$$\operatorname{Sym}(V) = \bigoplus_{n=0}^{\infty} \operatorname{Sym}^n(V) \subset \operatorname{Ten}(V),$$

where $\operatorname{Sym}^n(V) \subset V^{\otimes n}$ is the vector subspace of all symmetric tensors in $V^{\otimes n}$. One can easily see that $\operatorname{Sym}(V)$ is the maximal cocommutative subcoalgebra in $\operatorname{Ten}(V)$; in fact, for any cocommutative coalgebra C over k and coalgebra homomorphism $f: C \longrightarrow \operatorname{Ten}(V)$, the image of f is contained in $\operatorname{Sym}(V)$.

Let C be a conilpotent coalgebra over k. Assume that C is finitely cogenerated; so the vector space $V = \operatorname{Ext}^1_C(k, k) = \ker(C_+ \to C_+ \otimes_k C_+)$ is finite-dimensional. Choose a k-linear projection $g: C_+ \longrightarrow V$

onto the vector subspace $V \subset C_+$. Then, according to [26, Lemma 5.2(a)], the map g extends uniquely to coalgebra homomorphism $f\colon C \longrightarrow \mathrm{Ten}(V)$. Moreover, by [26, Lemma 5.2(b)], the map f is injective.

Assume additionally that C is cocommutative. Then, following the discussion above, the image of the map f is contained in the subcoalgebra $\mathrm{Sym}(V) \subset \mathrm{Ten}(V)$. Hence, C is a subcoalgebra in $\mathrm{Sym}(V)$.

Choose a k-vector space basis $x_1^*, \ldots, x_m^*$ in the vector space V. Then $x_1, \ldots, x_m$ is a basis in the dual vector space V^*. The choice of such bases identifies the k-vector space dual k-algebra $\mathrm{Sym}(V)^*$ to the symmetric coalgebra $\mathrm{Sym}(V)$ with the algebra of formal power series in the variables $x_1, \ldots, x_m$,

$$\mathrm{Sym}(V)^* \simeq k[[x_1, \ldots, x_m]].$$

Accordingly, the k-algebra C^* dual to C is a quotient algebra of the algebra of formal power series by an ideal $J \subset k[[x_1, \ldots, x_m]]$,

$$C^* \simeq k[[x_1, \ldots, x_m]]/J.$$

Denote by $s_1, \ldots, s_m \in C^*$ the images of the elements $x_1, \ldots, x_m$ under the surjective k-algebra homomorphism $k[[x_1, \ldots, x_m]] \longrightarrow C^*$ dual to the injective coalgebra map $C \longrightarrow \mathrm{Sym}(V)$. We have shown that C^* is a complete Noetherian commutative local ring with the maximal ideal $I = (s_1, \ldots, s_m) \subset C^*$ generated by the elements $s_1, \ldots, s_m \in C^*$. Indeed, the ring of formal power series $k[[x_1, \ldots, x_m]]$ is a complete Noetherian commutative local ring; hence so is any (nonzero) quotient ring of $k[[x_1, \ldots, x_m]]$.

Corollary 11.1. *All finitely cogenerated cocommutative conilpotent coalgebras are co-Noetherian. Consequently, all such coalgebras are weakly finitely Koszul.*

Proof. Let C be a finitely cogenerated cocommutative conilpotent coalgebra over k. Then the algebra C^* is Noetherian, as explained above. By Lemma 10.1(a–b), it follows that the coalgebra C is co-Noetherian. Now Proposition 10.2 tells us that C is weakly finitely Koszul. □

Corollary 11.1 says that the condition of Theorem 1.2(vii) is satisfied for any finitely cogenerated cocommutative conilpotent coalgebra C. Consequently, the triangulated functors $\Upsilon^+\colon \mathsf{D}^+(C{-}\mathsf{Comod}) \longrightarrow \mathsf{D}^+(C^*{-}\mathsf{Mod})$ and $\Theta^-\colon \mathsf{D}^-(C{-}\mathsf{Contra}) \longrightarrow \mathsf{D}^-(C^*{-}\mathsf{Mod})$ are fully faithful. In the rest of this section, our aim is to show, using the results of the paper [25], that the triangulated functors $\Upsilon^\varnothing\colon \mathsf{D}(C{-}\mathsf{Comod}) \longrightarrow \mathsf{D}(C^*{-}\mathsf{Mod})$ and $\Theta^\varnothing\colon \mathsf{D}(C{-}\mathsf{Contra}) \longrightarrow \mathsf{D}(C^*{-}\mathsf{Mod})$ between the unbounded derived categories are actually fully faithful in this case, too.

