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Li coefficients and the quadrilateral zeta function

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Abstract. In this note, we study the Li coefficients $\lambda_{n,a}$ for the quadrilateral zeta function. Furthermore, we give an arithmetic and asymptotic formula for these coefficients. Especially, we show that for any fixed $n \in \mathbb{N}$, there exists $a > 0$ such that $\lambda_{2n-1,a} > 0$ and $\lambda_{2n,a} < 0$.

1 Introduction and statement of main results

1.1 Li coefficients

The Riemann hypothesis (RH) is a critical question in analytic number theory. As such, it is interesting to examine different ways to frame it, which may shed more light on its resolution. In 1997, Xian-Jin Li has discovered a new positivity criterion for the RH. In [10], he defined the Li coefficients for the Riemann zeta function as

$$\lambda_n = \frac{1}{(n-1)!}\frac{d^n}{ds^n}\left[s^{n-1}\log\xi(s)\right]_{s=1},$$

where ξ is the completed Riemann zeta function defined by

$$\xi(s) = s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s),$$

which satisfies $\xi(s) = \xi(1-s)$ and gave a simple equivalence criterion for the RH: RH is true if and only if these coefficients are nonnegative for every positive integer n. The Li coefficients λ_n can be written as follows:

$$\lambda_n = \sum_{\rho}^{*}\left[1-\left(1-\frac{1}{\rho}\right)^n\right] = \lim_{T\to\infty}\sum_{\rho;|\mathrm{Im}(\rho)|\le T}\left[1-\left(1-\frac{1}{\rho}\right)^n\right],$$

where the sum runs over the nontrivial zeros of the Riemann zeta function counted with multiplicity. This criterion is generalized by Bombieri and Lagarias [4] for any arbitrarily multiset of numbers assuming certain convergence conditions. Voros [19, Section 3.3] has proved that the RH true is equivalent to the growth of λ_n as $\frac{1}{2}n\log n$ determined by its archimedean part, while the RH false is equivalent to the oscillations of λ_n with exponentially growing amplitude, determined by its finite part. The Li

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coefficients were generalized in two ways: by generalizing these coefficients to various sets of functions (the Selberg class, the class of automorphic L-functions, zeta function on function fields,. . .[8, 11, 17]) and by introducing new parameter in its definition (see [12]). The Li coefficients (and its generalizations) have generated a lot of research interest due to its applicability and simplicity.

1.2 Quadrilateral zeta function

Recall the definitions of Hurwitz and periodic zeta functions. The Hurwitz zeta function $\zeta(s,a)$ is defined by the series

$$\zeta(s,a) := \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}, \qquad \sigma > 1, \quad 0 < a \le 1.$$

The function $\zeta(s,a)$ is a meromorphic function with a simple pole at $s = 1$ whose residue is 1 (see, for example, [1, Section 12]). The periodic zeta function $\mathrm{Li}_s(e^{2\pi i a})$ is defined by

$$\mathrm{Li}_s(e^{2\pi i a}) := \sum_{n=1}^{\infty} \frac{e^{2\pi i n a}}{n^s}, \qquad \sigma > 1, \quad 0 < a \le 1$$

(see, for instance, [1, Exercise 12.2]). Note that the function $\mathrm{Li}_s(e^{2\pi i a})$ with $0 < a < 1$ is analytically continuable to the whole complex plane since $\mathrm{Li}_s(e^{2\pi i a})$ does not have any pole, that is shown by the fact that the Dirichlet series of $\mathrm{Li}_s(e^{2\pi i a})$ converges uniformly in each compact subset of the half-plane $\sigma > 0$ when $0 < a < 1$ (see, for example, [9, p. 20]). For $0 < a \le 1/2$, we define zeta functions

$$Z(s,a) := \zeta(s,a) + \zeta(s,1-a), \qquad P(s,a) := \mathrm{Li}_s(e^{2\pi i a}) + \mathrm{Li}_s(e^{2\pi i(1-a)}),$$
$$2Q(s,a) := Z(s,a) + P(s,a), \qquad \xi_Q(s,a) := s(s-1)\pi^{-s/2}\Gamma(s/2)Q(s,a).$$

We can see that $Q(s,a)$ is meromorphic functions with a simple pole at $s = 1$. In addition, we have $Q(0,a) = -1/2 = \zeta(0)$ and $\xi_Q(s,a) = \xi_Q(1-s,a)$, which is proved by

$$Q(1-s,a) = \Gamma_{\cos}(s)Q(s,a), \qquad \Gamma_{\cos}(s) := \frac{2\Gamma(s)}{(2\pi)^s}\cos\left(\frac{\pi s}{2}\right) \tag{1.1}$$