Let R be a commutative ring and $I \subset R$ be an ideal. An R-module M is said to be *I-torsion* if for every pair of elements $s \in I$ and $x \in M$ there exists an integer $n \geq 1$ such that $s^n x = 0$ in M. Equivalently, M is I-torsion if and only if for every $s \in I$ one has $R[s^{-1}] \otimes_R M = 0$. Here, $R[s^{-1}]$ is the ring obtained by inverting formally the element $s \in R$ (or equivalently, by localizing R at the multiplicative subset $S = \{1, s, s^2, s^3, \ldots\} \subset R$ spanned by s).

An R-module P is said to be an *I-contramodule* [25, 27] if for every element $s \in I$ one has

$$\mathrm{Hom}_R(R[s^{-1}], P) = 0 = \mathrm{Ext}_R^1(R[s^{-1}], P).$$

It is important to notice here that the projective dimension of the R-module $R[s^{-1}]$ can never exceed 1 [27, proof of Lemma 3.1].

The full subcategory $R{-}\mathsf{Mod}_{I{-}\mathsf{tors}}$ of all I-torsion R-modules is closed under submodules, quotients, extensions, and infinite direct sums in $R{-}\mathsf{Mod}$, as one can easily see [27, Theorem 1.1(b)]. The full subcategory $R{-}\mathsf{Mod}_{I{-}\mathsf{ctra}}$ of all I-contramodule R-modules is closed under kernels, cokernels, extensions, and infinite products in $R{-}\mathsf{Mod}$ [8, Proposition 1.1], [27, Theorem 1.2(a)]. Consequently, both the categories $R{-}\mathsf{Mod}_{I{-}\mathsf{tors}}$ and $R{-}\mathsf{Mod}_{I{-}\mathsf{ctra}}$ are abelian, and the inclusion functors $R{-}\mathsf{Mod}_{I{-}\mathsf{tors}} \longrightarrow R{-}\mathsf{Mod}$ and $R{-}\mathsf{Mod}_{I{-}\mathsf{ctra}} \longrightarrow R{-}\mathsf{Mod}$ are exact.

Theorem 11.2. *Let C be a finitely cogenerated cocommutative conilpotent coalgebra over a field k, and let $I \subset C^*$ be the maximal ideal of the complete Noetherian commutative local ring C^*. Then*

(a) the essential image of the comodule inclusion functor $\Upsilon\colon C-\mathsf{Comod} \longrightarrow C^*-\mathsf{Mod}$ *coincides with the full subcategory of I-torsion C^*-modules $C^*-\mathsf{Mod}_{I-\mathrm{tors}} \subset C^*-\mathsf{Mod}$, so the functor Υ induces a category equivalence $C-\mathsf{Comod} \simeq C^*-\mathsf{Mod}_{I-\mathrm{tors}}$;*

(b) the contramodule forgetful functor $\Theta\colon C-\mathsf{Contra} \longrightarrow C^*-\mathsf{Mod}$ *is fully faithful, and its essential image coincides with the full subcategory of I-contramodule C^*-modules $C^*-\mathsf{Mod}_{I-\mathrm{ctra}} \subset C^*-\mathsf{Mod}$, so the functor Θ induces a category equivalence $C-\mathsf{Contra} \simeq C^*-\mathsf{Mod}_{I-\mathrm{ctra}}$.*

Proof. The dual vector space W^* to any (discrete infinite-dimensional) vector space W is naturally endowed with a pro-finite-dimensional (otherwise known as linearly compact or pseudocompact) topology. The annihilators of finite-dimensional vector subspaces in W are precisely all the open vector subspaces in W^*, and they form a base of neighborhoods of zero in W^*. In the case of a coalgebra C, the vector space C^* with its pro-finite-dimensional topology is a topological algebra.

A left C^*-module N is said to be *discrete* (or "rational" in the terminology of [44]) if, for every element $x \in N$, the annihilator of x is an open left ideal in C^*. Equivalently, this means that the action map $C^* \times N \longrightarrow N$ is continuous in the pro-finite-dimensional topology on C^* and the discrete topology on N. The essential image of the comodule inclusion functor $\Upsilon\colon C-\mathsf{Comod} \longrightarrow C^*-\mathsf{Mod}$ consists precisely of all the discrete C^*-modules [44, Propositions 2.1.1–2.1.2]. (Cf. the proof of Theorem 1.2 in Section 9.) These assertions hold for any coalgebra C over k.

In the case of a finitely cogenerated cocommutative conilpotent coalgebra C, the surjective map $k[[x_1, \dots, x_m]] \longrightarrow C^*$ is open and continuous, because it is obtained by applying the dual vector space functor $W \longmapsto W^* = \mathrm{Hom}_k(W, k)$ to the inclusion of discrete vector spaces (coalgebras) $C \longrightarrow \mathrm{Sym}(V)$. The resulting pro-finite-dimensional topology on the algebra of formal power series $k[[x_1, \dots, x_m]]$ is the usual (adic) topology of the formal power series. One easily concludes that a C^*-module is discrete if and only if it is discrete over $k[[x_1, \dots, x_m]]$. By the definition of the adic topology on a formal power series ring $k[[x_1, \dots, x_m]]$, a module over this ring is discrete if and only if it is a torsion module for the maximal ideal $(x_1, \dots, x_m) \subset k[[x_1, \dots, x_m]]$. Hence, a C^*-module N is discrete if and only if it is a torsion module for the ideal $I = (s_1, \dots, s_m) \subset C^*$. One can also observe that the pro-finite-dimensional topology on C^* coincides with the I-adic topology, since this holds for the coalgebra $\mathrm{Sym}(V)$. This proves part (a).

To prove part (b), one needs to use the concept of a *contramodule over a topological ring* as an intermediate step. Without going into the (somewhat involved) details of this definition, which can be found in [22, Section 2.1], [24, Section 2.1], or [32, Sections 2.5–2.7], let us say that the category of left C-contramodules is naturally equivalent (in fact, isomorphic) to the category of left contramodules over the topological ring C^*. This equivalence agrees with the natural forgetful functors acting from the categories of left C-contramodules and left C^*-contramodules to the category of left C^*-modules. We refer to [22, Section 1.10] or [24, Section 2.3] for the details of the proof of this assertion.

The contramodule forgetful functor $C-\mathsf{Contra} \longrightarrow C^*-\mathsf{Mod}$ is fully faithful by Theorem 7.4. In the special case of *cocommutative* coalgebras C, one can also obtain this result as a particular case of the following theorems, which provide more information. By [22, Theorem B.1.1] or [24, Theorem 2.2], the forgetful functor from the category of C^*-contramodules to the category of C^*-modules is fully faithful, and its essential image is precisely the full subcategory of all I-contramodule C^*-modules. These results are actually applicable to any commutative Noetherian ring R with a fixed ideal I and the I-adic completion of R viewed as a topological ring with the I-adic topology. (Morever, the Noetherianity condition can be weakened and replaced with a certain piece of the weak proregularity condition; see [34, Proposition 1.5, Corollary 3.7, and Remark 3.8].) The combination of the references in this paragraph and in the previous one establishes part (b). $\square$

Theorem 11.3. *Let R be a commutative Noetherian ring and $I \subset R$ be an ideal. Then*
(a) the triangulated functor between the unbounded derived categories

$$\mathsf{D}(R-\mathsf{Mod}_{I-\mathrm{tors}}) \longrightarrow \mathsf{D}(R-\mathsf{Mod})$$

induced by the inclusion of abelian categories $R-\mathsf{Mod}_{I-\mathrm{tors}} \longrightarrow R-\mathsf{Mod}$ is fully faithful;