(see [13, Theorem 1.1]). Moreover, the function $Q(s,a)$ has the following properties. When $a = 1/6, 1/4, 1/3$, and $1/2$, the RH holds true if and only if all nonreal zeros of $Q(s,a)$ are on the line $\mathrm{Re}(s) = 1/2$ (see [14, Proposition 1.3]). Let $N_Q^{\mathrm{CL}}(T)$ be the number of the zeros of $Q(s,a)$ on the line segment from $1/2$ to $1/2 + iT$. In [13, Theorem 1.2], the third author proved that for any $0 < a \le 1/2$, there exist positive constants $A(a)$ and $T_0(a)$ such that

$$N_Q^{\mathrm{CL}}(T) \ge A(a)T \quad \text{whenever} \quad T \ge T_0(a).$$

Next, let $N_F(T)$ count the number of nonreal zeros of a function $F(s)$ having $|\mathrm{Im}(s)| < T$. Then, for any $0 < a \le 1/2$,

$$N_\zeta(T) - N_Q(T) = O_a(T),$$

and the third author [14, Proposition 1.8] proved that

$$N_Q(T) = \frac{T}{\pi}\log T - \frac{T}{\pi}\log(2e\pi a^2) + O_a(\log T).$$

Furthermore, he [14, Theorem 1.1] proved that there is a unique absolute $a_0 \in (0,1/2)$ such that

$$Q(1/2,a) > 0 \iff 0 < a < a_0.$$

In addition, it is proved in [14, Corollary 1.2] that all real zeros of $Q(s,a)$ are simple and are located only at the negative even integers just like $\zeta(s)$ if and only if $a_0 < a \le 1/2$. Let us note by Z_Q the set of all nontrivial zeros ρ_a of $\xi_Q(s,a)$. Since it is an entire function of order 1, one has

$$\xi_Q(s,a) = e^{A+Bs}\prod_{\rho_a\in Z_Q}\left(1-\frac{s}{\rho_a}\right)e^{\frac{s}{\rho_a}} = \xi_Q(0,a)\prod_{\rho_a\in Z_Q}\left(1-\frac{s}{\rho_a}\right), \tag{1.2}$$

where $e^A = 1/2$, $B = \frac{Q'}{Q}(0,a) - 1 - \frac{\gamma+\log\pi}{2}$, and γ denotes the Euler constant. Note that $Q'(0,a)$ is given explicitly in [14, Theorem 1.5].

1.3 Main results

Recall that $\zeta(1-s) = \Gamma_{\cos}(s)\zeta(s)$ and $Q(1-s,a) = \Gamma_{\cos}(s)Q(s,a)$ by (1.1). However, the function $Q(s,a)$ does not have an Euler product except for $a = 1/6, 1/4, 1/3$, and $1/2$. Hence, the function $Q(s,a)$ is a suitable object to consider the influence of not Riemann's functional equation but an Euler product to zeros of zeta functions. We show a criterion for nonvanishing of $Q(s,a)$ in terms of the positivity of the Li coefficients, an arithmetic and asymptotic formula for these coefficients in Theorems 1.1, 1.2, and 1.4, respectively. It should be emphasized that $\lambda_{n,a}$ defined in (1.3) are the first Li coefficients that we can explicitly give $n \in \mathbb{N}$ such that $\lambda_{n,a} < 0$. There is a possibility that this fact would give an idea to find negative Li coefficients for $\zeta(s)$ if they would exist.

For $n \neq 0$, the Li coefficients attached to $Q(s,a)$ nonvanishing at zero are defined by the sum

$$\lambda_{n,a} := \sum_{\rho_a\in Z_Q}^{*}\left(1-\left(1-\frac{1}{\rho_a}\right)^n\right) = \lim_{T\longmapsto\infty}\sum_{|\mathrm{Im}(\rho_a)|\le T}^{*}\left(1-\left(1-\frac{1}{\rho_a}\right)^n\right).$$

The symmetry $\rho_a \longmapsto 1-\rho_a$ in the set Z_Q of nontrivial zeros of $Q(s,a)$ implies that $\lambda_{-n,a} = \overline{\lambda_{n,a}} = \lambda_{n,a}$ for all $n \in \mathbb{N}$. So, $\lambda_{n,a}$ are real. We have also

$$\lambda_{n,a} := \frac{1}{(n-1)!}\frac{d^n}{ds^n}\left[s^{n-1}\log\xi_Q(s,a)\right]_{s=1}. \tag{1.3}$$

Moreover, from (1.2), we have (see [4, Equations (2.3) and (2.4)] or [17, Appendix A])

$$\sum_{n=0}^{\infty}\lambda_{n+1,a}s^n = \frac{d}{ds}\log\left[\xi_Q\left(\frac{1}{1-s},a\right)\right].$$

As an analogue of Li coefficients for the Riemann zeta function, we have the following.