(b) the triangulated functor between the unbounded derived categories

$$\mathsf{D}(R-\mathsf{Mod}_{I-\mathsf{ctra}}) \longrightarrow \mathsf{D}(R-\mathsf{Mod})$$

induced by the inclusion of abelian categories $R-\mathsf{Mod}_{I-\mathsf{ctra}} \longrightarrow R-\mathsf{Mod}$ *is fully faithful.*

Proof. These assertions actually hold for any *weakly proregular* finitely generated ideal I in a (not necessarily Noetherian) commutative ring R; while in a Noetherian commutative ring, any ideal is weakly proregular. See [25, Theorems 1.3 and 2.9] for the details. The proofs are based on the observations that (the derived functors of) the right adjoint functor $\Gamma: R-\mathsf{Mod} \longrightarrow R-\mathsf{Mod}_{I-\mathsf{tors}}$ and the left adjoint functor $\Delta: R-\mathsf{Mod} \longrightarrow R-\mathsf{Mod}_{I-\mathsf{ctra}}$ to the inclusions of abelian categories in question have finite homological dimensions, which go back to [18, Corollaries 4.28 and 5.27] and were also mentioned in [25, Lemmas 1.2(b) and 2.7(b)]. See [28, Theorem 6.4] (cf. [31, Proposition 6.5]) for an abstract formulation. □

Corollary 11.4. *Let C be a finitely cogenerated cocommutative conilpotent coalgebra over a field k. Then*
(a) the triangulated functor between the unbounded derived categories

$$\Upsilon^{\varnothing}: \mathsf{D}(C-\mathsf{Comod}) \longrightarrow \mathsf{D}(C^*-\mathsf{Mod})$$

induced by the comodule inclusion functor $\Upsilon: C-\mathsf{Comod} \longrightarrow C^*-\mathsf{Mod}$ *is fully faithful;*
(b) the triangulated functor between the unbounded derived categories

$$\Theta^{\varnothing}: \mathsf{D}(C-\mathsf{Contra}) \longrightarrow \mathsf{D}(C^*-\mathsf{Mod})$$

induced by the contramodule forgetful functor $C-\mathsf{Contra} \longrightarrow C^*-\mathsf{Mod}$ *is fully faithful.*

Proof. For part (a), compare the results of Theorems 11.2(a) and 11.3(a). For part (b), similarly compare Theorems 11.2(b) and 11.3(b). □

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Appendix. Standard Category-Theoretic Observations

In this appendix we collect several elementary category-theoretic homological algebra lemmas, which are used in Sections 5–8. The following lemma tells how far one can go computing the Ext groups with a resolution which is only *initially* projective.

Lemma A.1. *Let* A *be an abelian category and* $0 \longleftarrow X \longleftarrow P_0 \longleftarrow P_1 \longleftarrow P_2 \longleftarrow \cdots$ *be an exact sequence in* A. *Then for every object* $Y \in A$ *there are natural maps of abelian groups*

$$H^n(\operatorname{Hom}_A(P_\bullet, Y)) \longrightarrow \operatorname{Ext}_A^n(X, Y) \tag{47}$$

defined for all $n \geq 0$. *The map* (47) *is an isomorphism whenever the objects* P_i *are projective in* A *for all* $0 \leq i \leq n-1$.

Proof. The point of the lemma is that one does *not* need the objects P_n or P_{n+1} (but only the objects $P_0, \ldots, P_{n-1}$) to be projective in order to compute $\operatorname{Ext}_A^n(X, Y)$ using a resolution $P_\bullet$ of an object $X \in A$. The proof is standard. Denote by X_i the image of the differential $P_i \longrightarrow P_{i-1}$ (so $X = X_0$). Then the connecting homomorphisms in the long exact sequences of groups $\operatorname{Ext}_A^*(-, Y)$ related to the short exact sequences $0 \longrightarrow X_{i+1} \longrightarrow P_i \longrightarrow X_i \longrightarrow 0$ in A provide natural maps of abelian groups

$$H^n(\operatorname{Hom}_A(P_\bullet, Y)) = \operatorname{coker}(\operatorname{Hom}_A(P_{n-1}, Y) \longrightarrow \operatorname{Hom}_A(X_n, Y)) \longrightarrow \operatorname{Ext}_A^1(X_{n-1}, Y)$$
$$\longrightarrow \operatorname{Ext}_A^2(X_{n-2}, Y) \longrightarrow \cdots \longrightarrow \operatorname{Ext}_A^{n-1}(X_1, Y) \longrightarrow \operatorname{Ext}_A^n(X_0, Y),$$

whose composition is the desired map (47). Furthermore, the map $H^n(\operatorname{Hom}_A(P_\bullet, Y)) \longrightarrow \operatorname{Ext}_A^1(X_{n-1}, Y)$ is an isomorphism whenever $\operatorname{Ext}_A^1(P_{n-1}, Y) = 0$; the map $\operatorname{Ext}_A^1(X_{n-1}, Y) \longrightarrow \operatorname{Ext}_A^2(X_{n-2}, Y)$ is an isomorphism whenever $\operatorname{Ext}_A^1(P_{n-2}, Y) = 0 = \operatorname{Ext}_A^2(P_{n-2}, Y)$, etc.; and the map $\operatorname{Ext}_A^{n-1}(X_1, Y) \longrightarrow \operatorname{Ext}_A^n(X_0, Y)$ is an isomorphism whenever $\operatorname{Ext}_A^{n-1}(P_0, Y) = 0 = \operatorname{Ext}_A^n(P_0, Y)$. $\square$

The next lemma is a Tor version of Lemma A.1.

Lemma A.2. *Let* R *be an associative ring and* $0 \longleftarrow X \longleftarrow F_0 \longleftarrow F_1 \longleftarrow F_2 \longleftarrow \cdots$ *be an exact sequence of right* R-*modules. Then for every left* R-*module* Y *there are natural maps of abelian groups*

$$\operatorname{Tor}_n^R(X, Y) \longrightarrow H_n(F_\bullet \otimes_R Y) \tag{48}$$

defined for all $n \geq 0$. *The map* (48) *is an isomorphism whenever the* R-*modules* F_i *are flat for all* $0 \leq i \leq n-1$.

The following lemma is very easy.

Lemma A.3. *Let* $\Phi \colon B \longrightarrow A$ *be a fully faithful exact functor of abelian categories, and let* $X, Y \in B$ *be two objects. Then the map*

$$\operatorname{Ext}_B^1(X, Y) \longrightarrow \operatorname{Ext}_A^1(\Phi(X), \Phi(Y))$$

induced by the functor Φ *is injective. This map is surjective if and only if, for any short exact sequence* $0 \longrightarrow \Phi(Y) \longrightarrow A \longrightarrow \Phi(X) \longrightarrow 0$ *in* A, *the object* A *belongs to the essential image of* Φ.

Proof. The proof is left to the reader. $\square$

The next lemma is essentially well-known, but we give it a more precise formulation than it usually receives.

Lemma A.4. *Let* A *and* B *be abelian categories, and let* $\Phi \colon B \longrightarrow A$ *be an exact functor. Let* $n \geq 1$ *be an integer and* $Y \in B$ *be a fixed object.*
(a) Assume that the map of groups Ext^{n-1}

$$\operatorname{Ext}_B^{n-1}(X, Y) \longrightarrow \operatorname{Ext}_A^{n-1}(\Phi(X), \Phi(Y))$$

induced by the functor Φ *is an isomorphism for all objects* $X \in B$. *Then the map of groups* Ext^n

$$\operatorname{Ext}_B^n(X, Y) \longrightarrow \operatorname{Ext}_A^n(\Phi(X), \Phi(Y))$$

induced by the functor Φ *is injective for all objects* $X \in \mathsf{B}$.