Theorem 1.1 *The function* $Q(s,a)$ *does not vanish when* $\mathrm{Re}(s) > 1/2$ *if and only if* $\lambda_{n,a} \geq 0$ *for all* $n \in \mathbb{N}$.

An arithmetic formula for $\lambda_{n,a}$ is stated in the following theorems.

Theorem 1.2 *We have*

$$\lambda_{n,a} = 1 - \frac{n}{2}(\log(4\pi) + \gamma) + \sum_{k=2}^{n}(-1)^k \binom{n}{k}\left(1 - 2^{-k}\right)\zeta(k) + \sum_{k=1}^{n}\binom{n}{k}\gamma_Q(k-1),$$

where $\gamma_Q(n)$ *are defined as follows:*

$$\frac{Q'}{Q}(s+1,a) + \frac{1}{s} = \sum_{n=0}^{\infty}\gamma_Q(n)s^n.$$

Theorem 1.3 *For* $a = 1/2, 1/3, 1/4, 1/6$, *under the RH, we have*

$$\lambda_{n,a} = \frac{n}{2}\log n + \frac{n}{2}(\gamma - 1 - \log 2\pi) + O(\sqrt{n}\log n).$$

For a fixed $l \in \mathbb{N}$, we have the following asymptotic formula of $\lambda_{l,a}$ when $a \to +0$. We can see that there exists $n \in \mathbb{N}$ such that $\lambda_{n,a} < 0$ by Theorem 1.1 and the fact that $Q(s,a)$ does not satisfy an analogue of the RH when $a \in \mathbb{Q} \cap (0,1/2) \setminus \{1/6, 1/4, 1/3\}$ (see [14, Proposition 1.4]). Clearly, this argument gives no information on the frequency of $n \in \mathbb{N}$, the smallest $n \in \mathbb{N}$ such that $\lambda_{n,a} < 0$ and so on. However, the next theorem implies that $\lambda_{2n,a} < 0$ if we fix any $n \in \mathbb{N}$ and then we take $a > 0$ sufficiently small.

Theorem 1.4 *Fix* $l \in \mathbb{N}$. *Then it holds that*

$$\lambda_{l,a} = \frac{(-1)^{l+1}}{(2a)^l} + O_l\left(a^{1-l}|\log a|\right), \qquad a \to +0.$$

Especially, for any fixed $n \in \mathbb{N}$, *there are* $a > 0$ *such that*

$$\lambda_{2n-1,a} > 0 \quad \text{and} \quad \lambda_{2n,a} < 0.$$

2 Proofs

2.1 Proof of Theorem 1.1

Since $\lambda_{-n,a} = \overline{\lambda_{n,a}} = \lambda_{n,a}$ for all $n \in \mathbb{N}$, then $\mathrm{Re}(\lambda_{-n,a}) = \mathrm{Re}(\lambda_{n,a}) = \lambda_{n,a}$. Using that $\xi_Q(s,a)$ is an entire function of order 1, and its zeros lie in the critical strip $0 < \mathrm{Re}(s) < 1$, we obtain that the series $\sum_{\rho \in Z_Q} \frac{1+|\mathrm{Re}(\rho)|}{(1+|\rho|)^2}$ is convergent. Application of [4, Theorem 1] to the multiset Z_Q of zeros of $Q(s,a)$ gives that $\mathrm{Re}(\rho) \leq 1/2$ if and only if $\lambda_{n,a} \geq 0$ for all $n \in \mathbb{N}$. Now, the application of the same theorem to the multiset $1 - Z_Q = Z_Q$ gives $\mathrm{Re}(\rho) \geq 1/2$ if and only if $\lambda_{n,a} \geq 0$. This completes the proof.

Theorem 1.1 can also be proved by the same argument used in [5, Theorem 1], which is due to Oesterlé.

2.2 Proof of Theorem 1.2

From the expression of $\xi_Q(s,a)$, one has

$$\frac{\xi_Q'}{\xi_Q}(s,a)=\frac{1}{s}+\frac{1}{s-1}-\frac{1}{2}\log\pi+\frac{1}{2}\frac{\Gamma'}{\Gamma}(s/2)+\frac{Q'}{Q}(s,a),$$

which is rewritten as

$$\frac{\xi_Q'}{\xi_Q}(s+1,a)=\frac{1}{s+1}+\frac{1}{s}-\frac{1}{2}\log\pi+\frac{1}{2}\frac{\Gamma'}{\Gamma}((s+1)/2)+\frac{Q'}{Q}(s+1,a). \tag{2.1}$$