(b) *More generally, let* $0 \longrightarrow X \longrightarrow P \longrightarrow T \longrightarrow 0$ *be a short exact sequence in* B. *Assume that the map* $\mathrm{Ext}_{\mathsf{B}}^{n-1}(P, Y) \longrightarrow \mathrm{Ext}_{\mathsf{A}}^{n-1}(\Phi(P), \Phi(Y))$ *is surjective, while the map* $\mathrm{Ext}_{\mathsf{B}}^{n-1}(X, Y) \longrightarrow \mathrm{Ext}_{\mathsf{A}}^{n-1}(\Phi(X), \Phi(Y))$ *is injective. Then the intersection of the kernel of the map* $\mathrm{Ext}_{\mathsf{B}}^{n}(T, Y) \longrightarrow \mathrm{Ext}_{\mathsf{B}}^{n}(P, Y)$ *induced by the epimorphism* $P \longrightarrow T$ *with the kernel of the map* $\mathrm{Ext}_{\mathsf{B}}^{n}(T, Y) \longrightarrow \mathrm{Ext}_{\mathsf{A}}^{n}(\Phi(T), \Phi(Y))$ *induced by the functor* Φ *is the zero subgroup in* $\mathrm{Ext}_{\mathsf{B}}^{n}(T, Y)$.

Proof. This observation goes back, at least, to [2, Remarque 3.1.17(i)]. Part (b) is provable by a straightforward diagram chase of the commutative diagram of a morphism of long exact sequences

$$
\begin{array}{ccccccc}
\mathrm{Ext}_{\mathsf{B}}^{n-1}(P, Y) & \longrightarrow & \mathrm{Ext}_{\mathsf{B}}^{n-1}(X, Y) & \longrightarrow & \mathrm{Ext}_{\mathsf{B}}^{n}(T, Y) & \longrightarrow & \mathrm{Ext}_{\mathsf{B}}^{n}(P, Y) \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{Ext}_{\mathsf{A}}^{n-1}(\Phi(P), \Phi(Y)) & \longrightarrow & \mathrm{Ext}_{\mathsf{A}}^{n-1}(\Phi(X), \Phi(Y)) & \longrightarrow & \mathrm{Ext}_{\mathsf{A}}^{n}(\Phi(T), \Phi(Y)) & &
\end{array}
$$

induced by the functor Φ.

To deduce (a) from (b), let $T \in \mathsf{B}$ be an object and $\beta \in \mathrm{Ext}_{\mathsf{B}}^{n}(T, Y)$ be an extension class annihilated by the map $\mathrm{Ext}_{\mathsf{B}}^{n}(T, Y) \longrightarrow \mathrm{Ext}_{\mathsf{A}}^{n}(\Phi(T), \Phi(Y))$. Choose an epimorphism $P \longrightarrow T$ in B such that β is annihilated by the induced map $\mathrm{Ext}_{\mathsf{B}}^{n}(T, Y) \longrightarrow \mathrm{Ext}_{\mathsf{B}}^{n}(P, Y)$. Then part (b) implies that $\beta = 0$. $\square$

The final series of lemmas concerns abelian categories with enough injective or projective objects.

Lemma A.5. *Let* A *and* B *be abelian categories, and let* $\Phi \colon \mathsf{B} \longrightarrow \mathsf{A}$ *be a fully faithful exact functor. Assume that there are enough injective objects in the abelian category* B, *and denote by* $\mathsf{B}_{\mathrm{inj}} \subset \mathsf{B}$ *the class of all such injective objects. Let* $n \geq 1$ *be an integer. Then the following two conditions are equivalent:*

(1) the map of Ext groups

$$
\mathrm{Ext}_{\mathsf{B}}^{i}(X, Y) \longrightarrow \mathrm{Ext}_{\mathsf{A}}^{i}(\Phi(X), \Phi(Y))
$$

induced by the functor Φ *is an isomorphism for all objects* $X, Y \in \mathsf{B}$ *and integers* $0 \leq i \leq n$;

(2) $\mathrm{Ext}_{\mathsf{A}}^{i}(\Phi(X), \Phi(J)) = 0$ *for all objects* $X \in \mathsf{B}$ *and* $J \in \mathsf{B}_{\mathrm{inj}}$, *and all* $1 \leq i \leq n$.

Proof. (1) $\Longrightarrow$ (2) Condition (2) is clearly the particular case of (1) for $Y = J \in \mathsf{B}_{\mathrm{inj}}$.

(2) $\Longrightarrow$ (1) For $i = 0$, the map of Ext groups in question is an isomorphism by the assumption that the functor Φ is fully faithful. For $1 \leq i \leq n$, we proceed by increasing induction on i.