Note that $Q(s,a)$ is a meromorphic function on the whole complex plane, which is holomorphic everywhere except for a simple pole at $s=1$ with residue 1 (see [13, Section 2.1]). Let us define the coefficients $\gamma_Q(n)$ and $\tau_Q(n)$ as follows:

$$\frac{Q'}{Q}(s+1,a)+\frac{1}{s}=\sum_{n=0}^{\infty}\gamma_Q(n)s^n \tag{2.2}$$

and

$$-\frac{1}{2}\log\pi+\frac{1}{2}\frac{\Gamma'}{\Gamma}((s+1)/2)=\sum_{n=0}^{\infty}\tau_Q(n)s^n. \tag{2.3}$$

By Equation (1.2), one has

$$\log\xi_Q(s,a)=\log\xi_Q(0,a)-\sum_{\rho_a\in Z_Q}\sum_{m=1}^{\infty}\frac{1}{m\rho^m}s^m.$$

From the functional equation for the function $\xi_Q(s,a)$, in the neighborhood of $s=0$, we have

$$\frac{\xi_Q'}{\xi_Q}(s+1,a)=-\frac{\xi_Q'}{\xi_Q}(-s,a)=\sum_{m=0}^{\infty}(-1)^m\sum_{\rho_a\in Z_Q}\frac{1}{\rho^{m+1}}s^m. \tag{2.4}$$

Comparing Equations (2.1)–(2.4), we get

$$(-1)^m\sum_{\rho_a\in Z_Q}\frac{1}{\rho^{m+1}}=(-1)^m+\gamma_Q(m)+\tau_Q(m),$$

for $m\geq 0$. Hence, the definition of $\lambda_{n,a}$ yields

$$\lambda_{n,a}=\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\sum_{\rho_a\in Z_Q}\frac{1}{\rho^k}=1+\sum_{k=1}^{n}\binom{n}{k}\gamma_Q(k-1)+\sum_{k=1}^{n}\binom{n}{k}\tau_Q(k-1),$$

where

$$\tau_Q(0)=-\frac{1}{2}\log\pi+\frac{1}{2}\psi(1/2)\text{ and }\tau_Q(k-1)=(-1)^k\sum_{m=0}^{\infty}\frac{1}{(2m+1)^k}$$

using that $\psi(z) = -\gamma - \frac{1}{z} + \sum_{k=1}^{\infty} \frac{z}{k(k+z)}$. Here, $\psi(s) = \frac{\Gamma'}{\Gamma}(s)$ is the logarithmic derivative of the Gamma function. Since $\psi(1/2) = -\gamma - 2\log 2$, we obtain

$$\begin{aligned}\lambda_{n,a} &= 1 - \frac{n}{2}(\log(4\pi) + \gamma) + \sum_{k=2}^{n}(-1)^k \binom{n}{k} \sum_{m=0}^{\infty} \frac{1}{(2m+1)^k} + \sum_{k=1}^{n} \binom{n}{k} \gamma_Q(k-1) \\ &= 1 - \frac{n}{2}(\log(4\pi) + \gamma) + \sum_{k=2}^{n}(-1)^k \binom{n}{k} \left(1 - 2^{-k}\right) \zeta(k) + \sum_{k=1}^{n} \binom{n}{k} \gamma_Q(k-1).\end{aligned}$$

The equality above implies Theorem 1.2.

2.3 Proof of Theorem 1.3

Let us note that

$$\sum_{k=2}^{n}(-1)^k \binom{n}{k} \left(1 - 2^{-k}\right) \zeta(k) = \sum_{k=2}^{n}(-1)^l \binom{n}{k} \frac{\zeta(k, 1/2)}{2^k},$$

where $\zeta(s, a)$ is the Hurwitz zeta function defined in Section 1.2. With the notation of Flajolet and Vespas [7, Lines 2–4, p. 70], this is $A_n(1,2)$ and which is equal to

$$\frac{n}{2}\psi(n) + n\left(\gamma - \frac{1}{2} + \frac{1}{2}\log 2\right) + o(1),$$

where the o(1) error term above is exponentially small and oscillating and equal to

$$\frac{1}{2}\left(\frac{n}{\pi}\right)^{1/4} \exp(-\sqrt{2\pi n}) \cos\left(\sqrt{2\pi n} - \frac{5\pi}{8}\right) + O\left(n^{-1/4} e^{-\sqrt{2\pi n}}\right).$$

Then we have

$$\lambda_{n,a} = \frac{n}{2}\log n + \frac{n}{2}(\gamma - 1 - \log 2\pi) + \sum_{k=1}^{n} \binom{n}{k} \gamma_Q(k-1) + O(1).$$

It remains to prove that

$$\sum_{k=1}^{n} \binom{n}{k} \gamma_Q(k-1) = O(\sqrt{n}\log n). \tag{2.5}$$

To do so, we follow very closely the lines of the proof of the corresponding result in [8, Theorem 6.1] or [16, Lemma 3.3] and it will be shortened. We use the following kernel function:

$$k_n(s) := \left(1 + \frac{1}{s}\right)^n - 1 = \sum_{k=1}^{n} \binom{n}{k} \frac{1}{s^k}.$$

The residue theorem gives

$$\sum_{k=1}^{n} \binom{n}{k} \gamma_Q(k-1) = \frac{1}{2i\pi} \int_C k_n(s) \left(-\frac{Q'}{Q}(s+1, a)\right) ds,$$

where C is a contour enclosing the point $s = 0$ counterclockwise on a circle of small enough positive radius. The residue comes entirely from the singularity at $s = 0$, as no other singularities lie inside the contour. Let $T = \sqrt{n} + \varepsilon_n$, for some $0 < \varepsilon_n < 1$.

Now we follow very closely the lines in [16, pp. 1106–1107] using that the function $\frac{Q'}{Q}(s,a)$ satisfies the properties[1]

$$\frac{Q'}{Q}(s,a) = \sum_{\rho_a;\ |\mathrm{Im}(\rho_a - s)|<1} \frac{1}{s-\rho_a} + O(\log(1+|s|)),$$

for $-2 < \mathrm{Re}(s) < 2$ and

$$\left|\frac{Q'}{Q}(s+1,a)\right| = O(\log^2 T),$$

for $-2 \le \mathrm{Re}(s) \le 2$, and we get

$$\sum_{k=1}^{n} \binom{n}{k} \gamma_Q(k-1) = \lambda_{-n,a,T} + O(\sqrt{n}\log n),$$

where

$$\lambda_{-n,a,T} = \sideset{}{^*}\sum_{\rho_a \in Z_Q; |Im(\rho_a| \le T} \left(1 - \left(1 - \frac{1}{\rho_a}\right)^n\right),$$

with $T = \sqrt{n} + \varepsilon_n$. For $a = 1/2, 1/3, 1/4, 1/6$, under the RH, since $\left|1 - \frac{1}{\rho_a}\right| = 1$ and using formula of $N_Q(T)$ given in Section 1.2, we obtain $\lambda_{n,a,T} = O(T\log T + 1)$. Therefore, Equation (2.5) follows from that $\lambda_{-n,a,\sqrt{n}} = \lambda_{-n,a,\sqrt{n}} = O(\sqrt{n}\log n)$.

Remark Since $2Q(s,a) := Z(s,a) + P(s,a)$, from Corollary 2.3 below and [6, Equation (1.18)], we obtain

$$\gamma_Q(n) = \frac{1}{2}\left(\delta_n(a) + \frac{(-1)^n}{n!}(l_n(a) + l_n(1-a))\right),$$

where $\delta_n(a) = \frac{|\log a|^n}{an!} + O(1)$ and $l_n(a)$ are the coefficients in the expansion of $\mathrm{Li}_s(e^{2\pi i a})$ at $s = 1$; for $a \notin \mathbb{Z}$, one has

$$\mathrm{Li}_s(e^{2\pi i a}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} l_n(a)(s-1)^n.$$

2.4 Proof of Theorem 1.4

To show Theorem 1.4, we quote the following lemmas from [2, 3].

Lemma 2.1 [3, Theorem 1] *We set*

$$(s-1)\zeta(s,a) = 1 + \sum_{n=0}^{\infty} \gamma_n(a)(s-1)^{n+1}, \qquad 0 < a \le 1.$$

[1] These properties are well known for the Riemann zeta function. The proof for the function $Q(s,a)$ is exactly the same since the Riemann–von Mangoldt formula holds for $Q(s,a)$ (see [14, Proposition 2.5] or [18, p. 217]).

Then it holds that

$$\gamma_n(a) = \frac{(-1)^n}{n!} \lim_{m\to\infty} \left(\sum_{k=0}^{m} \frac{\log^n(k+a)}{k+a} - \frac{\log^{n+1}(m+a)}{n+1} \right).$$

Lemma 2.2 [2, Equation (26)] *Let $0 < a \leq 1$, and let n be a nonnegative integer. Then one has*

$$\begin{aligned}\zeta^{(n)}(0,a) = \left(\frac{1}{2} - a\right)|\log a|^n - n! + n!a \sum_{m=n}^{\infty} \frac{|\log a|^m}{m!} \\ + (-1)^n n \int_0^\infty \frac{\varphi(x)\log^{n-1}(x+a)}{(x+a)^2} dx \\ - (-1)^n n(n-1) \int_0^\infty \frac{\varphi(x)\log^{n-2}(x+a)}{(x+a)^2} dx,\end{aligned}$$

where $\varphi(x) = \int_0^x (y - \lfloor y \rfloor - 1/2) dy$ is periodic with period 1 and satisfies $2\varphi(x) = x(x-1)$ if $0 \leq x \leq 1$.