Given an object $Y \in \mathsf{B}$, choose a short exact sequence $0 \longrightarrow Y \longrightarrow J \longrightarrow Y' \longrightarrow 0$ in B with $J \in \mathsf{B}_{\mathrm{inj}}$. Then the exact functor Φ induces a commutative diagram of a morphism of long exact sequences

$$
\begin{array}{ccccccccc}
\mathrm{Ext}_{\mathsf{B}}^{i-1}(X, J) & \longrightarrow & \mathrm{Ext}_{\mathsf{B}}^{i-1}(X, Y') & \longrightarrow & \mathrm{Ext}_{\mathsf{B}}^{i}(X, Y) & \longrightarrow & 0 \\
\| & & \| & & \downarrow & & \\
\mathrm{Ext}_{\mathsf{A}}^{i-1}(\Phi(X), \Phi(J)) & \longrightarrow & \mathrm{Ext}_{\mathsf{A}}^{i-1}(\Phi(X), \Phi(Y')) & \longrightarrow & \mathrm{Ext}_{\mathsf{A}}^{i}(\Phi(X), \Phi(Y)) & \longrightarrow & 0
\end{array}
$$

Here $\mathrm{Ext}_{\mathsf{B}}^{i}(X, J) = 0$ since $i \geq 1$ and $J \in \mathsf{B}_{\mathrm{inj}}$, while $\mathrm{Ext}_{\mathsf{A}}^{i}(\Phi(X), \Phi(J)) = 0$ by (2). The leftmost and the middle vertical morphisms are isomorphisms by the induction assumption. It follows that the rightmost vertical morphism is an isomorphism, too (as desired). $\square$

The following lemma is the dual version of Lemma A.5, but with a more precise claim.

Lemma A.6. *Let* A *and* B *be abelian categories, and let* $\Phi \colon \mathsf{B} \longrightarrow \mathsf{A}$ *be a fully faithful exact functor. Assume that there are enough projective objects in the abelian category* B, *and denote by* $\mathsf{B}_{\mathrm{proj}} \subset \mathsf{B}$ *the*

class of all such projective objects. Let $n \geq 1$ be an integer. Then, for any fixed object $Y \in \mathsf{B}$, the following two conditions are equivalent:

(1) the map of Ext groups

$$\mathrm{Ext}^i_\mathsf{B}(X, Y) \longrightarrow \mathrm{Ext}^i_\mathsf{A}(\Phi(X), \Phi(Y))$$

induced by the functor Φ is an isomorphism for all objects $X \in \mathsf{B}$ and integers $0 \leq i \leq n$;

(2) $\mathrm{Ext}^i_\mathsf{A}(\Phi(P), \Phi(Y)) = 0$ for all objects $P \in \mathsf{B}_\mathrm{proj}$ and all $1 \leq i \leq n$.

Proof. Observe that the proof of Lemma A.5 works for a fixed object X, and then dualize. □

Our last lemma is the $n = 0$ counterpart of Lemma A.6.

Lemma A.7. *Let A and B be abelian categories, and let $\Phi \colon \mathsf{B} \longrightarrow \mathsf{A}$ be an exact functor. Assume that there are enough projective objects in the abelian category B, and denote by $\mathsf{B}_\mathrm{proj} \subset \mathsf{B}$ the class of all such projective objects. Then, for any fixed object $Y \in \mathsf{B}$, the following two conditions are equivalent:*

(1) the map of Hom groups

$$\mathrm{Hom}_\mathsf{B}(X, Y) \longrightarrow \mathrm{Hom}_\mathsf{A}(\Phi(X), \Phi(Y))$$

induced by the functor Φ is an isomorphism for all objects $X \in \mathsf{B}$;

(2) the map of Hom groups

$$\mathrm{Hom}_\mathsf{B}(P, Y) \longrightarrow \mathrm{Hom}_\mathsf{A}(\Phi(P), \Phi(Y))$$

induced by the functor Φ is an isomorphism for all objects $P \in \mathsf{B}_\mathrm{proj}$.

Proof. Represent the object X as the cokernel of a morphism of projective objects in the category B. □