By using the lemmas above, we immediately obtain the following.

Corollary 2.3 *When $a > 0$ is sufficiently small,*

$$(s-1)Z(s,a) = 2 + \sum_{n=0}^{\infty} \delta_n(a)(s-1)^{n+1}, \qquad \delta_n(a) = \frac{|\log a|^n}{an!} + O(1),$$

$$Z(s,a) = \sum_{n=1}^{\infty} \varepsilon_n(a)s^n, \qquad \varepsilon_n(a) = O(|\log a|^n).$$

Proof The first formula and estimation are easily proved by Lemma 2.1 (see also [3, Theorem 2]). For the first integral in Lemma 2.2, one has

$$\int_0^1 \frac{\varphi(x)\log^{n-1}(x+a)}{(x+a)^2} dx \ll \int_0^1 \frac{\log^{n-1}(x+a)}{x+a} dx = O(|\log a|^n),$$

$$\int_1^\infty \frac{\varphi(x)\log^{n-1}(x+a)}{(x+a)^2} dx \ll \int_1^\infty \frac{\log^{n-1}(x+a)}{(x+a)^2} dx = O(1)$$

from $x < x + a$ when $x, a > 0$. In addition, we have

$$a \sum_{m=n}^{\infty} \frac{|\log a|^m}{m!} \leq a \sum_{m=0}^{\infty} \frac{|\log a|^m}{m!} = ae^{|\log a|} = ae^{-\log a} = 1, \qquad 0 < a < 1/2.$$

Hence, we obtain

$$\zeta(s,a) = \sum_{n=0}^{\infty} \frac{\zeta^{(n)}(0,a)}{n!} s^n, \qquad \zeta^{(n)}(0,a) = O(|\log a|^n).$$

Therefore, we have $\varepsilon_n(a) = O(|\log a|^n)$ and the second formula in this corollary by the definition of $Z(s,a)$ and $Z(0,a) = \zeta(0,a) + \zeta(0, 1-a) = 0$ (see [15, Equation (4.11)]). ■

Proof of Theorem 1.4 Recall the functional equation

$$Z(1-s,a)=\Gamma_{\cos}(s)P(s,a), \qquad \Gamma_{\cos}(s):=\frac{2\Gamma(s)}{(2\pi)^s}\cos\Big(\frac{\pi s}{2}\Big)$$

(see [15, Lemma 4.11]). By using $\Gamma_{\cos}(s)\Gamma_{\cos}(1-s)=1$, we have

$$2Q(s,a)=Z(s,a)+P(s,a)=Z(s,a)+\Gamma_{\cos}(1-s)Z(1-s,a).$$

Let $|s-1|$ be sufficiently small. Then, by $\lim_{s\to1}(s-1)Q(s,a)=1$, the equation above, and the definitions of $Q(s,a)$ and $\xi_Q(s,a)$, we have

$$\begin{aligned}
\frac{d^l}{ds^l}\big[s^{l-1}\log\xi_Q(s,a)\big]_{s=1} &= \frac{d^l}{ds^l}\Big[s^{l-1}\log\big((s-1)Q(s,a)\big)+s^{l-1}\log\big(s\pi^{-s/2}\Gamma(s/2)\big)\Big]_{s=1}\\
&= \frac{d^l}{ds^l}\left[s^{l-1}\log\left(\frac{s-1}{2}\Big(Z(s,a)+\Gamma_{\cos}(1-s)Z(1-s,a)\Big)\right)\right]_{s=1}+O_l(1)\\
&= \frac{d^l}{ds^l}\left[s^{l-1}\log\left(1+\sum_{n=0}^{\infty}\big(\delta'_n(a)+\varepsilon'_n(a)\big)(s-1)^{n+1}\right)\right]_{s=1}+O_l(1),
\end{aligned}$$

where $\delta'_n(a)$ and $\varepsilon'_n(a)$ are defined by

$$\delta'_n(a):=\frac{\delta_n(a)}{2}, \qquad (s-1)\Gamma_{\cos}(1-s)Z(1-s,a)=2\sum_{n=0}^{\infty}\varepsilon'_n(a)(s-1)^{n+1}.$$

Clearly, the second estimation in Corollary 2.3 implies

$$Z(1-s,a)=\sum_{n=1}^{\infty}\varepsilon_n(a)(1-s)^n, \qquad \varepsilon_n(a)=O(|\log a|^n).$$

Thus, we can see that $\varepsilon'_n(a)=O(|\log a|^{n+1})$ from $\lim_{s\to1}(s-1)\Gamma_{\cos}(1-s)=-2$ and the fact that the function $(s-1)\Gamma_{\cos}(1-s)$ does not depend on a. Put $\eta_n(a):=\delta'_n(a)+\varepsilon'_n(a)$. Then, for $n\ge0$, we have

$$\eta_n(a)=\frac{1}{n!}\frac{|\log a|^n}{2a}+O(|\log a|^{n+1}), \qquad a\to+0 \tag{2.6}$$

by Corollary 2.3. By virtue of

$$\begin{aligned}
(a_0x+a_1x^2+a_2x^3+\cdots)^m &= a_0^m x^m+\binom{m}{1}a_0^{m-1}a_1x^{m+1}+\cdots\\
(a_0x+a_1x^2+a_2x^3+\cdots)^{m-1} &= a_0^m x^{m-1}+\binom{m-1}{1}a_0^{m-2}a_1x^{m}+\cdots\\
&\vdots\\
(a_0x+a_1x^2+a_2x^3+\cdots)^{1} &= \cdots+a_mx^m+\cdots,
\end{aligned}$$

where $m\in\mathbb{N}$ and $a_m,x\in\mathbb{C}$, the coefficient of $(s-1)^l$ in the function

$$f(s,a):=\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}\left(\sum_{n=0}^{\infty}\eta_n(a)(s-1)^{n+1}\right)^m$$

is expressed as

$$(2.7)\qquad \frac{(-1)^{l+1}}{l}\big(\eta_0(a)\big)^l + \frac{(-1)^l}{l-1}\binom{l-1}{1}\eta_0(a)^{l-2}\eta_1(a) + \cdots + \frac{(-1)^{1+1}}{1}\eta_{l-1}(a).$$

Note that the function above is estimated by

$$(2.8)\quad \frac{(-1)^{l+1}}{l}\big(\eta_0(a)\big)^l + O_l\big(\eta_0(a)^{l-2}\eta_1(a)\big) = \frac{(-1)^{l+1}}{l}(2a)^{-l} + O_l\big(a^{1-l}|\log a|\big)$$

from (2.6) when $a \to +0$. We can find that

$$(s-1)\Big(Z(s,a) + \Gamma_{\cos}(1-s)Z(1-s,a)\Big) = 1 + \sum_{n=0}^{\infty}\eta_n(a)(s-1)^{n+1}$$

is analytic when $|s-1| < 1$ form the poles of $Z(s,a)$ and $\Gamma_{\cos}(1-s)$. So we can choose $|s-1| > 0$ such that

$$\sum_{n=0}^{\infty}|\eta_n(a)||s-1|^{n+1} < \frac{1}{2}.$$

Then, from (2.7), the Leibniz product rule, the definition of $\eta_n(a)$, and the Taylor expansion of $\log(1+x)$ with $|x| < 1$, one has

$$\begin{aligned}
&\frac{d^l}{ds^l}\Big[s^{l-1}\log\xi_Q(s,a)\Big]_{s=1} = \frac{d^l}{ds^l}\Big[s^{l-1}f(s,a)\Big]_{s=1} + O_l(1)\\
(\flat)\quad &= \binom{l}{l}\frac{(-1)^{l+1}}{l}l!\big(\eta_0(a)\big)^l + O_l\big(\eta_0(a)^{l-2}\eta_1(a)\big)\\
(\natural)\quad &\quad + \binom{l}{l-1}(l-1)\frac{(-1)^l}{l-1}(l-1)!\big(\eta_0(a)\big)^{l-1} + O_l\big(\eta_0(a)^{l-3}\eta_1(a)\big)\\
(\sharp)\quad &\quad + \cdots + \binom{l}{1}(l-1)!\frac{(-1)^{1+1}}{1}\big(\eta_0(a)\big)^1 + O_l(1).
\end{aligned}$$

Note that $(\flat)$ comes from $f^{(l)}(s,a)$, $(\natural)$ is deduced by $f^{(l-1)}(s,a)$, and $(\sharp)$ derives from $f^{(1)}(s,a)$, $f^{(0)}(s,a)$, and $O_l(1)$ in the left-hand side of the formula above. Therefore, by (2.8), we obtain

$$\begin{aligned}
\frac{d^l}{ds^l}\Big[s^{l-1}\log\xi_Q(s,a)\Big]_{s=1} &= (-1)^{l+1}(l-1)!\big(\eta_0(a)\big)^l + O_l\big(\eta_0(a)^{l-2}\eta_1(a)\big)\\
&= (-1)^{l+1}\frac{(l-1)!}{(2a)^l} + O_l\big(a^{1-l}|\log a|\big),
\end{aligned}$$

which implies Theorem 1.4. ∎

At the end of the paper, we give numerical computation for $\lambda_{n,a}$ by Mathematica 13.0. Let

$$\lambda_{n,a}^{[k]} := \frac{1}{(n-1)!}\frac{d^n}{ds^n}\big[s^{n-1}\log\xi_Q(s,a)\big]_{s=1-10^{-k}}, \qquad \lambda_{n,a}^{*} := \frac{(-1)^{n+1}}{(2a)^n}.$$

Then, we have the following.

For $n = 1$, we have

$$\begin{array}{lll}
a := 2^{-17} & \lambda_{1,a}^{[10]} = 65,537\ldots & \lambda_{1,a}^{[10]}/\lambda_{1,a}^{*} = 1.00001\ldots \\
a := 2^{-18} & \lambda_{1,a}^{[10]} = 131,074\ldots & \lambda_{1,a}^{[10]}/\lambda_{1,a}^{*} = 1.00002\ldots \\
a := 2^{-19} & \lambda_{1,a}^{[10]} = 262,151\ldots & \lambda_{1,a}^{[10]}/\lambda_{1,a}^{*} = 1.00003\ldots \\
a := 2^{-17} & \lambda_{1,a}^{[11]} = 65,536.6\ldots & \lambda_{1,a}^{[11]}/\lambda_{1,a}^{*} = 1.00001\ldots \\
a := 2^{-18} & \lambda_{1,a}^{[11]} = 131,073\ldots & \lambda_{1,a}^{[11]}/\lambda_{1,a}^{*} = 1.00001\ldots \\
a := 2^{-19} & \lambda_{1,a}^{[11]} = 262,145\ldots & \lambda_{1,a}^{[11]}/\lambda_{1,a}^{*} = 1.00000\ldots \\
a := 2^{-17} & \lambda_{1,a}^{[12]} = 655,365\ldots & \lambda_{1,a}^{[12]}/\lambda_{1,a}^{*} = 1.00001\ldots \\
a := 2^{-18} & \lambda_{1,a}^{[12]} = 131,073\ldots & \lambda_{1,a}^{[12]}/\lambda_{1,a}^{*} = 1.00000\ldots \\
a := 2^{-19} & \lambda_{1,a}^{[12]} = 262,145\ldots & \lambda_{1,a}^{[12]}/\lambda_{1,a}^{*} = 1.00000\ldots
\end{array}$$

For $n = 2$, we have

$$\begin{array}{lll}
a := 2^{-17} & \lambda_{2,a}^{[10]} = -4.29352\ldots \times 10^{9} & \lambda_{2,a}^{[10]}/\lambda_{2,a}^{*} = 0.999663\ldots \\
a := 2^{-18} & \lambda_{2,a}^{[10]} = -1.7177\ldots \times 10^{10} & \lambda_{2,a}^{[10]}/\lambda_{2,a}^{*} = 0.999836\ldots \\
a := 2^{-19} & \lambda_{2,a}^{[10]} = -6.87162\ldots \times 10^{10} & \lambda_{2,a}^{[10]}/\lambda_{2,a}^{*} = 0.999952\ldots \\
a := 2^{-17} & \lambda_{2,a}^{[11]} = -4.29478\ldots \times 10^{9} & \lambda_{2,a}^{[11]}/\lambda_{2,a}^{*} = 0.999956\ldots \\
a := 2^{-18} & \lambda_{2,a}^{[11]} = -1.71753\ldots \times 10^{10} & \lambda_{2,a}^{[11]}/\lambda_{2,a}^{*} = 0.999736\ldots \\
a := 2^{-19} & \lambda_{2,a}^{[11]} = -6.87149\ldots \times 10^{10} & \lambda_{2,a}^{[11]}/\lambda_{2,a}^{*} = 0.999933\ldots \\
a := 2^{-17} & \lambda_{2,a}^{[12]} = -4.29477\ldots \times 10^{9} & \lambda_{2,a}^{[12]}/\lambda_{2,a}^{*} = 0.999955\ldots \\
a := 2^{-18} & \lambda_{2,a}^{[12]} = -1.6911\ldots \times 10^{10} & \lambda_{2,a}^{[12]}/\lambda_{2,a}^{*} = 0.984353\ldots \\
a := 2^{-19} & \lambda_{2,a}^{[12]} = -6.87187\ldots \times 10^{10} & \lambda_{2,a}^{[12]}/\lambda_{2,a}^{*} = 0.999989\ldots
\end{array}$$

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